MIDTERM

M559 – LINEAR ALGEBRA – MARCH 21, 2024

Solve all problems in class. [I hope you can do all of this in class, as it is comparable to half of a diagnostic exam.] Then, you may take this sheet home and solve all remaining problems, or problems that you think you've missed, upload it to Canvas by Saturday (03/24) by 11:59pm. I will consider it for some partial credit. [At *most* half of the original number of points you've missed in the question.]

Important: After you take them home, you should treat these problems as a *take-home exam*, not as a homework. So, you should not discuss *anything* about these problems with *anyone* (except me), nor use any texts or the internet.

Note: All vector spaces here are over an arbitrary field F, but you can assume that $F = \mathbb{R}$ if you want.

1. Let V be a vector space over the field F. Prove that if $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$ is such that $V = \operatorname{span}(S)$, but for all $i \in \{1, 2, \ldots, n\}$ we have that $V \neq \operatorname{span}(S \setminus \{v_i\})$, then S is a basis of V.

2. Let V and W be vector spaces over the field F of [finite] dimensions n and m respectively, and $T : V \to W$ and $S : W \to V$ be linear transformations such that $T \circ S$ and $S \circ T$ are the identity maps of W and V respectively.

(a) Show that both T and S are onto.

(b) Show that $m = n$.

3. Let V be a vector space over F [possibly infinite dimensional] and $f \in V^* \setminus \{0\}$. Let v_0 such that $f(v_0) \neq 0$ and $N \stackrel{\text{def}}{=} \ker(f)$. Prove that for all $v \in V$ there are *unique* $c \in F$ and $w \in N$ such that $v = cv_0 + w$.

[**Hint:** For a given $v \in V$, find some $c \in F$ such that $f(v - cv_0) = 0$.]