1) Solve the systems $A\mathbf{x} = \mathbf{b}$, given that:

(a) $$\left[\, A \,\middle|\, \mathbf{b} \,\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Solution. No solution, since the last row gives $0 = -1$. □

(b) $$\left[\, A \,\middle|\, \mathbf{b} \,\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution. $(x_1, x_2, x_3, x_4) = (2, -1, 3, 0)$. □

(c) $$\left[\, A \,\middle|\, \mathbf{b} \,\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

Solution. $(x_1, x_2, x_3, x_4, x_5) = (4 - 2s + t, 2 + s - 2t, s, -1 - t, t)$. □

2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, find $(AB)_{2,2}$ and $(AB)_{1,3}$. [**Remember:** $(A)_{i,j}$ is the entry of A in the i-th row and j-th column.]

Solution. We have

$$(AB)_{2,2} = 4 \cdot 2 + 5 \cdot 1 + 6 \cdot 1 = 19$$

and

$$(AB)_{1,3} = 1 \cdot 3 + 2 \cdot 3 + 3 \cdot 2 = 15.$$

□

3) If A is an 3×3 matrix with $\det(A) = 2$, then what is $\det((2A^{\mathrm{T}})^{-1})$? [Show steps!]

Solution. We have:

$$\begin{aligned} \det((2A^{\mathrm{T}})^{-1}) &= \det((2A^{\mathrm{T}}))^{-1} \\ &= [2^3 \cdot \det(A^{\mathrm{T}})]^{-1} \\ &= [2^3 \cdot \det(A)]^{-1} \\ &= [2^3 \cdot 2]^{-1} \\ &= 1/16. \end{aligned}$$

□

4) Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find D^{-1}.

Solution. $D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

□

(b) If B and C are square matrices of the same size, then complete the formula:

$$(B \cdot C)^{-1} = \boxed{C^{-1} \cdot B^{-1}}.$$

(c) If $A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, then, with the D above, find $(DA)^{-1}$.

Solution. From the above:

$$(DA)^{-1} = A^{-1}D^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1/3 \\ 1/2 & -1 & 1/3 \\ 0 & -3 & 0 \end{bmatrix}.$$

Note that the last multiplication can be done quickly using the properties of multiplication by diagonal matrices. □

5) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 5 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}$. Given that $\det(A) = 15$, what is $(A^{-1})_{3,2}$? [**Hint:** You do *not* need to compute the whole inverse to find this!]

Solution. We use the formula $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$. So, $(A^{-1})_{3,2} = \frac{1}{\det(A)} \cdot (\operatorname{adj}(A))_{3,2}$. Now,

$$(\operatorname{adj}(A))_{3,2} = C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 0 \end{vmatrix}.$$

But, since we have two rows which are equal, we have that the determinant above is zero. Therefore, $(A^{-1})_{3,2} = \frac{1}{15} \cdot 0 = 0$. □

6) Let $A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 1 & 5 & 6 \\ 0 & 2 & -1 & 8 \\ 2 & 0 & 4 & 3 \end{bmatrix}$. Find the reduced row echelon form of A, $\det(A)$, and, if possible, A^{-1}. [**Hint:** You can compute all these together!]

Solution. We have:

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 1/2 & 0 & 0 & 0 \\ 3 & 1 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3/2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -3/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -17/2 & 4 & -2 & 3 \\ 0 & 1 & 0 & 0 & 13/2 & -1 & 1 & 5 \\ 0 & 0 & 1 & 0 & 5 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

Hence, the reduces row echelon form of A is I_3, $\det(A) = 2$ [only operation that changes the determinant was dividing a row by 2, and so $\det(A) = 2 \cdot \det(I_n) = 2$], and

$$A^{-1} = \begin{bmatrix} -17/2 & 4 & -2 & 3 \\ 13/2 & -1 & 1 & -5 \\ 5 & -2 & 1 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

□