# Two-sided $SLE_{8/3}$ and the Infinite Self-Avoiding Polygon

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**Abstract**

In this paper we construct two-sided $SLE_{8/3}$ and describe why it is a model of the infinite self-avoiding polygon.

# 1 Introduction

The Schramm-Loewner evolution, $SLE_\kappa$, as introudced in [9], is a candidate for scaling limits of random paths at criticality in two dimensions. Different values of $\kappa$ correpond to different systems. One value of particular importance is $\kappa = 8/3$, and the corresponding system is conjectured to be the limit of the self-avoiding walk. Trying to understand this led to the definition of the restriction property [6], and then the (nonrigorous) identification of the limit for self-avoiding walks.

The scaling limit can be considered a probability measure on curves $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(0) = 0$. The point 0 is special on the curve. If we look locally at any other point on the curve, then locally we see two curves at that point (the "past" and the "future"). To understand this, one might consider the limit as $r \to \infty$ of the curves $\gamma^{(r)}(s) = \gamma(s+r) - \gamma(r)$. Assuming this limit exists, we should have a limiting measure on curves $\gamma : (-\infty, \infty) \to \mathbb{C}$ with $\gamma(0) = 0$. Equivalently, we can consider this as a measure on pairs of nonintersecting (one-sided) curves.

In this paper, we complete the picture in [7] by describing the measure on two-sided curves. We can think of a two-sided curve as a simple loop that goes through both the origin and infinity. For this reason, we conjecture that this measure is the scaling limit for self-avoiding polygons.

Let us outline this paper. We start by discussing the discrete model, the infinite self-avoiding polygon (ISAP). While we do not know how to prove the scaling limit of ISAP exists, we do use the heuristics from this model to derive the defintion of the two-sided $SLE_{8/3}$. In particular, our approach is to make precise the idea that two-sided radial $SLE_{8/3}$ is obtained by taking two independent radial $SLE_{8/3}$ and conditioning them not to intersect. This is conditioning on an event of probability zero, so one must take a limit. Much of this paper deals with the justifying this limit.

In Section 4, we consider the probability that two radial $SLE_{8/3}$ paths do not intersect, one running to time $\infty$ and the other to time $t$. This tends to zero as $t \to \infty$, and we give the asymptotic behavior. The calculation uses the restriction property which reduces the problem to a derivative estimate for radial $SLE_{8/3}$. We do this calculation in detail

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although the argument is similar to arguments that have appeared in previous papers. After doing this, we weight a path by a corresponding martingale to give a process that we call one-side of radial two-sided $SLE_{8/3}$. It is a radial analogue of the $SLE_{8/3}(\kappa.\rho)$ processes as introduced in [6].

This definition is not obviously symmetric in the two paths. In the next section, we describe an alternate defintion that is obviously symmetric. This shows that the two-sided radial $SLE_{8/3}$ can be grown in any order, that is, we grow one side for a while and then the other. The fact that the order does not make a difference is an example of commutation. See [1] for a much more detailed discussion of commutation properties of $SLE_\kappa$ paths. In Section 6, we prove the restriction property for two-sided radial $SLE_{8/3}$.

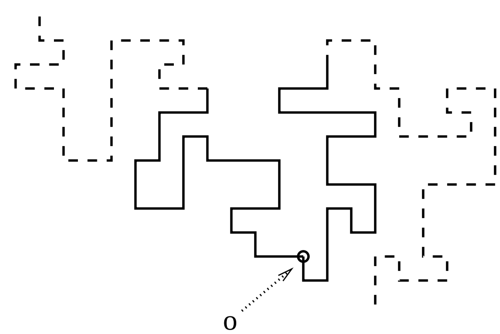
We then consider the chordal analogues of two-sided $SLE_{8/3}$. This is an example of a chordal $SLE(\kappa, \rho)$ process and is also an example of the kind of processes discussed in [1].

Our final section gives a proof that chordal $SLE_\kappa$ is the limit of radial $SLE_\kappa$ in the following sense. Suppose that $\eta : (0, \infty) \to \mathbb{D}$ is a simple curve with $\eta(0+) \in \partial\mathbb{D} \setminus \{1\}$ and $\eta(t) \to 0$ as $t \to \infty$. For every $t < \infty$, we can consider radial $SLE_\kappa$ from 1 to 0 in $\mathbb{D} \setminus \eta(0, t]$. As $t \to \infty$, this measure approaches that of chordal $SLE_\kappa$ from 1 to 0 in $\mathbb{D} \setminus \eta(0, \infty]$. This fact, which we prove for $\kappa \leq 4$, is used in one of our constructions of two-sided radial $SLE_{8/3}$. We give a more preicse formulation of the limit in this section.

Except for the final section, we restrict our consideration to $\kappa = 8/3$ for simplicity. The ideas can be generalized to other $\kappa \leq 4$, but in these cases the measure is not obtained by "two $SLE_\kappa$ paths conditioned not to intersect".

We assume that the reader knows the basic facts about complex variables and conformal transformations (Schwarz lemma, Schwarz reflection, Koebe-(1/4) lemma, Bieberbach estimate, Beurling estimate). See [2] or [4] for references.

## 2 The infinite self-avoiding polygon




In this section we will describe the discrete model whose limit we are trying to describe. A *two-sided self-avoiding walk (2-SAW)* in $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ of lengths $j, k$ (centered at the origin) is a nearest neighbor path

$$\omega = [\omega_{-j}, \omega_{-j+1}, \ldots, \omega_{k-1}, \omega_k], \quad \omega_l \in \mathbb{Z}^2,$$

with $\omega_0 = 0$ and $\omega_l \neq \omega_m, -j \leq l < m \leq k$. Let $\mathcal{L}_{j,k}$ denote the set of such walks and write just $\mathcal{L}_k$ for $\mathcal{L}_{0,k}$. Walks in $\mathcal{L}_k$ are called *(one-sided) self-avoiding walks (SAW)* of length $k$ (rooted at the origin). Note that there is a natural one-to-one correspondence between $\mathcal{L}_{j,k}$ and $\mathcal{L}_{j+k}$. For any $n \geq j, m \geq k$, there is a probability measure $\mu_{j,k,n,m}$ on $\mathcal{L}_{j,k}$ given by

$$\mu_{j,k,n,m}(\omega) = \frac{\#\{\tilde{\omega} \in \mathcal{L}_{n,m} : [\tilde{\omega}_{-j}, \ldots, \tilde{\omega}_k] = \omega\}}{\#(\mathcal{L}_{n,m})}.$$

