Proposition. *Let* $F = \mathbb{F}_p(s^p, t^p)$ *and* $K = \mathbb{F}_p(s, t)$*, where* s *and* t *are variables. Then* $[K : F] = p^2$ *and there are infinitely many intermediate extensions between* K *and* F*.*

Proof. One can easily prove that $[\mathbb{F}_p(s, t^p) : \mathbb{F}_p(s^p, t^p)] = p$ and $[\mathbb{F}_p(s, t) : \mathbb{F}_p(s, t^p)] = p$, and so $[K : F] = p^2$.

Now, let $f \in K$ and define $E_f = F[s + f^p \cdot t]$. Then, clearly $F \subseteq E_f$, but $F \neq E_f$. So, one can see that $[E_f : F] = p$.

Suppose now that $E_f = E_g$, and call this extension simply E. Then, $s + f^p t, s + g^p t \in E$, and so the difference $(f^p - g^p)t \in E$. Since $(f^p - g^p) \in F \subseteq E$, we get that $t \in E$. Then, since $t, s + f^p t \in E$, we get that $s \in E$. Therefore, $s, t \in E$ and hence $E = K$.

But $[K : F] = p^2$ and $[E : F] = p$, a contradiction. Thus, $E_f \neq E_g$. Since there are infinitely many $f \in K$, we have infinitely many intermediate extensions.

$\square$