1) Let $I = \{-1, 0, 1, 2\}$ and $A_i = \{0, i, i^2, -i^3\}$.

(a) [5 points] Write out explicitly the sets A_i for $i \in I$.

Solution. We have:

$$\begin{aligned} A_{-1} &= \{-1, 0, 1\}, \\ A_0 &= \{0\}, \\ A_1 &= \{-1, 0, 1\}, \\ A_2 &= \{-8, 0, 2, 4\}. \end{aligned}$$

□

(b) [5 points] Compute $\bigcup_{i \in I} A_i$.

Solution. We have $\bigcup_{i \in I} A_i = \{-8, -1, 0, 1, 2, 4\}$. □

(c) [5 points] Compute $\bigcap_{i \in I} A_i$.

Solution. We have $\bigcap_{i \in I} A_i = \{0\}$. □

2) [15 points] Express $\exists! x \in A\ (P(x))$ without using $\exists!$.

Solution. We have

$$\exists! x \in A\ (P(x)) \sim \exists x \in A\, [P(x) \wedge \forall y \in A\, (P(y) \to x = y)]\,.$$

□

3) [15 points] Show that

$$\exists x\,(P(x) \to Q(x)) \sim [\forall x(P(x))] \to [\exists x(Q(x))]\,.$$

Solution. We have:

$$\begin{aligned}
\exists x\,(P(x) \to Q(x)) &\sim \exists x\,(\neg P(x) \vee Q(x)) \\
&\sim \exists x\,(\neg P(x)) \vee \exists x\,(Q(x)) \\
&\sim \neg\,[\forall x\,(\neg\neg P(x))] \vee \exists x\,(Q(x)) \\
&\sim \neg\,[\forall x\,(P(x))] \vee \exists x\,(Q(x)) \\
&\sim [\forall x\,(P(x))] \to [\exists x\,(Q(x))]\,.
\end{aligned}$$

□

4) Analyze the following statements:

(a) [10 points] Someone likes a person who doesn't like anyone.

Solution. Let $L(x, y)$ be "x likes y". Then, the statement is:

$$\exists x\,[\exists y\,(L(x, y) \wedge (\forall z(\neg L(y, z))))]\,.$$

□

(b) [10 points] No father is happy unless all his children are happy.

Solution. Let $F(x, y)$ be "x is y's father" and $H(x)$ be "x is happy". Then, the statement is:

$$\forall x\,[\forall y\,(F(x, y) \to (\neg H(y) \to \neg H(x)))]\,.$$

□

5) Let $A = \{x \in \mathbb{R} \mid x > 0\}$ and consider the statement

$$\neg\left[\forall \epsilon \in A \left(\exists \delta \in A \left(\forall x(|x-a| < \delta \to |x^n - L| < \epsilon)\right)\right)\right].$$

[**Note:** Although irrelevant for your solution, but simply for your information, the statement above is the negation of $\lim_{x \to a} x^n = L$.]

(a) [5 points] State which are the free and which are the bound variables of the statement.

Solution. ϵ, δ and x are bound. a, n and L are free. [A is given, so it is not a variable.] □

(b) [15 points] Reexpress the statement as its equivalent positive statement.

Solution. We have:

$$\begin{aligned}
&\neg\left[\forall \epsilon \in A \left(\exists \delta \in A \left(\forall x(|x-a| < \delta \to |x^n - L| < \epsilon)\right)\right)\right] \\
&\quad \sim \exists \epsilon \in A \neg \left(\exists \delta \in A \left(\forall x(|x-a| < \delta \to |x^n - L| < \epsilon)\right)\right) \\
&\quad \sim \exists \epsilon \in A \left(\forall \delta \in A \neg \left(\forall x(|x-a| < \delta \to |x^n - L| < \epsilon)\right)\right) \\
&\quad \sim \exists \epsilon \in A \left(\forall \delta \in A \left(\exists x \neg(|x-a| < \delta \to |x^n - L| < \epsilon)\right)\right) \\
&\quad \sim \exists \epsilon \in A \left(\forall \delta \in A \left(\exists x(|x-a| < \delta \wedge |x^n - L| \geq \epsilon)\right)\right).
\end{aligned}$$

□

6) [15 points] Analyze the logical form of the following statement. [You may use $\in$, $\notin$, $=$, $\neq$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$, but not $\subseteq$, $\nsubseteq$, $\mathscr{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$ or $\neg$.]

$$\bigcap_{i\in I} A_i \subseteq \mathscr{P}(B) \setminus \bigcup_{i\in I} C_i.$$

Solution. We have:

$$\begin{aligned}
\bigcap_{i\in I} A_i \subseteq \mathscr{P}(B) \setminus \bigcup_{i\in I} C_i &\sim \forall x \left[\left(x \in \bigcap_{i\in I} A_i\right) \rightarrow \left(x \in \mathscr{P}(B) \setminus \bigcup_{i\in I} C_i\right)\right] \\
&\sim \forall x \left[(\forall i \in I(x \in A_i)) \rightarrow \left(x \in \mathscr{P}(B) \wedge \neg \left(x \in \bigcup_{i\in I} C_i\right)\right)\right] \\
&\sim \forall x\,[(\forall i \in I(x \in A_i)) \rightarrow (x \subseteq B \wedge \neg\,(\exists i \in I(x \in C_i)))] \\
&\sim \forall x\,[(\forall i \in I(x \in A_i)) \rightarrow ((\forall y \in x(y \in B)) \wedge (\forall i \in I(x \notin C_i)))]
\end{aligned}$$

□