It is conjectured but has not been proven that the limits

$$\mu_k^*(\omega) = \lim_{m\to\infty} \mu_{0,k,0,m}(\omega),$$

$$\mu_{j,k}(\omega) = \lim_{n,m\to\infty} \mu_{j,k,n,m}(\omega),$$

exist and the second limit is independent of the way that $n, m$ go to infinity. Assuming this conjecture, the measures $\{\mu_k^* : 0 \leq k < \infty\}$ and $\{\mu_{j,k} : 0 \leq j, k < \infty\}$ must be consistent. Hence we get a probability measure $\mu^*$ on infinite (one-sided) SAWs

$$\omega = [\omega_0 = 0, \omega_1, \omega_2, \ldots]$$

and a probability measure $\mu$ on infinite two-sided self-avoiding walks

$$\omega = [\ldots, \omega_{-2}, \omega_{-1}, \omega_0 = 0, \omega_1, \omega_2, \ldots],$$

with projection measures $\mu_k^*, \mu_{j,k}$. We call these measures the *(whole plane) infinite self-avoiding walk (ISAW)* and *(whole plane) infinite self-avoiding polygon (ISAP)*, respectively. Assuming the conjecture, the measure $\mu$ must be stationary, i.e., if

$$\theta_n\omega = [\ldots, \omega_{n-2} - \omega_n, \omega_{n-1} - \omega_n, 0, \omega_{n+1} - \omega_n, \omega_{n+2} - \omega_n, \cdots],$$

then for each integer $n$, $\theta_n\mu = \mu$. We can also consider $\mu$ as a measure on one-sided infinite self-avoiding walks, by looking at the projection

$$\omega \mapsto [\omega_0 = 0, \omega_1, \omega_2, \ldots].$$

We call this *one side of ISAP*. Note that this is *not* the same measure as the ISAW $\mu^*$.

There are two important *critical exponents* for SAWs. While these are usually defined in terms of uniform measures on $\mathcal{L}_n$, they can also be defined in terms of the measure $\mu$. The *mean-square displacement exponent* $\nu$ is defined by saying that under the measure $\mu$, the expected value of $|\omega_n|^2$ grows like $n^{2\nu}$ as $n \to \infty$. The intersection exponent $\zeta$ (this is the same as the exponent $\gamma - 1$ as in [8]) is defined by saying that if two one-sided walks

$$\omega^1 = [\omega_0^1, \omega_1^1, \ldots], \quad \omega^2 = [\omega_0^2, \omega_1^2, \ldots],$$

are chosen independently using $\mu^*$, then the probability that

$$\{\omega_1^1, \ldots, \omega_n^1\} \cap \{\omega_2^1, \ldots, \omega_n^2\} = \emptyset$$

decays like $n^{-\zeta}$. The existence of these exponents has not been proved but there is very strong evidence for the values $\nu = 3/4, \zeta = 11/32$ (see [8]). Combining these two conjectures, we can say that the probability that two independent SAWs reach distance $R$ without intersecting decays like $R^{-11/24}$.

We will consider continuum limits of these measures. Assuming that the exponent $\nu$ exists, we could expect that we can scale the walks by $n^{-\nu}$ to get a measure on continuous curves. There are actually four measure on continuous curves:

- $m^*$: the scaling limit of $\mu^*$ which gives a measure on simple curves $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(0) = 0$.
- $m$: the scaling limit of $\mu$ which gives a measure on simple curves $\gamma : (-\infty, \infty) \to \mathbb{C}$ with $\gamma(0) = 0$. Equivalently, it can be considered as a measure on ordered pairs of curves $(\gamma^1, \gamma^2)$ where $\gamma^j : [0, \infty) \to \mathbb{C}$ with $\gamma^j(0) = 0$ and

$$\gamma^1(0, \infty) \cap \gamma^2(0, \infty) = \emptyset.$$

- The marginal measure on $\gamma^1$ in the measure $m$.
- The conditional measure on $\gamma^2$ given $\gamma^1$ in $m$.

The first and fourth of these measures were considered in [7] where it was shown that there is only one possibility for the scaling limit assuming that the limit exists and is conformally invariant. The fourth measure was considered first. Under the assumption of conformal invariance, given $\gamma^1$ we can map $\mathbb{C} \setminus \gamma^1[0, \infty)$ to the upper half-plane $\mathbb{H}$ mapping 0 to 0 and $\infty$ to $\infty$. Then the stationarity property of the ISAP implies that the measure on $\gamma^2$, appropriately parametrized, satisfies the conformal Markov property. From this it was derived that the distribution must be that of chordal $SLE_\kappa$ as introduced by Schramm [9]. The nature of the scaling limit also implied that the limit would satisfy a certain property which was denoted the *restriction property.* In [6] it was shown that this implies that $\kappa$ must be $8/3$. A similar argument established that the only possibility for $m^*$ is that of whole-plane $SLE_{8/3}$, which is really a version of radial $SLE_{8/3}$.

At the moment there is no proof of the existence of the scaling limit or of its conformal invariance. However, there is strong numerical evidence [3] that the limit of SAWs is given by $SLE_{8/3}$. Moreover, the analogues of the exponents $\nu = 4/3, \zeta = 11/32$ can be computed for $SLE_{8/3}$ which gives very strong evidence for their correctness.

In this paper we will complete the picture by considering the other two measures. This requires considering two $SLE_{8/3}$ at one time. In summary the conjectured scaling limits of the four measures above are

- whole-plane $SLE_{8/3}$, in other words, the distribution of $\gamma[t,\infty)$ given $\gamma[0,t]$ is radial $SLE_{8/3}$.
- two-sided whole-plane $SLE_{8/3}$, in other words, the distribution of $\gamma(-\infty,\infty)$ given $\gamma[-s,t]$ is two-sided radial $SLE_{8/3}$.
- one side of two-sided whole-plane $SLE_{8/3}$,
- chordal $SLE_{8/3}$.

# 3 Radial $SLE_{8/3}$ and restriction

In this section, we remind the reader of the definition of radial $SLE_\kappa$ and the restriction property, which $SLE_{8/3}$ satisfies. The restriction property will be crucial in our development of two-sided $SLE_{8/3}$. See [4] for more details.

Let $B_t$ be a standard Brownian motion. Then radial $SLE_\kappa$ is the solution to the Loewner equation with driving function $\sqrt{\kappa}B_t$. That is to say, it is the family of conformal maps $\tilde{g}_t$ solving the initial value problem

$$\partial_t \tilde{g}_t(z) = \tilde{g}_t(z)\frac{e^{i\sqrt{\kappa}B_t}+\tilde{g}_t(z)}{e^{i\sqrt{\kappa}B_t}-\tilde{g}_t(z)}, \quad \tilde{g}_0(z)=z,$$

for $z \in \mathbb{D}$. These maps satisfy the normalization that $\tilde{g}'(0)=e^t$. It will be convenient for us to change this parametrization by a factor of $1/\kappa$. This is equivalent to considering solutions to the initial value problem

$$\partial_t g_t(z) = \frac{a}{2}\, g_t(z)\, \frac{e^{iB_t}+g_t(z)}{e^{iB_t}-g_t(z)}, \quad g_0(z)=z, \tag{1}$$

for $z\in\mathbb{D}$, where $a=2/\kappa$. Here the conformal maps $g_t$ are normalized so that $g'(0)=e^{at/2}$. Although this change may make some of the exponents in our computations a little less friendly, we prefer to work with a standard Brownian motion rather one multiplied by $\sqrt{\kappa}$.

