Extra Credit 1

Math 456

February 19, 2007

Be careful in Problem 1(a) in the Midterm I:

1. Let R be a ring and I be an ideal of R.

(a) Prove that if J is an ideal of R containing I, then $\bar{J} \stackrel{\text{def}}{=} \{\bar{a} \in R/I \ : \ a \in J\}$ is an ideal of R/I.

(b) Prove that if $\bar{J}'$ is an ideal of R/I, then $J' \stackrel{\text{def}}{=} \{a \in R \ : \ \bar{a} \in \bar{J}'\}$ is an ideal of R containing I.

If, in your solution you say: "Let $\bar{a} \in \bar{J}$. Then, by defintion of $\bar{J}$, we have that $a \in J$.", you will need to justify it (or say something else)!!! It does *not* follow from the definition! (Why not?)

In fact, for extra credit, give an example where I and J are ideals of a commutative ring R with unity, in which there is an $\bar{a} \in \bar{J} \stackrel{\text{def}}{=} \{\bar{a} \in R/I \ : \ a \in J\}$, with $a \notin J$.

Solution. Note first that in the case of the exam, one can get around the problem by saying someting like: "Let $\bar{a} \in \bar{J}$. Then, there exists $b \in J$ such that $\bar{b} = \bar{a}$ in $\bar{J}$." And then use, $\bar{b}$ instead of $\bar{a}$.

On the other hand, since $I \subseteq J$, we have that a indeed has to be in J, since [as you can see in the solution of the exam], if $\bar{b} = \bar{a}$, then $a - b \in I \subseteq J$. Since $b \in J$ and J is closed under addition, $a \in J$.

Now, for the extra credit: Let $R = \mathbb{Z}$, $I = (3)$ and $J = (5)$. Then, $\bar{J} = \mathbb{Z}/(3)$, since, in $\mathbb{Z}/3\mathbb{Z}$, we have $\bar{5} = \bar{2}$, $\overline{10} = \bar{1}$ and $\overline{15} = \bar{0}$. But then $\bar{2} \in \bar{J}$, but $2 \notin J = 5\mathbb{Z}$.

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