

Research Statement

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My research interests lie in the intersection of complex analysis, probability, and graph theory, and my work splits in two directions. First, I explore questions pertaining to the geometry of solutions to the Loewner differential equation. Secondly, I study questions relating to discrete modulus on graphs, with a special interest in relating discrete objects to continuous objects in complex analysis. I will begin by discussing my recent work related to the Loewner equation, and then I will discuss my work regarding discrete modulus.

Introduction to the Loewner equation

The Loewner differential equation, introduced by C. Loewner in the 1920s, became an integral part of the random processes called Schramm-Loewner Evolution (SLE) that was developed by O. Schramm in 2000 [S1]. SLE utilized the Loewner equation in a new way, and as a deep understanding of SLE emerged, it became apparent that the deterministic understanding lagged behind. My research explores the nature of the correspondence created by the Loewner equation between increasing families of 2-dimensional sets (called *hulls*) and continuous 1-dimensional functions (called *driving functions*).

For a continuous driving function $\lambda : [0, T] \rightarrow \mathbb{R}$ and for $z \in \mathbb{H} = \{z : \operatorname{Re} z > 0\}$, the chordal Loewner differential equation in \mathbb{H} is the following initial value problem:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \quad (1)$$

In solving (1), we consider z to be a fixed complex number with t the (time) variable. (Our notation gives away the fact that we will switch our view momentarily.) For each $z \in \mathbb{H}$, a unique solution to (1) exists on some time interval, and we let $[0, T_z]$ be the largest such interval. That is, $T_z = \inf \{t : \lambda(t) = g_t(z)\}$ is the “killing time” for the point z . Collecting together all the “killed points,” we define the hull $K_t = \{z : T_z \leq t\}$. Then from the classical theory, $\mathbb{H} \setminus K_t$ is a simply connected domain and g_t (the solution to (1)) is a conformal map from $\mathbb{H} \setminus K_t$ onto \mathbb{H} . Often it is possible to define a continuous curve γ , called the *trace*, so that K_t is the curve $\gamma[0, t]$ with the holes filled in (or more precisely, K_t is the complement of the unbounded component of $\mathbb{H} \setminus \gamma[0, T]$.) Notice that we started with the function λ , a one-dimensional object, and through the Loewner equation created K_t , an increasing family of two-dimensional sets.

The process can be reversed. Suppose we start with an appropriate family K_t of increasing sets in the plane. (The easiest case is when $K_t = \gamma[0, t]$, where γ is a simple curve in \mathbb{H} with $\gamma(0) \in \mathbb{R}$.) Then for each time t , there is a unique conformal map g_t from $\mathbb{H} \setminus K_t$ onto \mathbb{H} with the hydrodynamic normalization at infinity. We may ask how this family of maps

varies as t varies. After reparameterizing if needed, the maps g_t will satisfy (1) for some continuous function $\lambda(t)$. Further, we can recover λ from the maps g_t . (In the simple curve case, $\lambda(t) = g_t(\gamma(t))$.) See [La] for further background.

For $\kappa \geq 0$, chordal SLE $_{\kappa}$ is the random family of hulls created by the Loewner equation when the driving term is $\lambda(t) = \sqrt{\kappa}B_t$, where B_t is standard Brownian motion. For SLE, it is possible to define the trace. That is, there is an almost surely continuous path $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ so that the hull K_t generated by $\lambda(t) = \sqrt{\kappa}B_t$ is the curve $\gamma[0, t]$ with holes filled in. Further, SLE $_{\kappa}$ has three distinct phases of geometric behavior: (i) $\gamma(t)$ is almost surely a simple curve for $\kappa \leq 4$, (ii) $\gamma(t)$ is almost surely a non-simple, non-space-filling curve for $4 < \kappa < 8$, and (iii) $\gamma(t)$ is almost surely a space-filling curve for $\kappa \geq 8$. See [RS] and, for the case $\kappa = 8$, [LSW].

A natural deterministic class of functions for the Loewner equation is Lip(1/2), that is, functions λ which satisfy

$$|\lambda(t) - \lambda(s)| \leq c|t - s|^{1/2} \tag{2}$$

for all s and t in the domain of λ . The smallest c for which (2) is satisfied is called the Lip(1/2) norm of λ , denoted $\|\lambda\|_{1/2}$. In [L1, MR], it is shown that deterministic functions also have a phase transition. In particular, for $\|\lambda\|_{1/2} < 4$, the Loewner hull driven by λ is a simple curve, and for each $c \geq 4$ there is an example of driving function λ with $\|\lambda\|_{1/2} = c$ whose Loewner hull is non-simple.

Recent work regarding the Loewner equation

Loewner Curvature, with S. Rohde, [LRoh]

In this work, we interpret the deterministic phase transition in the Loewner theory as an analog of the hyperbolic variant of the Schur theorem about curves of bounded curvature. To accomplish this, we introduce a new notion, called Loewner curvature, and establish some of its properties. In particular, we show that if the Loewner curvature of a curve is bounded above by 8, then the curve can never hit back on itself or on the boundary of the domain. Additionally, if we have bounds on the Loewner curvature, then we can specify a region of the domain that must contain the curve.

To define Loewner curvature, we first introduce a family of curves that have a certain conformal self-similarity property. This family will be the curves of constant Loewner curvature. We then show that every sufficiently smooth curve has a unique best-approximating curve in this family. Thus we can define Loewner curvature by comparison with the curves of constant Loewner curvature. As a tool, we show that the Loewner curvature can be expressed as the following nice formula of the driving function: $\lambda'(t)^3/\lambda''(t)$.

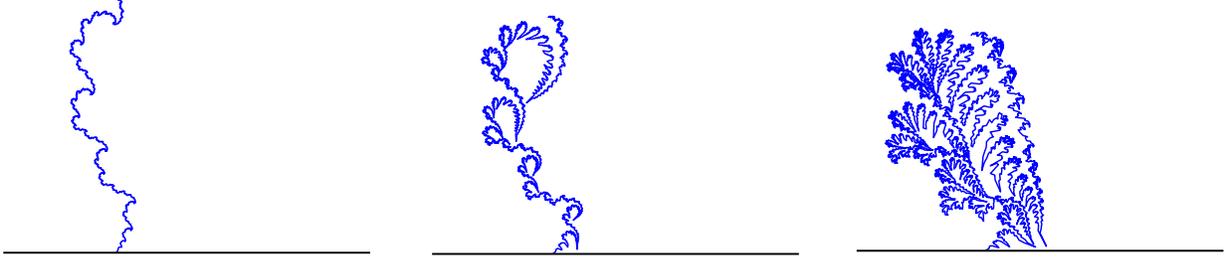


Figure 1: Simulations of the hulls generated by $cW(t)$ for $c = 0.8$ (left), $c = 1.2$ (middle), and $c = 1.6$ (right).

Regularity of Loewner Curves, with H. Tran, [LT]

This work relates the regularity of the driving function λ to the regularity of the trace γ . To state it, we use the notation $f \in C^{n+\alpha}$ to mean that $f \in C^n$ and $f^{(n)}$ is Hölder continuous with exponent α . We show that $\lambda \in C^\beta[0, T]$ implies $\gamma \in C^{\beta+1/2}(0, T]$ for all $\beta > 2$ (with a slightly weaker statement when $\beta + 1/2 \in \mathbb{N}$), and further, if λ is analytic on $[0, T]$, then γ is analytic on $(0, T]$. This extends [W], in which C. Wong proved the result for $1/2 < \beta \leq 2$.

Additionally, we analyze γ near $t = 0$ for $\lambda \in C^\beta[0, T]$. The problem with the smoothness of γ at $t = 0$ is a feature of the halfplane-capacity parametrization, illustrated by our result that for t near zero,

$$\gamma(t) = 2it^{1/2} + a_2t + ia_3t^{3/2} + a_4t^2 + \cdots + a_{2n}t^n + o(t^n),$$

where the real-valued coefficients a_k depend on $\lambda'(0), \dots, \lambda^{(n)}(0)$ and $n \in \mathbb{N}$ satisfies $n < \beta$. However, under the simple change of parametrization $\Gamma(t) = \gamma(t^2)$ we prove that the smoothness does extend to $t = 0$.

This work depends on a detailed study of the solutions $f(u) = f(u, s, \epsilon)$ to the ODE

$$f'(u) = \frac{-2}{f(u)} + \lambda'(s - u), \quad f(0) = i\epsilon \in \mathbb{H}$$

where $0 \leq u \leq s$. This variant of the backward Loewner equation provides the connection between λ and γ since $f(s, s, \epsilon)$ converges uniformly to $\gamma(s)$ as $\epsilon \rightarrow 0+$.

Loewner deformations driven by the Weierstrass function, with J. Robins, [LRob]

In this work, we study the Loewner hulls driven by a multiple of the Weierstrass function $W(t) = \sum_{n=0}^{\infty} 2^{-n/2} \cos(2^n t)$, which is a deterministic analog of Brownian motion. In comparison with SLE, we prove that this family exhibits a phase transition, as illustrated in Figure 1. In particular, when c is small enough, the hull generated by $cW(t)$ is a simple curve in $\mathbb{H} \cup \{cW(0)\}$, and this is not the case when c is large enough.

To establish this result, we prove a lower bound on the growth of the Weierstrass function near its local maxima, and then we utilize a comparison with the driving function $\kappa\sqrt{1-t}$. Although the result is formulated for the Weierstrass function, it applies more generally to any $\text{Lip}(1/2)$ function with a lower $\text{Lip}(1/2)$ bound near a local extremum.

Effect of random time changes on Loewner hulls, with K. Kobayashi & A. Starnes, [KLS]

This work examines the geometric effect on the Loewner hulls when the driving function is composed with a random time change, such as the inverse of an α -stable subordinator. In contrast to SLE, we show that for a large class of random time changes, the time-changed Brownian motion process does not generate a simple curve Loewner hull. We also develop criteria which can be applied in many situations to determine whether the Loewner hull generated by a time-changed driving function is a simple curve or not. In addition to utilizing the developed criteria, the result about the time-changed Brownian motion relies on a connection between Brownian motion and the 3-dimensional Bessel process.

Although generalizations of SLE_κ to the case of the time-changed Brownian motion are considered and numerically analyzed in [NRR, CMHA], our investigation in this paper provides the first theoretical account of geometric properties of random curves associated with a large class of time-changed functions.

To further understand the effect of the random time-change, we also explore some examples of Loewner hulls generated by time-changed deterministic functions, including a time-changed Weierstrass function. To aid our analysis of these examples, we prove a deterministic result that a driving function that moves faster than at^r for $r \in (0, 1/2)$ generates a hull that leaves the real line tangentially.

Tangential Loewner hulls, [L2]

In this work, we analyze driving functions that approach 0 at least as fast as $a(T-t)^r$ as $t \rightarrow T$, where $r \in (0, 1/2)$, and show that the corresponding Loewner hulls have tangential behavior at time T . This final-time result is a counterpoint to an initial-time result in [KLS]. The final-time question, however, is slightly harder to analyze due to the influence of the past on hull growth.

We also prove a result about trace existence and apply it to show that the Loewner hulls driven by $a(T-t)^r$ for $r \in (0, 1/2)$ have a tangential trace curve. The result about trace existence utilizes the notion of Loewner curvature introduced in [LRoh]. Although the needed condition forces the driving function to be monotone, we give an example to show that monotonicity alone is not enough to guarantee trace existence.

Phase transition for a family of complex-driven Loewner hulls, with J. Utley, [LU]

In [T], H. Tran extends the Loewner theory to the situation when the driving function is complex-valued. In this case, the Loewner map $g_t : \mathbb{C} \setminus L_t \rightarrow \mathbb{C} \setminus R_t$ relates two Loewner hulls, the left hull L_t and the right hull R_t . Tran proves that there exists $\sigma > 0$ so that when the $\text{Lip}(1/2)$ norm of the driving function is bounded by σ , then L_t is a simple curve. However, his work does not identify the optimal value of σ , and he raises the question whether it matches the optimal value in the real-valued case of $\sigma = 4$ from [L1].

Our work in [LU] answers this question, showing that the optimal value of σ is strictly less than 4. In fact, we show that it is bounded above by 3.73. We accomplish this through a careful study of the left and right hulls generated by driving functions of the form $c\sqrt{1-t}$ and $c\sqrt{\tau+t}$ for $c \in \mathbb{C}$. Although we calculate implicit solutions in a similar fashion to the real-valued case studied in [KNK], analyzing these equations is more complicated in our situation and we utilize the notion of holomorphic motion to aid our analysis.

Additionally, we show that the complex-valued case allows for geometric behavior that is not possible in the real-valued case. In particular, it is possible for the domain of the Loewner map $\mathbb{C} \setminus L_t$ to be disconnected, and this behavior arises at the phase transition for the driving function $c\sqrt{1-t}$, except when c is real-valued.

Introduction to discrete modulus

Let $G = (V, E, \sigma)$ be a finite weighted graph, which is also referred to as a *network*, with vertex set V , edge set E , and edge weights $\sigma : E \rightarrow (0, \infty)$. Let Γ be a family of objects in G , such as a collection of paths in G or a collection of spanning trees in G . (A spanning tree is a connected subgraph that contains every vertex and has no cycles.) The *usage matrix* \mathcal{N} is a $|\Gamma| \times |E|$ matrix with $\mathcal{N}(\gamma, e)$ giving the usage of edge e in object γ . In the case that γ is a path or a spanning tree, then $\mathcal{N}(\gamma, e) = \mathbb{1}_{\{e \in \gamma\}}$.

A density $\rho : E \rightarrow [0, \infty)$ on G is called *admissible* for Γ if

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e) \rho(e) \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

When Γ is family of paths or spanning trees we consider $\ell_\rho(\gamma)$ to be the length of γ under ρ . In this setting ρ is admissible for Γ if every path or spanning tree has length at least 1. For $1 \leq p < \infty$, the (*discrete*) p -*modulus* of Γ is defined as

$$\text{Mod}_p(\Gamma) := \inf \sum_{e \in E} \sigma(e) \rho(e)^p,$$

where the infimum is taken over all admissible densities ρ . Further background on the discrete modulus can be found in [ABPPCW, ACFPC, APC].

There is a close connection between the discrete 2-modulus and discrete harmonic functions. For a network $G = (V, E, \sigma)$, when $v, w \in V$ are adjacent, we will use the notation vw to refer to the edge incident to v and w . A function $f : V \rightarrow \mathbb{R}$ is *discrete harmonic* at $v \in V$ if

$$f(v) \sum_{w: vw \in E} \sigma(vw) = \sum_{w: vw \in E} f(w) \sigma(vw),$$

or in other words, the value of f at v is a weighted average of the values of f at the neighbors of v . Viewed from the electric network perspective, a discrete harmonic function is called the *voltage function*, and from its discrete gradient, we obtain the *current flow*.

Recent work regarding discrete modulus

The scaling limit of fair Peano curves, with N. Albin & P. Poggi-Corradini, [ALPC1]

It was shown in [LSW] that the scaling limit of random Peano curves arising from uniform spanning trees of planar grids exists and is SLE_8 . In this work, we study random Peano curves which are generated by random spanning trees that are not necessarily uniform. In particular, we are interested in studying the limiting behavior of laws on spanning trees that arise in the context of spanning tree modulus. The latter random trees are called *fair trees*, because rather than having the same probability of being sampled, they are sampled in such a way to yield the same (if possible) edge probabilities. This work draws on the two papers [ACHPCST, ALPC2] that initiated the study of fair trees.

Fair Peano curves are random Peano curves arising from fair trees. In our main result, we show that if we simply follow the same construction as in [LSW], then the resulting fair Peano curves have a deterministic scaling limit, and further, the limit is not a continuous spacefilling curve. This is partly due to the fact that the number of fair trees is much smaller than the number of uniform trees in this case. As a result, when mesh size of the grid is small we show that the fair trees have a diagonal structure with high probability, as illustrated in Figure 2.

Minimizing the determinant of the graph Laplacian, with N. Albin & P. Poggi-Corradini, [ALPC2]

In this work, we study extremal values for the determinant of the weighted graph Laplacian under simple nondegeneracy conditions on the weights. The weighted graph Laplacian is the $|V| \times |V|$ matrix $D - W$, where D is the diagonal degree matrix, with D_{kk} equal to the sum of the edge weights incident to vertex v_k and W is the symmetric weight matrix with $W_{jk} = \sigma(v_j v_k)$. Kirchhoff's matrix tree theorem gives a way to compute the determinant of the weighted graph Laplacian using weighted spanning trees.

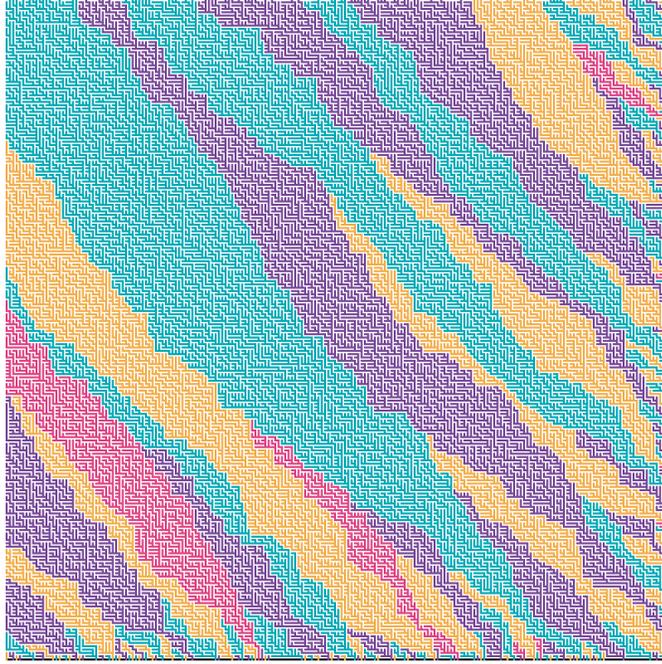


Figure 2: A sample of a fair tree when mesh size is $1/200$, with coloring to show the diagonal nature of the tree branches. Image created by David White.

