Midterm 2

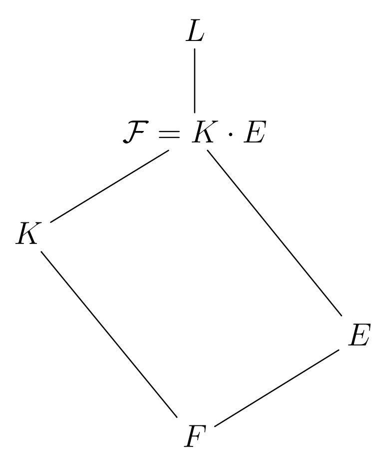
This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Walker), class notes and solutions to *our* HW problems *posted by me* or done by yourself. No other reference, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: noon on Wednesday (04/09). If you cannot bring it to class or to me, a scanned/typed copy by e-mail would be OK.

1) [20 points] Let K/F be a field extension of characteristic different from 2 with $[K : F] = 2$. Show that K/F is a normal extension. [Note it is also true in characteristic 2, but it needs, quite likely, a different proof.]

2) [20 points] Let $F = \mathbb{Q}[\sqrt[3]{2}]$. Prove that there is no root of unity ζ such that $F \subseteq \mathbb{Q}[\zeta]$.

3) [20 points] Let L/F be a *finite* field extension and K and E be intermediate extensions. [I.e., K and E are fields with $F \subseteq K \subseteq L$ and $F \subseteq E \subseteq L$. See the diagram below.] Assume that E/F is Galois with Galois group $G \stackrel{\text{def}}{=} \text{Gal}(E/F)$ and let $\mathcal{F} \stackrel{\text{def}}{=} K \cdot E$ be the compositum [as in the first midterm] of these extensions.



(a) Prove that there exists $\alpha \in E$ such that $E = F[\alpha]$. [Note that this implies that $\mathcal{F} = K[\alpha]$.]

(b) Show that $\mathcal{F}/K$ is also Galois. [**Careful:** The base field is K, not F!]

(c) If $H = \mathrm{Gal}(\mathcal{F}/K)$ and $\sigma \in H$, then show that $\sigma|_E \in G$.

(d) Show that, in the situation above, $\sigma|_E = \mathrm{id}_E$ [the identity function on E] if and only if $\sigma = \mathrm{id}_{\mathcal{F}}$.

[**Note:** This shows that H is isomorphic to a subgroup of G via restrictions.]

4) [20 points] Let L/F be a *finite* field extension and K_1 and K_2 be intermediate extensions, *both* Galois over F with $G_i \stackrel{\text{def}}{=} \mathrm{Gal}(K_i/F)$, for $i = 1, 2$. Let $\mathcal{F} \stackrel{\text{def}}{=} K_1 \cdot K_2$ be the compositum of the extensions.
[Here you can assume that $K_1 = F[\alpha]$, $K_2 = F[\beta]$ and $\mathcal{F} = F[\alpha, \beta]$, without proof, if you wish.]

(a) Prove that $\mathcal{F}/F$ is Galois. [Here the base field *is* F.]

(b) Let $G \stackrel{\text{def}}{=} \mathrm{Gal}(\mathcal{F}/F)$ and $\sigma \in G$. Show that, for $i = 1, 2$, we have $\sigma|_{K_i} \in G_i$.

(c) Let then $\phi : G \to G_1 \times G_2$ defined as $\phi(\sigma) = (\sigma|_{K_1}, \sigma|_{K_2})$. Show that the kernel [you don't need to check it is a homomorphism, but it clearly is] of ϕ is simply $\{\mathrm{id}_{\mathcal{F}}\}$.

(d) Let $[K_i : F] = d_i$ and assume that $[\mathcal{F} : F] = d_1 \cdot d_2$. Prove that ϕ is onto. [Hence, in this case, ϕ is an isomorphism!]

5) [20 points] Let $f = (x^2 - 2)(x^2 - 3) \in \mathbb{Q}[x]$. Find its Galois group [over $\mathbb{Q}$] and draw the diagram of its subfields of the splitting field. Which ones are normal over $\mathbb{Q}$?