Exam 4

1) [50 points] Let $\sigma, \tau \in S_{11}$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 10 & 4 & 11 & 6 & 2 & 7 & 1 & 3 & 9 \end{pmatrix} \quad \text{and} \quad \tau = (1\ 5\ 10)(3\ 11\ 2\ 4\ 7)(6\ 8\ 9).$$

(a) Write the complete factorization of σ into disjoint cycles.

Solution. $\sigma = (1\ 5\ 11\ 9)(2\ 8\ 7)(3\ 10)(4)(6)$. □

(b) Write τ is matrix form.

Solution.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 4 & 11 & 7 & 10 & 8 & 3 & 9 & 6 & 1 & 2 \end{pmatrix}.$$

□

(c) Compute τ^{-1}. [Your answer *must* be in *disjoint cycles form*!]

Solution. $\tau^{-1} = (1\ 10\ 5)(3\ 7\ 4\ 2\ 11)(6\ 9\ 8)$. □

(d) Compute $\sigma\tau$. [Your answer *must* be in *disjoint cycles form*!]

Solution. $\sigma\tau = (1\ 11\ 8)(2\ 4)(3\ 9\ 6\ 7\ 10\ 5)$. □

(e) Compute $\sigma\tau\sigma^{-1}$. [Your answer *must* be in *disjoint cycles form*!]

Solution. $\sigma\tau\sigma^{-1} = (5\ 11\ 3)(10\ 9\ 8\ 4\ 2)(6\ 7\ 1)$. □

(f) Write τ as a product of transpositions.

Solution. $\tau = (1\ 10)(1\ 5)(3\ 7)(3\ 4)(3\ 2)(3\ 11)(6\ 9)(6\ 8)$ □

(g) Compute $\text{sign}(\tau)$.

Solution. $\text{sign}(\tau) = (-1)^8 = 1$ or $(-1)^{11-3} = 1$. □

(h) Compute $|\tau|$.

Solution. $|\tau| = \text{lcm}(3, 5, 3) = 15$. □

(i) Give an element $\alpha \in S_{11}$, with $\alpha \neq 1$, such that $\alpha \cdot \tau = \tau \cdot \alpha$ [i.e., α must commute with τ].

Solution. That $\alpha = \tau$. □

(j) Give an element $\beta \in S_{11}$ such that $\beta \cdot \tau \neq \tau \cdot \beta$ [i.e., β does *not* commute with τ]. *Show work!* [**Hint:** You need $\beta \cdot \tau \cdot \beta^{-1} \neq \tau$.]

Solution. Take $\beta = (5\ 10)$. Then, $\beta\tau\beta^{-1} = (1\ 10\ 5)(3\ 11\ 2\ 4\ 7)(6\ 8\ 9) \neq \tau$. □

2) [15 points] Let G be an *Abelian* group. Define then:

$$\operatorname{Tor}(G) = \{x \in G \ : \ x^n = e \text{ for } \textit{some } n \in \mathbb{Z}_{>0}\}.$$

[Here we are using the multiplicative notation and e is the identity element, which we could also denote simply by "1".] Prove that $\operatorname{Tor}(G)$ is a subgroup of G. *Make clear where you use the fact the G is Abelian!* [If you never do, say so.]

[**Careful:** Different x's in $\operatorname{Tor}(G)$ might have different powers that give e, like maybe $x_1^5 = e$, while $x_2^{71} = e$. So, there might not be a common power n that works for every $x \in \operatorname{Tor}(G)$.]

Solution. First note that $e \in \operatorname{Tor}(G)$, as $e^1 = e$. So, $\operatorname{Tor}(G) \neq \varnothing$.

Let $x, y \in \operatorname{Tor}(G)$. Then, by definition, there are $m, n \in \mathbb{Z}_{>0}$ such that $x^m = y^n = e$. In particular $(y^{-1})^n = y^{-n} = (y^n)^{-1} = e^{-1} = e$. So, we have that $y^{-1} \in \operatorname{Tor}(G)$ and hence $\operatorname{Tor}(G)$ is closed under inverses.

Also, $(xy)^{mn} = x^{mn}y^{mn} = (x^m)^n(y^n)^m = e^n e^m = e^{n+m} = e$. We use the fact that G is Abelian in this first equality. But this shows that $\operatorname{Tor}(G)$ is closed under multiplication.

Therefore, $\operatorname{Tor}(G)$ is a subgroup of G. □

3) The sets below are *not* groups. Justify why not.

(a) [10 points] The set O of all odd integers and 0 with addition. [So, $O = \{0, 1, -1, 3, -3, 5, -5, \ldots\}$.]

Solution. We have $1 \in O$, but $1 + 1 = 2 \notin O$, so it is not closed under the operation, and hence not a group. □

(b) [10 points] $S = \left\{ \begin{pmatrix} 2 & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \ : \ 2d - bc \neq 0 \right\}$ with the multiplication of matrices.

Solution. Since the identity matrix is not in S [as the $(1,1)$-entry of the identity is 1 and not 2], it is not a group [as if it were, it would be a subgroup of $\mathrm{GL}_2(\mathbb{R})$]. □

4) [15 points] Show that

$$S_3 = \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

is not cyclic, but every proper subgroup [i.e., subgroup different of S_3 itself] is cyclic.

Solution. Note that $|S_3| = 6$, so if it were cyclic, there would be an element of order 6. But the elements are 1, which has order 1, two cycles, which have order 3, or 3-cycles, which have order 3. So, no element of order 6, and hence it cannot be cyclic.

[Alternatively, it suffices to observe that S_3 is not Abelian, as $(2\ 3)(1\ 2\ 3)(2\ 3) = (1\ 3\ 2) \neq (1\ 2\ 3)$, and hence it cannot be cyclic [as every cyclic group is Abelian].]

Now, if H is a subgroup of G with $H \neq G$, then $|H| \mid 6$ with $|H| \neq 6$. So, $|H|$ is 1, 2, 3. If 2 or 3, then by Corollary 2.87, H is cyclic. If 1, then $H = \{1\} = \langle 1 \rangle$. □