1) [12 points] Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that

$$\left(\bigcup \mathcal{F}\right) \setminus \left(\bigcup \mathcal{G}\right) \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G}) .$$

Partial credit: If you can't do this or are stuck, I will give some credit [10 points] for the definitions of $x \in X \setminus Y$, $x \in \bigcup \mathcal{F}$ and $\neg(x \in \bigcup \mathcal{G})$.

Proof. Let $x \in (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$. Then, $x \in \bigcup \mathcal{F}$ and $x \notin \bigcup \mathcal{G}$. The former means that there is $A \in \mathcal{F}$ such that $x \in A$. The latter means that for all $B \in \mathcal{G}$, we have that $x \notin B$. So, we have that $A \notin \mathcal{G}$ [as $x \in A$]. Hence, $A \in \mathcal{F} \setminus \mathcal{G}$. Thus, $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$. □

2) [12 points] Suppose R is a partial order on A, $B_1 \subseteq A$, $B_2 \subseteq A$, x_1 the least upper bound of B_1, and x_2 the least upper bound of B_2. Prove that if $B_1 \subseteq B_2$, then $x_1 R x_2$ [or $x_1 \preccurlyeq x_2$, as I usually write for ordering relations].

[**Hint:** Prove that x_2 is an upper bound of B_1.]

Partial credit: If you can't do this or are stuck, I will give some credit [10 points] for the definitions of upper bound and least upper bound.

[This was a homework problem.]

Proof. Let $y \in B_1$. Since $B_1 \subseteq B_2$, we have that $y \in B_2$. Also, as x_2 is an upper bound of B_2, we have that $y \preccurlyeq x_2$. Since $y \in B_1$ was arbitrary, x_2 is an upper bound of B_1.

Now, x_1 is the *least* upper bound of B_1 and x_2 is an upper bound of B_1, so $x_1 \preccurlyeq x_2$.

□

3) [12 points] Let R be an equivalence relation on a set A. Prove that $[x] \subseteq [y]$ iff xRy. [Remember that $[a]$ denotes the equivalence class of a.]
[This was done in class.]
Partial credit: If you can't do this or are stuck, I will give half credit [10 points] for the definitions of equivalence relation and equivalence class.

Proof. [$\rightarrow$] Suppose $[x] \subseteq [y]$. Since R is reflexive [as R is an equivalence relation], we have that xRx and hence $x \in [x]$. So, since $[x] \subseteq [y]$, we get $x \in [y]$. By definition of equivalence class, this means that xRy.
[$\leftarrow$] Suppose that xRy and let $a \in [x]$. Then, aRx. Since R is transitive [as it is an equivalence relation] and we have aRx and xRy, we have that aRy. By definition of equivalence class again, we have $a \in [y]$. So, $[x] \subseteq [y]$. $\square$

4) [12 points] Let $f : A \to C$ and $g : B \to C$. Prove that if A and B are disjoint, then $(f \cup g) : A \cup B \to C$.
[This was a homework problem.]
Partial credit: If you can't do this or are stuck, I will give some credit [10 points] for the definition of a function.

Proof. Let $x \in A \cup B$. [We need to show that there is a unique $y \in C$ such that $(x, y) \in f \cup g$.] Then, $x \in A$ or $x \in B$.
Case 1: [Existence] Assume $x \in A$, since $f : A \to C$, there is $y \in C$ such that $(x, y) \in f$, so $(x, y) \in f \cup g$.
[Uniqueness] If also $(x, y') \in f \cup g$, then $(x, y') \in f$ or $(x, y') \in g$. If $(x, y') \in g$, then $x \in B$ [as $g : B \to C$], which is impossible as $x \in A$ and $A \cap B = \varnothing$. So, $(x, y') \in f$. Since $f : A \to C$, and $(x, y), (x, y') \in f$, we have that $y = y'$.
Case 2: [Existence] Assume $x \in B$, since $g : B \to C$, there is $y \in C$ such that $(x, y) \in g$, so $(x, y) \in f \cup g$.
[Uniqueness] If also $(x, y') \in f \cup g$, then $(x, y') \in f$ or $(x, y') \in g$. If $(x, y') \in f$, then $x \in A$ [as $f : A \to C$], which is impossible as $x \in B$ and $A \cap B = \varnothing$. So, $(x, y') \in g$. Since $g : B \to C$, and $(x, y), (x, y') \in g$, we have that $y = y'$. $\square$

5) [13 points] Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is onto, then g is onto. [This was done in class.]

Partial credit: If you can't do this or are stuck, I will give some credit [10 points] for the definition of an onto function.

Proof. Let $c \in C$. [Need $b \in B$ such that $g(b) = c$.] Since $g \circ f : A \to C$ is onto, there is $a \in A$ such that $g \circ f(a) = c$, i.e., $g(f(a)) = c$. Since $f(a) \in B$ we have that $g(b) = c$ for $b = f(a)$. $\square$

6) [13 points] Prove that for any $n \in \mathbb{Z}_{\geq 1}$ we have

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof. We prove it by induction on n.

For $n = 1$ we have that:

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \frac{1^2 \cdot 2^2}{4}.$$

Now assume

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

for some $n \geq 1$. Then,

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left[\sum_{i=1}^{n} i^3\right] + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{[by the IH]} \\
&= (n+1)^2 \left[\frac{n^2}{4} + (n+1)\right] \\
&= (n+1)^2 \left[\frac{n^2 + 4n + 4}{4}\right] \\
&= (n+1)^2 \left[\frac{(n+2)^2}{4}\right] \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

$\square$

7) [13 points] Prove that for any $n \in \mathbb{Z}_{\geq 0}$ we have $(n+4)! \geq 4^n$.

Proof. We prove it by induction on n.
For $n = 0$ we have that $(0+4)! = 24 \geq 1 = 4^0$.
Now assume that $(n+4)! \geq 4^n$ for some $n \geq 0$. Then,

$$\begin{aligned}
(n+5)! &= (n+5) \cdot (n+4)! & \\
&\geq (n+5) \cdot 4^n & \text{[by the IH]} \\
&\geq 5 \cdot 4^n & \text{[as } n \geq 0\text{]} \\
&> 4 \cdot 4^n = 4^{n+1} & \text{[as } 5 > 4\text{].}
\end{aligned}$$

$\square$

8) [13 points] Consider the sequence a_n defined as follows:

$$a_0 = 1,$$
$$a_{n+1} = 1 + \frac{1}{a_n}, \text{ for } n \geq 0.$$

Prove that for all $n \geq 0$ we have

$$a_n = \frac{F_{n+2}}{F_{n+1}},$$

where F_n is the n-th Fibonacci number.
[**Remember:** $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.]
[This was done in a video.]

Proof. We prove it by induction on n.
For $n = 0$ we have

$$a_0 = 1 = \frac{1}{1} = \frac{F_2}{F_1}.$$

Now assume that

$$a_n = \frac{F_{n+2}}{F_{n+1}}$$

for some $n \geq 0$. Then,

$$\begin{aligned} a_{n+1} &= 1 + \frac{1}{a_n} \\ &= 1 + \frac{1}{F_{n+2}/F_{n+1}} \\ &= 1 + \frac{F_{n+1}}{F_{n+2}} \\ &= \frac{F_{n+2} + F_{n+1}}{F_{n+2}} \\ &= \frac{F_{n+3}}{F_{n+2}}. \end{aligned}$$

□