If $\kappa \le 4$, then radial $SLE_\kappa$ gives a measure on simple curves. The radial $SLE_\kappa$ path is the function $\gamma:[0,\infty)\to\overline{\mathbb{D}}$ with the following properties: $\gamma(0)=e^{iB_0}$, $\gamma(0,\infty)\subset \mathbb{D}\setminus\{0\}$, $\lim_{t\to\infty}\gamma(t)=0$, and $g_t$ is the unique conformal transformation of $\mathbb{D}\setminus\gamma[0,t]$ onto $\mathbb{D}$ with $g_t(0)=0$ and $g_t'(0)>0$. We will often refer to this path as radial $SLE_{8/3}$ (starting at $e^{iB_0}$) rather than the family of maps $g_t$.

Simply stated, the restriction property says that $SLE_{8/3}$ in a subdomain of $\mathbb{D}$ is $SLE_{8/3}$ in $\mathbb{D}$ conditioned to stay in the subdomain. To describe this more fully, let $\mathcal{A}$ denote the set of $A\in\mathbb{D}$ such that $\mathbb{D}\setminus A$ is a simply connected domain containing the origin. If $x\in\mathbb{R}$, let $\mathcal{A}(x)=\mathcal{A}_x=\{A\in\mathcal{A}:\mathrm{dist}(e^{ix},A)>0\}$. For $A\in\mathcal{A}$, let $D_A=\mathbb{D}\setminus A$ and let $\Psi_A:D_A\to\mathbb{D}$ be the unique conformal transformation with $\Psi_A(0)=0$ and $\Psi_A'(0)>0$. If $A\in\mathcal{A}_x$, then $\Psi_A$ has an analytic extension in a neighborhood of $e^{ix}$, and hence $\Psi_A'(e^{ix})$ is well defined. Suppose $A\in\mathcal{A}_x$ and $\gamma$ is radial $SLE_{8/3}$ started at $e^{ix}$. On the event $\{\gamma(0,\infty)\cap A=\emptyset\}$ let

$\eta(t) = \Psi_A \circ \gamma(t)$. Then the restriction property states that the conditional distribution of $\eta$ given $\{\gamma(0,\infty) \cap A = \emptyset\}$ is the same (modulo time reparametrization) as radial $SLE$ from $\Psi_A(e^{ix})$ to 0 in $\mathbb{D}$.

The following computation is at the heart of the restriction property:

$$\mathbb{P}\{\gamma[0,\infty) \cap A = \emptyset\} = |\Psi_A'(e^{iB_0})|^{5/8}\, \Psi_A'(0)^{5/48}, \tag{2}$$

and we will often refer to this simply as the restriction property. To establish (2), one must show that $M_t$, as defined below, is a bounded martingale with $\lim_{t\to\infty} M_t = 1\{\gamma(0,\infty) \cap A = \emptyset\}$. This will allow us to conclude that $M_t = \mathbb{P}[\gamma(0,\infty) \cap A = \emptyset | \mathcal{F}_t]$, since the latter is also a bounded martingale with the same limit at infinity. We define

$$M_t = 1\{\gamma(0,t] \cap A = \emptyset\} |\Psi_{A_t}'(e^{iB_t})|^{5/8}\, \Psi_{A_t}'(0)^{5/48},$$

where $A_t = g_t(A) \cap \mathbb{D}$. Notice that $M_0 = |\Psi_A'(e^{iB_0})|^{5/8}\, \Psi_A'(0)^{5/48}$. For the details, see Section 6.5 of [4]. In Section 5, we will prove the restriction property for two-sided radial $SLE_{8/3}$, and the proof of this will follow the same general argument.

### Whole-plane $SLE_\kappa$

If $D$ is a simply connected domain, $z \in \partial D, w \in D$, then radial $SLE_\kappa$ from $z$ to $w$ in $D$ is the conformal image of radial $SLE_\kappa$ in $\mathbb{D}$ from 1 to 0 by the conformal transformation of $\mathbb{D}$ onto $D$ mapping 1 to $z$ and 0 to $w$. This is considered a measure on paths modulo time reparametrization.

Whole plane $SLE_\kappa$, $0 < \kappa \leq 4$, is the measure on simple curves $\gamma : [0,\infty) \to \mathbb{C}$ with $\gamma(0) = 0$ that has the property that given $\gamma[0,t]$ the conditional distribution of $\gamma(t,\infty)$ is that of radial $SLE_\kappa$ from $\gamma(t)$ to $\infty$ in $\mathbb{C} \setminus \gamma[0,t]$. Standard results about conformal maps can be used to see that this is well defined; see, e.g., [4, Section 6.6].

## 4 One side of two-sided radial $SLE_{8/3}$

The measure on two-sided SAWs of lengths $n, n$ is exactly the same as the measure of two independent (one-sided) SAWs of length $n$ conditioned not to intersect. Hence, we can think of the infinite ISAP as the measure on two independent ISAWs given by condtioning that they do not intersect. This description does not make precise sense since this is conditioning on an event of probability zero. However, we could hope to make rigorous sense by a limiting argument.

Using this as an analogy, we will try to build up two-sided radial $SLE_{8/3}$ by taking two (one-sided) radial $SLE_{8/3}$ paths and conditioning them not to intersect. Again, this is conditioning on an event of probability zero so we must take a limiting argument. We begin our study of two-sided radial $SLE_{8/3}$ by using the restriction property to understand the probability that a $SLE_{8/3}$ path will avoid the beginning of another, independent $SLE_{8/3}$ path. We will obtain a particular martingale $M_s$, and then then we will define one side of two-sided $SLE_{8/3}$ by weighting a $SLE_{8/3}$ path by $M_s/M_0$. The process so obtained is also referred to as "$SLE_{8/3}$ conditioned to avoid another $SLE_{8/3}$." We conclude the section by discussing an alternate definition of the process, derived using Girsanov's Theorem.

Let $B_t$ and $\hat{B}_t$ be independent standard Brownian motions with $z := e^{iB_0} \neq e^{i\hat{B}_0} := \hat{z}$, and let $\gamma$ and $\hat{\gamma}$ denote the $SLE_{8/3}$ paths generated by these Brownian motions (with corresponding functions $g_t$ and $\hat{g}_t$) by solving (1) (with $a = 3/8$). The restriction property tells us that

$$\mathbb{P}\{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset \mid \gamma[0,t]\} = e^{5t/128}\, |g_t'(\hat{z})|^{5/8}.$$

(Recall that $g_t'(0) = e^{3t/8}$ and hence $e^{5t/128} = g_t'(0)^{5/48}$.) Hence,

$$\mathbb{P}\{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset\} = e^{5t/128}\, \mathbb{E}[|g_t'(\hat{z})|^{5/8}], \tag{3}$$

where the expectation is over the first Brownian motion $B_t$. As we wish to understand what happens with (3) as $t$ approaches infinity, we must examine $\mathbb{E}[|g_t'(\hat{z})|^{5/8}]$.

Our first step is to introduce $h_t(z) = -i \log g_t(e^{iz})$. Here the branch of the logarithm is chosen so that $-i\log e^{iB_0} = B_0$. For fixed $t < \infty$, this is well defined in a neighborhood of $\gamma[0,t]$. Note that $|g_t'(\hat{z})| = h_t'(\hat{B}_0)$, allowing us to study $\phi(t,x) = \mathbb{E}[h_t'(x)^{5/8}]$ instead. Equation (1) implies that

$$\dot{h}_t(z) = \frac{3}{8}\, \cot\left(\frac{h_t(z) - B_t}{2}\right), \quad h_0(z) = z. \tag{4}$$

Differentiating this gives

$$h_t'(z) = \exp\left\{-\frac{3}{16}\int_0^t \csc^2\left(\frac{h_s(z) - B_s}{2}\right)\, ds\right\}.$$

If we let $x = \hat{B}_0$ and $V_t = h_t(x) - B_t$, then $V_t$ satisfies

$$dV_t = \frac{3}{8}\cot\left(\frac{V_t}{2}\right)\, dt - dB_t,$$

and

$$h_t'(x) = \exp\left\{-\frac{3}{16}\int_0^t \csc^2\left(\frac{V_s}{2}\right)\, ds\right\}.$$

We will assume for ease that $B_0 = 0$. Let $\mathcal{F}_t$ denote the filtration generated by $\{B_s : 0 \leq s \leq t\}$. Then if $s < t$,

$$\mathbb{E}[h_t'(x)^{5/8} \mid \mathcal{F}_s] = h_s'(x)^{5/8}\, \phi(t-s, V_s).$$

Since this is a martingale, Itô's formula implies that

$$-\dot{\phi}(t,x) + \frac{1}{2}\,\phi''(t,x) + \frac{3}{8}\,\cot\left(\frac{x}{2}\right)\,\phi'(t,x) - \frac{15}{128}\,\csc^2\left(\frac{x}{2}\right)\,\phi(t,x) = 0. \tag{5}$$

We could also have obtained the differential equation for $\phi(t,x)$ by appealing to the Feynman-Kac formula.

Let $L$ be the differential operator described by (5), that is

$$L(f) := -\dot{f} + \frac{1}{2} f'' + \frac{3}{8} \cot\left(\frac{x}{2}\right) f' - \frac{15}{128} \csc^2\left(\frac{x}{2}\right) f.$$

To solve a differential equation like $L(f) = 0$, one often attempts to find a solution of the form $e^{-\beta t} F(x)$, by solving an ordinary differential equation for $F$. In our case, we would need to solve

$$\frac{1}{2} F''(x) + \frac{3}{8} \cot\left(\frac{x}{2}\right) F'(x) + \left[\beta - \frac{15}{128} \csc^2\left(\frac{x}{2}\right)\right] F(x) = 0. \tag{6}$$

We notice that

$$F(x) = c \sin^{3/4}(x/2),$$

is a solution when $\beta = 27/128$.

Although we now know $c_* \, e^{-27t/128} \sin^{3/4}(x/2)$, is a solution to $L(f) = 0$, this cannot be equal $\phi(t,x)$ since they do not have the same initial conditions. In particular, $\phi(0,x) = 1$. We wish to compare $\phi(t,x)$ to this solution, however, and we will be especially interested in the behavior of these two functions as $t$ goes to infinity. Although the choice seems arbitrary at this point, we will take

$$c_* = \frac{\int_0^{2\pi} \sin^{9/4}(x/2)\, dx}{\int_0^{2\pi} \sin^3(x/2)\, dx}.$$

In the subsection below, we explain how we obtained the constant $c_*$. However, the exact value of this constant will not matter for our development of two-sided $SLE_{8/3}$.

To show that $\phi(t,x) \sim c_* \, e^{-27t/128} \sin^{3/4}(x/2)$ as $t \to \infty$, we construct functions $F_\epsilon(t,x)$ and $G_\epsilon(t,x)$ as follows. To begin, for $\epsilon > 0$ set

$$F_\epsilon(t,x) = c_* \, e^{-27t/128} \sin^{3/4}(x/2) + e^{-at/128} \left(1 - c_* \sin^{3/4}(x/2)\right) + \epsilon - \phi(t,x),$$

where $a > 27$. Then

$$L(F_\epsilon(x,t)) = \frac{e^{-at}}{128} \left(a - 15 \csc^2(x/2) - (a-27) c_* \sin^{3/4}(x/2)\right) - \frac{15\epsilon}{128} \csc^2(x/2),$$

which is negative for an appropriate choice of $a$, such as $a = 54$. Looking at the boundary conditions, note that $F_\epsilon(0,x) = \epsilon > 0$, and $F_\epsilon(t,0) = F_\epsilon(t,2\pi) > 0$. Suppose that $F_\epsilon(t,x) < 0$ for some $(t,x) \in [0,\infty) \times [0,2\pi]$. Then there is some point $(t_0,x_0) \in [0,\infty) \times [0,2\pi]$ with $F_\epsilon(t_0,x_0) = 0$ and with $F_\epsilon(t,x) - \phi(t,x) > 0$ for all $t < t_0$. It follows that we must have $\dot{F}_\epsilon(t_0,x_0) \le 0$, $F'_\epsilon(t_0,x_0) = 0$, and $F''_\epsilon(t_0,x_0) \ge 0$. This, however, contradicts the fact that $L(F_\epsilon(t_0,x_0)) < 0$. Therefore, $F_\epsilon \geq 0$, and by letting $\epsilon$ go to zero, we have that $c_*\, e^{-27t/128} \sin^{3/4}(x/2) + c\, e^{-at/128} \geq \phi(t,x)$. In particular, notice that we have

$$\lim_{t\to\infty} e^{27t/128}\phi(t,x) \leq c_*\, \sin^{3/4}(x/2).$$

We also wish to obtain the opposite inequality. We can accomplish this with a similar argument in which we utilize

$$G_\epsilon(t,x) = c_*\, e^{-27t/128} \sin^{3/4}(x/2) \left(1 + e^{-(a_1-27)t/128}\right) - 2c_*\, e^{-a_2 t/128} - \epsilon - \phi(t,x),$$

with $a_2 > a_1 > 27$ appropriately chosen so that $L(G_\epsilon) > 0$. In this way, we obtain

$$\lim_{t\to\infty} e^{27t/128}\phi(t,x) = c_*\, \sin^{3/4}(x/2) = 2^{-3/4} c_*\, |e^{ix} - 1|^{3/4}. \tag{7}$$

We have now established the following proposition.

**Proposition 4.1.** Suppose $\gamma, \hat\gamma$ are independent radial $SLE_{8/3}$ curves started at $z, \hat z$ respectively. Then,

$$\lim_{t\to\infty} e^{11t/64}\, \mathbb{P}\{\gamma[0,t] \cap \hat\gamma[0,\infty) = \emptyset\} = C_*\, |z - \hat z|^{3/4},$$

where

$$C_* = 2^{-3/4} c_* = 2^{-3/4} \frac{\int_0^{2\pi} \sin^{9/4}(x/2)\; dx}{\int_0^{2\pi} \sin^3(x/2)\; dx}.$$

*Proof.* Assume without loss of generality that $z = 1, \hat z = e^{ix}$, and as before, let $\phi(t,x) = \mathbb{E}[h'(x)^{5/8}]$. Then,

$$\mathbb{P}\{\gamma[0,t] \cap \hat\gamma[0,\infty) = \emptyset\} = e^{5t/128}\, \phi(t,x).$$

The proposition then follows from (7).

$\square$

We continue to use the notation $x = \hat B_0$ and $V_s = h_s(x) - B_s$, and now we wish to consider

$$M_s = C_*\, e^{27s/128}\, |g_s'(e^{ix})|^{5/8} |e^{iB_s} - g_s(e^{ix})|^{3/4} = e^{27s/128}\, h_s'(x)^{5/8} F(V_s),$$

where $F(x) = c_* \sin^{3/4}(x/2)$ satisfies (6) with $\beta = 27/128$. Using this, we can compute that

$$dM_s = -\frac{3}{8} \cot\left(\frac{V_s}{2}\right)\, M_s\, dX_s,$$

where $X_s = B_s$. Therefore $M_s$ is a martingale with $|M_s| \leq c'\, e^{27s/128}$.

We claim that

$$M_s = \lim_{t\to\infty} e^{11t/64}\,\mathbb{P}\{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset \mid \mathcal{F}_s\}.$$

By using the restriction property, we have

$$\begin{aligned}
&\mathbb{P}\{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset \mid \mathcal{F}_s\} \\
&\quad = \mathbb{P}\{\gamma[0,s] \cap \hat{\gamma}[0,\infty) = \emptyset \text{ and } g_s \circ \gamma[s,t] \cap g_s \circ \hat{\gamma}[0,\infty) = \emptyset \mid \mathcal{F}_s\} \\
&\quad = e^{5s/128}\, h_s'(x)^{5/8}\,\mathbb{P}\{g_s \circ \gamma[s,t] \cap g_s \circ \hat{\gamma}[0,\infty) = \emptyset \mid \mathcal{F}_s\}
\end{aligned}$$

Proposition 4.1 implies that

$$\lim_{t\to\infty} e^{11(t-s)/64}\,\mathbb{P}\{g_s \circ \gamma[s,t] \cap g_s \circ \hat{\gamma}[0,\infty) = \emptyset \mid \mathcal{F}_s\} = C_* |e^{iB_s} - g_s(e^{ix})|,$$

which proves the claim.

This second view of $M_s$ leads us to define one side of two-sided $SLE_{8/3}$, otherwise called $SLE_{8/3}$ conditioned to avoid another $SLE_{8/3}$, by weighting a $SLE_{8/3}$ path by $M_s/M_0$. In particular, let $\mathbf{Q}$ denote the probability measure on paths induced by this positive martingale, and let $\mathbf{Q}_s$ denote this measure restricted to $X_t, 0 \leq t \leq s$. Then $d\mathbf{Q}_s/d\mathbf{P} = M_s/M_0$.

By making use of Girsanov's Theorem, we obtain an alternate viewpoint of this object: we can consider one side of two-sided $SLE_{8/3}$ to be the solution to the Loewner equation where the driving term has an appropriate drift. Girsanov's Theorem states that with respect to the measure $\mathbf{Q}$,

$$W_s := X_s + \frac{3}{8}\int_0^s \cot\left(\frac{V_t}{2}\right)\, dt,$$

is a standard Brownian motion, or in other words,

$$dX_s = -\frac{3}{8}\, \cot\left(\frac{V_s}{2}\right) ds + dW_s.$$

Thus one side of two-sided $SLE_{8/3}$ is the process generated by the Loewner equation with driving term $X_s$, where $W_s$ is a Brownian motion,

$$dX_s = -\frac{3}{8}\, \cot\left(\frac{\hat{X}_s - X_s}{2}\right) ds + dW_s,$$

and

$$d\hat{X}_s = \frac{3}{8} \cot\left(\frac{\hat{X}_s - X_s}{2}\right)\, ds.$$

Note that we have replaced $h_s(x)$ by $\hat{X}_s$ in anticipation of the notation we will use in the next section, which will reflect the fact that the two sides of two-sided $SLE_{8/3}$ are symmetric.

## The constant $c_*$

In this subsection, we briefly describe how we obtained the constant $c_*$ found in our previous calculations. It arises naturally when finding a certain invariant density. The basic ideas used here are discussed in Section 1.11 of [4].

Recall that

$$dV_t = \frac{3}{8} \cot\left(\frac{V_t}{2}\right) dt - dB_t,$$

and

$$h_t'(x)^{5/8} = \exp\left\{-\frac{15}{128}\int_0^t \csc^2(V_s/2)\, ds\right\}.$$

Let $p(t,x,y)$ denote the transition probability density defined by

$$\mathbb{E}^x\left[f(V_t)\, h_t'(x)^{5/8}\right] = \int_0^{2\pi} p(t,x,y)\, f(y)\, dy,$$

and notice that for $\phi(t,x) = \mathbb{E}^x\left[h_t'(x)^{5/8}\right]$, we have that

$$\phi(t,x) = \int_0^{2\pi} p(t,x,y)\, dy.$$

There are two differential equations that $p(t,x,y)$ must satisfy, one for when $y$ is fixed and the other for $x$ fixed. If we had started in a simpler situation and were interested in the transition probability density for $\mathbb{E}^x[f(V_t)]$, we could easily obtain the two differential equations from Kolmogorov's backward and forward equations. Although our situation is slightly more complicated, it is not difficult to find the desired equations. We have already seen that $\phi(t,x)$ satisfies the differential equation (5), and one can show that $p$ must satisfy this as well. That is,

$$\dot{p} = \frac{1}{2} p_{xx} + \frac{3}{8} \cot(x/2)\, p_x - \frac{15}{128} \csc^2(\frac{x}{2})\, p.$$

By considering the adjoint, we can find our second differential equation for $p$:

$$\dot{p} = \frac{1}{2} p_{yy} - \partial_y\, [\frac{3}{8} \cot(y/2)\, p] - \frac{15}{128} \csc^2(y/2)\, p.$$

See section 1.11 of [4] for further details.

As we did for $\phi$, we can solve these differential equations to understand the behavior of $p$ as $t$ approaches infinity. In particular, we will find positive functions $\psi_1$ and $\psi_2$ satisfying the ordinary differential equations

$$\frac{1}{2}\, \psi_1''(x) + \frac{3}{8} \cot(x/2)\, \psi_1'(x) + [\beta - \frac{15}{128} \csc^2(x/2)]\, \psi_1(x) = 0,$$

$$\frac{1}{2}\,\psi_2''(y) - \partial_y\,[\frac{3}{8}\,\cot(y/2)\,\psi_2(y)] + [\beta - \frac{15}{128}\,\csc^2(y/2)]\,\psi_2(y) = 0,$$

so that $p(t,x,y) \sim c\,e^{-\beta t}\,\psi_1(x)\,\psi_2(y)$ as $t \to \infty$. The desired solutions are

$$\psi_1(x) = \sin^{3/4}(x/2), \qquad \psi_2(y) = \sin^{9/4}(y/2),$$

with $\beta = 27/128$. We therefore get

$$p(t,x,y) \sim c_1\,e^{-27t/128}\,\sin^{3/4}(x/2)\,\sin^{9/4}(y/2), \qquad t \to \infty. \tag{8}$$

We will now compute the constant $c_1$ as well as our previous constant $c_*$. The function $\sin^{9/4}(y/2)$ can be considered an invariant density in the sense that

$$\int_0^{2\pi} \sin^{9/4}(x/2)\,p(t,x,y)\,dx = e^{-27t/128}\,\sin^{9/4}(y/2).$$

Plugging in (8) gives

$$c_1 = \left[\int_0^{2\pi} \sin^3(x/2)\,dx\right]^{-1}.$$

Using (8) with

$$\phi(t,x) = \mathbb{E}^x\,[h_t'(x)^{5/8}] = \int_0^{2\pi} p(t,x,y)\,dy,$$

gives again that

$$\phi(t,x) \sim c_*\,e^{-27t/128}\,\sin^{3/4}(x/2),$$

and here it is clear that

$$c_* = \frac{\int_0^{2\pi} \sin^{9/4}(x/2)\,dx}{\int_0^{2\pi} \sin^3(x/2)\,dx}.$$

# 5 Two-sided radial $SLE_{8/3}$

In this section we will define two-sided radial $SLE_{8/3}$ by weighting two independent $SLE_{8/3}$ paths by a two-parameter martingale $N_{s,r}/N_{0,0}$. We will show that $N_{s,r}$ is symmetric in $s$ and $r$, and this will give commutation, meaning that we can "grow" the two curves in either order.

We begin by establishing the notation we will use. Let $\gamma$ and $\hat{\gamma}$ be independent radial $SLE_{8/3}$ paths starting at $z$ and $\hat{z}$ with corresponding conformal maps $g_t$ and $\hat{g}_t$. Let $\mathcal{F}_{s,r}$

denote the $\sigma$-algebra generated by $\{B_t : 0 \le t \le s\} \cup \{\hat{B}_t : 0 \le t \le r\}$, let $E_{t,r}$ denote the event

$$E_{t,r} = \{\gamma[0,t] \cap \hat{\gamma}[0,r] = \emptyset\},$$

and let $E_t = E_{t,\infty}$. In Proposition 4.1 we showed that

$$\lim_{t\to\infty} e^{11t/64}\,\mathbb{P}(E_t) = C_*\,|z - \hat{z}|^{3/4}.$$

On the event $E_{s,r}$, we let $v_{s,r}$ denote the unique conformal transformation of $\mathbb{D} \setminus (\gamma(0,s] \cup \hat{\gamma}(0,r])$ onto $\mathbb{D}$ with $v_{s,r}(0) = 0$ and $v'_{s,r}(0) > 0$. Note that $g_s = v_{s,0}$ and $\hat{g}_r = v_{0,r}$. For $U_{s,r} := v_{s,r}(\gamma(s))$ and $\hat{U}_{s,r} := v_{s,r}(\hat{\gamma}(r))$, we observe that $U_{s,0} = e^{iB_s}$, $\hat{U}_{s,0} = g_s(\hat{z})$, $U_{0,r} = \hat{g}_r(z)$, and $\hat{U}_{0,r} = e^{i\hat{B}_r}$. Finally, define $g_{s,r}$ and $\hat{g}_{s,r}$ by the relations $v_{s,r} = g_{s,r} \circ \hat{g}_r = \hat{g}_{s,r} \circ g_s$.

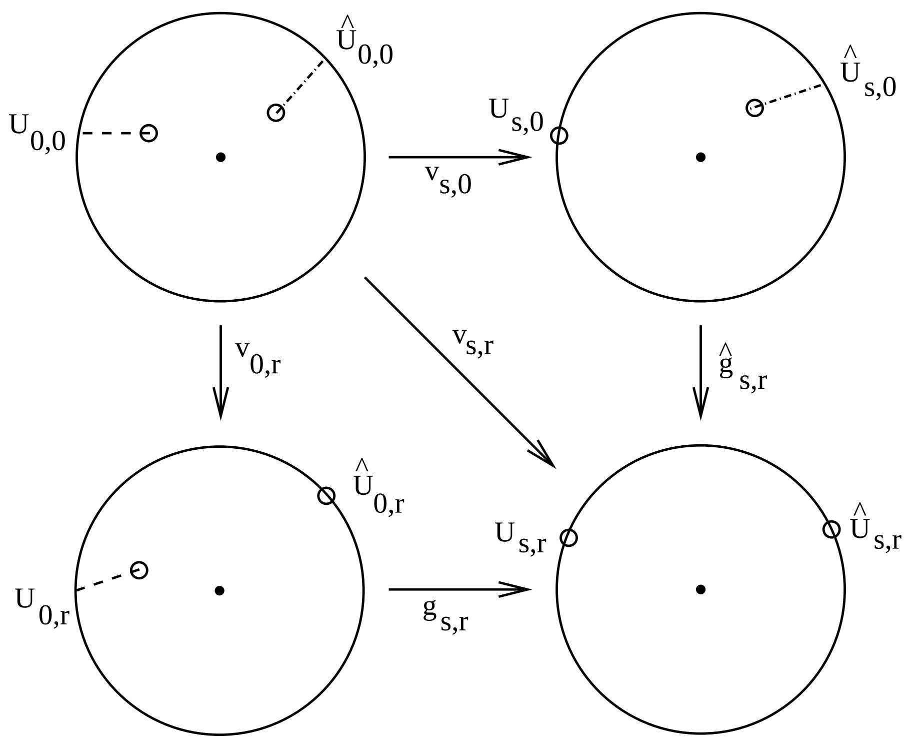



Figure 1: The maps $v_{s,r}, g_{s,r}, \hat{g}_{s,r}$. Note that $g_s = g_{s,0} = v_{s,0}$ and similarly $\hat{g}_r = \hat{g}_{0,r} = v_{0,r}$.

Suppose $0 \le s \le t, 0 \le r < \infty$, and let

$$N_{s,r} := \lim_{t\to\infty} e^{11t/64}\,\mathbb{P}(E_t \mid \mathcal{F}_{s,r}).$$

Although it is not immediately obvious that the limit exists and that the definition of $N_{s,r}$ is symmetric, that is, it does not depend the way we order the two $SLE_{8/3}$'s, the next lemma establishes these facts.

**Lemma 5.1.**

$$N_{s,r} = C_* \, 1_{E_{s,r}} \; g'_{s,r}(0)^{5/48} \, |g'_{s,r}(\hat{U}_{0,r})|^{5/8} \, \hat{g}'_{s,r}(0)^{5/48} \, |\hat{g}'_{s,r}(U_{s,0})|^{5/8} \cdot |U_{s,r} - \hat{U}_{s,r}|^{3/4} \, v'_{s,r}(0)^{11/24}.$$

In particular, $N_{s,r}$ is a two-parameter martingale in the sense that if $s \leq s'$ and $r \leq r'$,

$$\mathbb{E}[N_{s',r'} \mid \mathcal{F}_{s,r}] = N_{s,r}. \tag{9}$$

Moreover, there is a $c$ such that for all $s$ and $r$, $N_{s,r} \leq c \, e^{11(r+s)/64}$.

*Proof.* We first write

$$N_{s,r} = \lim_{t \to \infty} e^{11t/64} \, \mathbb{E}[\mathbb{P}(E_t \mid \mathcal{F}_{t,r}) \mid \mathcal{F}_{s,r}]$$

The restriction property implies that

$$\mathbb{P}[E_t \mid \mathcal{F}_{t,r}] = 1_{E_{t,r}} \; g'_{t,r}(0)^{5/48} \, |g'_{t,r}(\hat{U}_{0,r})|^{5/8},$$

and hence,

$$\begin{aligned} N_{s,r} &= \lim_{t \to \infty} e^{11t/64} \, \mathbb{E}[1_{E_{t,r}} \; g'_{t,r}(0)^{5/48} \, |g'_{t,r}(\hat{U}_{0,r})|^{5/8} \mid \mathcal{F}_{s,r}] \\ &= \lim_{t \to \infty} \mathbb{P}[E_{t,r} \mid \mathcal{F}_{s,r}] \, e^{11t/64} \, \mathbb{E}[g'_{t,r}(0)^{5/48} \, |g'_{t,r}(\hat{U}_{0,r})|^{5/8} \mid \mathcal{F}_{s,r}, \, E_{t,r}]. \end{aligned}$$

Another application of the restriction property gives

$$\lim_{t \to \infty} \mathbb{P}[E_{t,r} \mid \mathcal{F}_{s,r}] = \mathbb{P}[E_{\infty,r} \mid \mathcal{F}_{s,r}] = 1_{E_{s,r}} \; \hat{g}'_{s,r}(0)^{5/48} \, |\hat{g}'_{s,r}(U_{s,0})|^{5/8}.$$

On the event $E_{t,r}$, let $u_{t,s,r} = v_{t,r} \circ v_{s,r}^{-1}$, so that $g_{t,r} = u_{t,s,r} \circ g_{s,r}$. Therefore,

$$\begin{aligned} N_{s,r} = & 1_{E_{s,r}} \; \hat{g}'_{s,r}(0)^{5/48} \, |\hat{g}'_{s,r}(U_{s,0})|^{5/8} \, g'_{s,r}(0)^{5/48} \, |g'_{s,r}(\hat{U}_{0,r})|^{5/8} \\ & \cdot \lim_{t \to \infty} e^{11t/64} \, \mathbb{E}[u'_{t,s,r}(0)^{5/48} \, |u'_{t,s,r}(\hat{U}_{s,r})|^{5/8} \mid \mathcal{F}_{s,r}, \, E_{t,r}]. \end{aligned}$$

Using the restriction property a third time, we can rewrite the above expectation as

$$\mathbb{P}\{\text{a } SLE_{8/3} \text{ path from } \hat{U}_{s,r} \text{ to } 0 \text{ avoids } v_{s,r} \circ \gamma[s,t]\}.$$

If we reparametrize the curve $v_{s,r} \circ \gamma[s,\tau]$ so that $u'_{\tau,s,r}(0) = e^{3\tau/8}$, we will be able to apply Proposition 4.1 to obtain

$$\lim_{\tau \to 0} e^{11\tau/64} \, \mathbb{P}\{\text{a } SLE_{8/3} \text{ path from } \hat{U}_{s,r} \text{ to } 0 \text{ avoids } v_{s,r} \circ \gamma[s,\tau]\} = C_* \, |U_{s,r} - \hat{U}_{s,r}|^{3/4}.$$

Thus,

$$\begin{aligned}\lim_{t\to\infty} e^{11t/64}\, & \mathbb{E}[u'_{t,s,r}(0)^{5/48}\, |u'_{t,s,r}(\hat U_{s,r})|^{5/8} \mid \mathcal{F}_{s,r},\, E_{t,r}] \\ & = C_* \, |U_{s,r} - \hat U_{s,r}|^{3/4} \lim_{t\to\infty} e^{11t/64} \left(u'_{t,r,s}(0)\right)^{-11/24} \\ & = C_* \, |U_{s,r} - \hat U_{s,r}|^{3/4}\, v'_{s,r}(0)^{11/24},\end{aligned}$$

since $u'_{t,r,s}(0) = v'_{t,r}(0)/v'_{s,r}(0)$ and $v'_{t,r}(0) \sim g'_t(0) = e^{3t/8}$ as $t$ approaches infinity with $r$ fixed. The exponent $11/24$ results from $(11/64)/(3/8)$. We have now shown that $N_{s,r}$ has the form we claimed.

From the restriction property, we can see that

$$[g'_{s,r}(0)^{5/48}\, |g'_{s,r}(\hat U_{0,r})|^{5/8}]\, [\hat g'_{s,r}(0)^{5/48}\, |\hat g'_{s,r}(U_{s,0})|^{5/8}] \leq 1,$$

since the terms in brackets each represent probabilities of events. Also $v'_{s,r}(0) \leq c\, e^{3(r+s)/8}$, implying that $|N_{s,r}| \leq c\, e^{11(r+s)/64}$. The relation (9) is immediate.

□

Notice that if $r = 0$, $v_{s,0} = g_{s,0} = g_s$ and $\hat g_{s,0}$ is the identity. Hence,

$$N_{s,0} = C_* \, e^{27s/128}\, |g'_s(\hat z)|^{5/8}\, |e^{iB_s} - g_s(\hat z)|^{3/4},$$

which is equal to the martingale $M_s$ that we considered in the previous section.

With Lemma 5.1 behind us, we will now define two-sided radial $SLE_{8/3}$:

**Definition**. If $z, \hat z \in \partial\mathbb{D}$ are distinct points, then two-sided radial $SLE_{8/3}$ in $\mathbb{D}$ starting at $(z, \hat z)$ is the measure on ordered pairs of paths $(\gamma, \hat\gamma)$ such that for each $s, r < \infty$, the distribution of

$$\gamma(s'), 0 \leq s \leq s', \qquad \hat\gamma(r'), 0 \leq r \leq r'$$

is given by saying that the Radon-Nikodym derivative of this distribution with respect to that of *independent* radial $SLE$'s starting at $z, \hat z$ is $N_{s,r}/N_{0,0}$, which is equal to

$$1_{E_{s,r}}\; g'_{s,r}(0)^{5/48}\, |g'_{s,r}(\hat U_{0,r})|^{5/8}\; \hat g'_{s,r}(0)^{5/48}\, |\hat g'_{s,r}(U_{s,0})|^{5/8}\, \frac{|U_{s,r} - \hat U_{s,r}|^{3/4}}{|z - \hat z|^{3/4}}\; v'_{s,r}(0)^{11/24}.$$

