Claim: If $\alpha \in \mathbb{Q}_p$, then there is a sequence $\{a_i\}_{i \geq n_0}$ with $a_i \in \{0, 1, \ldots, (p-1)\}$ such that $\alpha = \sum_{i=n_0}^{\infty} a_i p^i$ [or, equivalently, $\lim_{n\to\infty} \left|\alpha - \sum_{i=n_0}^{n} a_i p^i\right|_p = 0$].

Proof. First, remember that if $\beta, \gamma \in \mathbb{Q}$, with $\beta = \sum_{i=k_0}^{\infty} b_i p^i$, $\gamma = \sum_{i=l_0}^{\infty} c_i p^i$ and $|\beta - \gamma|_p < p^{-N}$, for some $N > \max\{m_0, l_0\}$, then $m_0 = l_0$ and $b_i = c_i$ for all $i \in \{m_0, m_0+1, \ldots, N\}$.

So, let $\{\alpha_i\}$ be a Cauchy sequence in $\mathbb{Q}$ that converges to α [in $\mathbb{Q}_p$]. We will now define the a_i's as in the statement.

Let k be a given positive integer. Since $\alpha_n \longrightarrow \alpha$, there is N_k such that if $n \geq N_k$ we have $|\alpha_n - \alpha|_p < p^{-k-1}$. Possibly replacing N_k by the $\max\{N_k, N_{k-1}+1\}$, if $k > 1$, we may assume $N_1 < N_2 < N_3 < \cdots$.

We then define $\{a_{n_0}, \ldots, a_k\}$, with $a_i \in \{0, 1, \ldots, (p-1)\}$, as the [unique] coefficients such that

$$\alpha_{N_k} = \sum_{i=n_0}^{k} a_i p^i + p^{k+1}[\cdots].$$

[The omitted part is just the remaining of its power series.]

Are these well defined? More precisely, to get a_{k+1} we do the same as above, with k replaced by $(k+1)$, but these would define some a_i' for $i \in \{n_0, \ldots, k, k+1\}$. The question is if $a_i' = a_i$ for $i \in \{n_0, \ldots, k\}$, so that we are in fact only defining a "new" a_{k+1}?

This is, indeed the case: we have that $N_{k+1} > N_k$, and thus

$$\begin{aligned} \left|\alpha_{N_{k+1}} - \alpha_{N_k}\right|_p &= \left|(\alpha_{N_{k+1}} - \alpha) + (\alpha - \alpha_{N_k})\right|_p \\ &\leq \left|\alpha_{N_{k+1}} - \alpha\right|_p + |\alpha - \alpha_{N_k}|_p < 2p^{-k-1} \leq p^{-k}. \end{aligned}$$

Since $\alpha_{N_k}, \alpha_{N_{k+1}} \in \mathbb{Q}$, by our first observation, we have that if $\alpha_{N_{k+1}} = \sum_{n_0}^{k} a_i p^i + p^{k+1}[\cdots]$.

We now claim that $\lim_{k\to\infty} \left|\alpha - \sum_{i=n_0}^{k} a_i p^i\right|_p = 0$ [which will finish the proof]. Remember that

$$|\alpha - \alpha_{N_k}|_p < p^{-k-1} \quad \text{[by def. of } N_k\text{]}$$

and that

$$\left|\alpha_{N_k} - \sum_{i=n_0}^{k} a_i p^i\right|_p < p^{-k} \quad \text{[by def. of the } a_i\text{'s]}.$$

Then:

$$\begin{aligned} 0 \leq \left| \alpha - \sum_{i=n_0}^{k} a_i p^i \right|_p &= \left| \alpha - \alpha_{N_k} + \alpha_{N_k} - \left(\sum_{i=n_0}^{k} a_i p^i \right) \right|_p \\ &\leq |\alpha - \alpha_{N_k}|_p + \left| \alpha_{N_k} - \left(\sum_{i=n_0}^{k} a_i p^i \right) \right|_p \\ &< p^{-k-1} + p^{-k} < 2p^{-k}, \end{aligned}$$

which gives us the desired result. □