Midterm 2

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* on this exam and you can only look at *our* book (Velleman), videos and class notes. *No other reference is allowed*, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

As you will have a lot of time, I expect the solutions to be well written and clearly explained.

If you need *clarifications on any statement*, please use the "Q&A (Math Related)" forum on Blackboard. *Questions related to content/math of the exam should be submitted by e-mail!* Please use your best judgment on what is appropriate to ask in the forum.

Due date: Your solutions must be uploaded on Blackboard by Tuesday 06/24 by 11:59pm. Please send as a PDF and make sure your scanned/typed exam is clear and legible.

1) [15 points] Let A, B and C be sets. Prove that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

2) [15 points] Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets with $\mathcal{F} \subseteq \mathcal{G}$. Prove that $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$.

3) [15 points] Let R be a relation from A to B and S and T be relations from B to C. Prove that $(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$.

4) [15 points] Let R_1 and R_2 be symmetric relations on A. Prove that $R_1 \setminus R_2$ is also symmetric.

5) [20 points] Consider the ordering relation in $\mathbb{R}^2$ defined by $(a, b) \preccurlyeq (c, d)$ [the LaTeX code for this symbol is `\preccurlyeq`] if both $a \leq c$ and $b \leq d$. [You can assume without proving it that this is a partial order in $\mathbb{R}^2$.] Consider the set $B = \{(0, 0)\} \cup \{(1, y) \mid y \in \mathbb{R}\}$. [So, B is the origin together with the vertical line $x = 1$.]

(a) Show that $(0, 0)$ is a minimal element of B.

(b) Show that B has no other minimal element besides $(0, 0)$.

(c) Show that B has no smallest element. [In particular, $(0, 0)$ is the only minimal element, but not the smallest element.]

6) [20 points] Let $A = \mathbb{R}^2 \setminus \{(0, 0)\}$ [i.e., the Cartesian plane without the origin] and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ [i.e., the interval $(0, \infty)$]. Define a relation R on A by:

$$R = \{((a, b), (c, d)) \in A \times A \mid \exists x \in \mathbb{R}_{>0}\ (c = ax \wedge d = bx)\}.$$

[I.e., $(a, b)\, R\, (c, d)$ if $(c, d) = (ax, bx)$ for some positive real number x.]

(a) Prove that R is an equivalence relation on A.

(b) Draw on $A = \mathbb{R}^2 \setminus \{(0, 0)\}$ [or describe geometrically] the equivalence class $[(0, 1)]_R$.