Exam 3

You must upload the solutions to this exam (as a PDF file) on Canvas by 11:59pm on Sunday 07/01. Since this is a take home, I want all your solutions to be neat and well written.

You can look at *our* book only! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

1) Let A_i and B_i be indexed families with $I \neq \varnothing$. Prove that

$$\left(\bigcap_{i\in I} A_i\right) \times \left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I}(A_i \times B_i).$$

2) Let $A = \{1,2,3,4,5\}$, $B = \{a,b,c,d,e\}$, and R and S be relations on $A \times B$ and $B \times A$, respectively, given by:

$$R = \{(1,a),(1,d),(2,c),(2,e),(3,b),(3,d),(3,e),(5,a)\},$$
$$S = \{(a,2),(b,1),(b,4),(d,4),(d,5),(e,1),(e,4)\}.$$

(a) Give $\mathrm{Dom}(R)$.

(b) Give $\mathrm{Ran}(S)$.

(c) Give R^{-1}.

(d) Give $S \circ R$.

3) Let R be a non-empty relation on the non-empty set A.

(a) Prove that if R is reflexive, then $R \subseteq R \circ R$.

(b) Prove that if R is transitive, then $R \circ R \subseteq R$.

4) Suppose that R is partial order on A, $B \subseteq A$, and let b be the largest element of B. Prove that b is also a maximal element of B and that it is the only maximal element of B.

5) Let $A = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y \neq 0\}$ [i.e., $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$], and define for $(a, b), (c, d) \in A$ the relation R by $(a, b)R(c, d)$ if $ad = bc$.

(a) Prove that R is an equivalence relation. [**Note:** transitivity is a bit tricky, and requires some algebraic manipulations.]

(b) For any $b \in \mathbb{Z} \setminus \{0\}$, let $S_b = \{(k, kb) \mid k \in \mathbb{Z} \setminus \{0\}\}$. Prove that $[(1, b)]_R = S_b$.

6) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be partitions of A_1 and A_2 respectively, with $A_1 \cap A_2 = \varnothing$ [and $A_1, A_2 \neq \varnothing$]. Prove that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a partition of $A_1 \cup A_2$.