Some Algebraic Structures

Here are some algebraic structures we will study this *year*:

- Rings
- Fields
- Groups
- Vector Spaces (in more details)
- Modules

Rings

Perhaps the most familiar algebraic structure is *rings*.

Rings have two operations: sum and product. [Product here is not *scalar* product, but product between two elements!] Of course, we ask these operations to satisfy some common properties: associativity, distributive, commutativity [sometimes], etc.

Examples are:

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$;
- $R[X]$ [polynomials with variable X and coefficients in R] where R is one of the examples above [or a *commutative* ring]
- $M_n(R)$ [$n \times n$ matrices with entries in R] where R is a one of the examples above [or a *commutative* ring]

On the other hand, $\mathbb{N}$ is not a ring, as it lacks "negatives" of elements.

Commutative Rings

Note that in the last example [matrices] the *multiplication* is not commutative! We require the addition to *always* be commutative, but not multiplication. When multiplication is commutative, we call the ring a *commutative ring*.

$\mathbb{Z}/n\mathbb{Z}$

Another important example is $\mathbb{Z}/n\mathbb{Z}$: integers modulo n. [This is an example from Math 351.] Remember that in $\mathbb{Z}/n\mathbb{Z}$, you perform operations [sum and product] just as in $\mathbb{Z}$, but identify:

$$\cdots = -2n = -n = 0 = n = 2n = \cdots$$
$$\cdots = -2n+1 = -n+1 = 1 = n+1 = 2n+1 = \cdots$$
$$\cdots = -2n+2 = -n+2 = 2 = n+2 = 2n+2 = \cdots$$
$$\vdots$$
$$\cdots = -n-1 = -1 = n-1 = 2n-1 = 3n-1 = \cdots$$

[**Ex:** In $\mathbb{Z}/4\mathbb{Z}$, $3+3=6=2$ and $3\cdot 3=9=1$.]

Note that $\mathbb{Z}/n\mathbb{Z}$ has n elements.

Fields

Another familiar algebraic structure is *fields*.

Basically fields are commutative rings [“with 1”] for which every non-zero element has an *inverse*: if $a \neq 0$, then there is b [also in the field] such that $ab = 1$. [So, we can “divide” by non-zero elements.]

Examples are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. [Note that $\mathbb{Z}$, $R[X]$, $M_n(R)$ are *not* fields.]

Another example is $F(X)$, which is the set of all *rational functions* [i.e., quotient of polynomials, with non-zero denominator] with coefficients in some field F.

Finally $\mathbb{Z}/p\mathbb{Z}$ is a field if [and only if] p is *prime*.

Vector Spaces

Vector Spaces are the structures studied on Math 251: a set with two operations, sum and *scalar* multiplication. [You multiply an element of the vector space by a scalar, *not* by another element of the vector space.]

In Math 251 scalars were real numbers, but more generally scalars can be elements of any *field* [as above], such as $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}/7\mathbb{Z}$, etc.

In Math 251 you've seen diagonalization of matrices. You've seen that it is not always possible! One of the main topics will be to find out the "next best thing(s)": *rational and Jordan canonical forms*.

Modules

Modules are like vector spaces, except the "scalars" are not in a field, but in a *ring*. This makes things *much* more complicated, especially if the ring is non-commutative.

We will deal only very briefly with modules and only over [a special case of] commutative rings. We will only deal with them because they give a "natural" way to prove of the canonical forms [for vector spaces] results.

Algebras

Algebras are modules [or vector spaces] which area also rings. Thus, we have sum and both multiplication and scalar multiplication.

The main examples are:

- $R[X]$ [polynomials with coefficients in R and variable X];
- $M_n(R)$ [$n \times n$ matrices with entries in R];

where R is a commutative ring [and the scalars are the elements of R].

We will likely not deal with algebras [at least explicitly] in this course.