April 29th, 2015

Math 351

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Spring 2015

Name:..

Student ID (last 6 digits): XXX- ....

Final

You have two hours to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 8 questions and 11 printed pages (including this one and two pages for scratch work in the end).

No books or notes are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	25	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	10	
Total	100	

1) Let $\sigma, \tau \in S_9$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 7 & 2 & 8 & 3 & 9 & 1 & 6 & 4 \end{pmatrix} \quad \text{and} \quad \tau = (1\,5\,3)(2\,4\,8\,9).$$

(a) [5 points] Write the complete factorization of σ into disjoint cycles.

(b) [4 points] Compute σ^{-1}. [Your answer can be in any form.]

(c) [4 points] Compute $\tau\sigma$. [Your answer can be in any form.]

(d) [4 points] Compute $\sigma\tau\sigma^{-1}$. [Your answer can be in any form.]

(e) [4 points] Write τ as a product of transpositions.

(f) [4 points] Compute $\text{sign}(\tau)$ and $|\tau|$.

2) [10 points] Give the set of all integral solutions of the system

$$\begin{aligned} 3x &\equiv 2 \pmod{5}, \\ x &\equiv 3 \pmod{11}. \end{aligned}$$

[No need to do the *Extended Euclidean Algorithm* explicitly.]

3) [10 points] Let G be an *Abelian* group [with multiplicative notation] and $a, b \in G$. Prove that

$$\langle a, b \rangle \stackrel{\text{def}}{=} \{a^m \cdot b^n \ : \ m, n \in \mathbb{Z}\}$$

is a subgroup of G. [Make sure to say where, if anywhere, you use the hypothesis that G is Abelian.]

4) [10 points] Prove that if R is a domain, then $U(R[x]) = U(R)$. [Remember, $U(R)$ is the set of units of R, which I usually denote by $R^{\times}$. So, what you need to prove it that the units of the polynomial ring are the constant polynomials which are units of R.]

5) Examples:

(a) [5 points] Give an example of a finite *non-commutative* ring [with $1 \neq 0$]. [What examples of non-commutative rings do you know?]

(b) [5 points] Give an example of an *infinite* ring R for which $2015 \cdot a = 0$ for all $a \in R$.

6) [10 points] Let R be a [possibly non-commutative] ring [with $1 \neq 0$] and $a \in R$ such that there are $s, t \in R$ for which $sa = at = 1$. Prove that $s = t$.

7) Let $G = \{1, a, b\}$ be a group [with multiplicative notation] with exactly three elements. [So, 1, a and b are distinct and 1 is the identity of G.]

(a) [7 points] Prove that $ab = 1$. [**Hint:** Check that any other possibility would be impossible.]

(b) [8 points] Prove that $a^2 = b$. [**Hint:** Check that any other possibility would be impossible.]

8) [10 points] Let R be a ring [with $1 \neq 0$] and suppose there is $a \in R \setminus \{0\}$ such that $a^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$. Prove that there is $b \in R \setminus \{0\}$ such that $b^2 = 0$. [**Hint:** Break into cases: n even or odd.]

Scratch:

Scratch: