Errata 2

Math 652

February 9, 2016

In class we left the following unproven:

Lemma 1. *Let A be a finite commutative F-algebra and $e \neq 0$ an idempotent. Then e is a (finite) sum of* primitive *idempotents.*

Thomas showed me a proof at the end of the class, but this is slightly different. (Behind the curtains, it *is* the same.)
We will use the following lemma proved in class:

Lemma 2. *If A is finite commutative F-algebra which is* indecomposable*, then the only idempotents of A are 1 and 0.*

Proof of Lemma 1. Let $\{e_1, \ldots, e_n\}$ be the set of all primitive idempotents of A. [We've seen that this set must be finite. In fact $n \leq \dim_F A$.] Since this is system of orthogonal idempotents associates to the decomposition of A into indecomposable algebras, we have that $1 = \sum_{i=1}^{n} e_i$, and so $e = \sum_{i=1}^{n} ee_i$.
Now, $ee_i \in Ae_i$, which is indecomposable [since e_i is primitive] and ee_i is idempotent [since e and e_i are], so by Lemma 2 above we have that $ee_i = 0$ or $ee_i = 1_{Ae_i} = e_i$. Since $e \neq 0$, we must have $ee_i \neq 0$ for at least one index i. Assume, without loss of generality that $ee_i = e_i$ for $i = 1, \ldots, m$ and $ee_i = 0$ for $i = (m+1), \ldots, n$. Then, we have $e = \sum_{i=1}^{n} ee_i = \sum_{i=1}^{m} e_i$. $\square$