We derive necessary and sufficient conditions for the determinant of the Laplacian to be bounded away from zero and for the existence of a minimizing set of weights. These conditions are given both in terms of properties of random spanning trees and in terms of a type of density on graphs. This work is a partner project to the previous work, as it provides some of the background for fair trees needed in [ALPC1].

Convergence of the probabilistic interpretation of modulus, with N. Albin & P. Poggi-Corradini, [ALPC3]

Many objects in complex analysis are approximated by discrete analogues. In this work, we establish the convergence of three discrete objects, discrete modulus, discrete paths that are extremal for the modulus, and discrete harmonic functions, to their continuous counterparts, the continuous modulus, the extremal curves (also known as horizontal trajectories) for the modulus, and harmonic functions.

Given a Jordan domain $\Omega \in \mathbb{C}$ and two arcs A, B on $\partial\Omega$, the modulus of the curve family connecting A and B in Ω is famously related, via the conformal map ϕ mapping Ω to a rectangle $R = [0, L] \times [0, 1]$ so that A and B are sent to the vertical sides, to the corresponding modulus in R . Moreover, in the case of the rectangle the family of horizontal segments connecting the two sides has the same modulus as the entire connecting family.

Pulling these segments back to Ω via ϕ yields a family of extremal curves connecting A to B in Ω . We show that these extremal curves can be approximated by some discrete curves arising from an orthodiagonal approximation of Ω . Moreover, we show that these curves carry a natural probability mass function (pmf) deriving from the theory of discrete modulus and that these pmf's converge to the uniform distribution on the set of extremal curves.

The key ingredient is an algorithm that, for an embedded planar graph, takes the current flow between two sets of nodes A and B , and produces a unique path decomposition with non-crossing paths. Moreover, some care was taken to adapt the recent result [GJN] for harmonic convergence on orthodiagonal maps, to our context. As a consequence of this work, we also obtain a rectangle packing, analogous to the famous square uniformization of O. Schramm [S2].

Recent graduate and undergraduate mentoring

Since tenure, I have supervised two Ph.D. students, one Masters student, and five undergraduate research students. Doctoral student Andrew Starnes studied multiple Loewner hulls [St] and worked with postdoc Kei Kobayshi and myself on [KLS] (described above). David Horton's Ph.D thesis contained a further study of Loewner hulls driven by Weierstrass functions. In her Masters project, Lindsay Grinstead explored discrete modulus from various viewpoints. After she graduated, Jessica Robins and I completed the project [LRob] about the hulls driven by the Weierstrass function (described above). Undergraduate students Bridget Jones and Hannah Clark worked to create and numerically analyze examples of space-filling curves generated by Loewner equation. Gavin Glenn generalized the results in [LRob] for his undergraduate math honors thesis project. Most recently, Jeffrey Utley worked on the project [LU] involving complex-valued driving functions (described above), and he developed software to simulate left and right hulls driven by complex-valued functions.

Future Plans

I look forward to furthering my work with the Loewner equation and with the discrete modulus, as well as engaging in new directions. Future plans include

- **Deeper understanding of Loewner hulls driven by complex-valued drivers.** The work in [T, LU] only scratches the surface of our understanding of Loewner hulls driven by complex-valued driving functions and opens up numerous fascinating questions. Where does the phase transition identified in [T] occur? In [LU], we identify one possible difference between the real-valued setting and the complex setting; are there other differences? Further, the results identified so far are all deterministic; what behavior results from random driving functions?

- **A new setting for fair Peano curves.** One reason the scaling limit of the Peano curves in [ALPC1] turned out to be deterministic is because the family of fair spanning trees is much smaller than the family of all spanning trees. With N. Albin and P. Poggi-Corradini, we plan to study a different approach that is expected to give rise to a much richer family of spanning trees. This may result in a non-deterministic scaling limit for the associated Peano curve. In particular, we will utilize σ -weighted uniform spanning trees, which are random spanning trees γ whose probability is proportional to $\prod_{e \in \gamma} \sigma(e)$.
- **Explore new directions.** In Spring 2022, I will be participating in the semester program “The Analysis and Geometry of Random Spaces” at MSRI, for which I am a co-organizer. This experience will give me ample opportunity to explore new topics and possible collaborations.

Acknowledgement: The research in [LRoh], [LT] and [LRob] were supported in part by NSF grant DMS-1100714.

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