Prelim Exam – Spring 2008 – Problem I.2

October 22, 2014

Does there exist a simple group G of order 336 such that for some positive integer d, G has exactly 6 (distinct) subgroups of order d?

Solution. No! Suppose there is, and let

$$\mathcal{S} \stackrel{\text{def}}{=} \{H \leq G \;:\; |H| = d\}.$$

Then, by assumption, $|\mathcal{S}| = 6$. We have that G acts on $\mathcal{S}$ by conjugation, as for $H \in \mathcal{S}$ and $g \in G$, we have $|gHg^{-1}| = |H| = d$.

Thus, we have a [permutation] representation:

$$\phi : G \to S_6 = \mathrm{Sym}(\mathcal{S})$$

Since G is simple and $\ker\phi \trianglelefteq G$, we have that $\ker\phi = 1$ or $\ker\phi = G$.

If $\ker\phi = G$, then for all $g \in G$ and for any $H \in \mathcal{S}$, we have $gHq^{-1} = H$. But this would mean that $H \trianglelefteq G$, and hence $d = 1$ or $d = |G| = 336$. In either case, we have a contradiction, as there is only one subgroups of order 1 [namely $\{1\}$] and one subgroup of order 336 [namely, G itself].

Hence, $\ker\phi = 1$, and hence ϕ is injective. But then, $336 = |G| \mid |S_6| = 6!$ [by the *First Isomorphism Theorem*], which is a contradiction, as $7 \mid 336$, but $7 \nmid 6!$.

So, there can be no d as above. □