Note that we can write

$$\frac{N_{s,r}}{N_{0,0}} = \frac{N_{s,0}}{N_{0,0}}\,\frac{N_{s,r}}{N_{s,0}} = \frac{N_{0,r}}{N_{0,0}}\,\frac{N_{s,r}}{N_{0.r}}. \tag{10}$$

In other words we can grow the first path and then the second or the second path then the first and we get the same distribution. Since

$$N_{s,0} = M_s = C_* \, e^{27s/128}\, |g'_s(\hat z)|^{5/8}\, |e^{iB_s} - g_s(\hat z)|^{3/4},$$

growing one of the paths corresponds to growing one side of two-sided $SLE_{8/3}$, which was the topic of the previous section. Again, we can describe this in terms of the Loewner equation. Let $U_{s,0} = e^{iX_s}, \hat{U}_{s,0} = e^{i\hat{X}_s}$. If $X_s$ and $\hat{X}_s$ satisfy

$$dX_s = -\frac{3}{8}\cot\left(\frac{\hat{X}_s - X_s}{2}\right)\, ds + dB_s, \qquad d\hat{X}_s = \frac{3}{8}\cot\left(\frac{\hat{X}_s - X_s}{2}\right)\, ds, \tag{11}$$

then the Loewner chain driven by $X_s$ gives the two-sided $SLE_{8/3}$ up to time $(s, 0)$. To get the second path we map down by $g_s$ and then proceed similarly, interchanging the roles of the two paths. In this case we need to go until a time that depends on the path $\gamma$.

We will use the notation $(\gamma, \hat{\gamma})$ to denote two-sided $SLE_{8/3}$ starting at $(z, \hat{z})$. By a slight abuse of notation, we will write just $\gamma$ for $\gamma[0,\infty]$ and $(\gamma, \hat{\gamma})$ for $\gamma[0,\infty] \cup \hat{\gamma}[0,\infty]$. Note that these curves are defined modulo reparametrization.

## Conditional distribution of $\hat{\gamma}$ given $\gamma$

Suppose that $t$ is large and we have generated $\gamma[0,t]$ according to the distribution of one side of two-sided radial $SLE_{8/3}$. What is the conditional distribution of $\hat{\gamma}[0,1]$ given this? By (10), we can see that the Radon-Nikodym derivative of this conditional measure with respect to that of radial $SLE_{8/3}$ run until time 1 is $N_{t,1}/N_{t,0}$, i.e.,

$$1_{E_{t,1}}\, \frac{g_{t,1}'(0)^{5/48}\, |g_{t,1}'(\hat{U}_{0,1})|^{5/8}\, \hat{g}_{t,1}'(0)^{5/48}\, |\hat{g}_{t,1}'(U_{t,0})|^{5/8}\, |U_{t,1} - \hat{U}_{t,1}|^{3/4}\, v_{t,1}'(0)^{11/24}}{e^{27t/128}\, |g_t'(\hat{z})|^{5/8}\, |e^{iB_t} - g_t(\hat{z})|^{3/4},}. \tag{12}$$

Given $\gamma$, we can consider three distributions of $\hat{\gamma}[0,1]$:

- radial $SLE_{8/3}$ in $\mathbb{D}$ weighted by $N_{t,1}/N_{t,0}$;
- radial $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma[0,t]$ from $\hat{z}$ to 0;
- chordal $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma[0,\infty) \cup \{0\}$ from $\hat{z}$ to $\{0\}$.

What we would like to show is that as $t \to \infty$ these three distributions on $\hat{\gamma}[0,1]$ are asymptotically the same.

Let us first consider the second distribution. The restriction property states that the distribution of radial $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma[0,t]$ from $\hat{z}$ to 0 is exactly the same as that of radial $SLE_{8/3}$ in $\mathbb{D}$ from $\hat{z}$ to 0 conditoned on the event $E = \{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset\}$. From this we see that the Radon-Nikodym derivative of the second distribution with respect to that of radial $SLE_{8/3}$ in $\mathbb{D}$ from $\hat{z}$ to 0 is given by

$$\frac{1\{\hat{\gamma}[0,1] \cap \gamma[0,t] = \emptyset\}\; \mathbb{P}\{\hat{\gamma}[1,\infty) \cap \gamma[0,t] = \emptyset \mid \hat{\gamma}[0,1]\}}{\mathbb{P}\{\gamma[0,t] \cap \hat{\gamma}[0,\infty) = \emptyset\}}$$

which equals

$$1_{E_{t,1}}\, \frac{g_{t,1}'(0)^{5/48}\, |g_{t,1}'(\hat{U}_{0,1})|^{5/8}}{g_{t,0}'(0)^{5/48}\, |g_{t,0}'(\hat{z})|^{5/8}} = 1_{E_{t,1}}\, \frac{g_{t,1}'(0)^{5/48}\, |g_{t,1}'(\hat{U}_{0,1})|^{5/8}}{e^{5t/128}\, |g_{t,0}'(\hat{z})|^{5/8}}.$$

If we recall that $v'_{t,1}(0) = \hat{g}'_{t,1}(0)\, g'_{t,0}(0) = e^{3t/8}\, \hat{g}'_{t,1}(0)$, we can see that the quantity in (12) equals this times

$$\hat{g}'_{t,1}(0)^{27/48}\, |\hat{g}'_{t,1}(U_{t,0})|^{5/8}\, \frac{|U_{t,1} - \hat{U}_{t,1}|^{3/4}}{|U_{t,0} - \hat{U}_{t,0}|^{3/4}}.$$

**Lemma 5.2.** For every $\epsilon > 0$, there exists an $r > 0$ such that the following holds. Suppose $\gamma : (0,1] \to \mathbb{D}, \hat{\gamma} : (0,1] \to \mathbb{D}$ are two simple curves with $\gamma(0+), \hat{\gamma}(0+) \in \partial\mathbb{D}$ and $\gamma(0,1] \cap \hat{\gamma}(0,1] = \emptyset$. Suppose $|\hat{\gamma}(t)| \geq 1/4$ for all $t$; $0 \notin \gamma(0,1]$ and $|\gamma(1)| \leq r$. Let $g$ denote the unique conformal transformation of $\mathbb{D} \setminus \gamma(0,1]$ onto $\mathbb{D}$ with $g(0) = 0, g'(0) > 0$. Let $A = g(\hat{\gamma}(0,1])$. Let $h$ denote the unique conformal transformation of $\mathbb{D} \setminus A$ onto $\mathbb{D}$ with $h(0) = 0, h'(0) > 0$. Let $z = g(\gamma(1)), w = g(\hat{\gamma}(0)), z^* = h(z), w^* = h(g(\hat{\gamma}(1)))$. Then,

$$1 \leq h'(0) \leq 1 + \epsilon, \qquad |\,|h'(z)| - 1\,| \leq \epsilon,$$

$$\left| \frac{|z - w|}{|z^* - w^*|} - 1 \right| \leq \epsilon.$$

We will not give the details of this proof, but the key estimate is the Beurling estimate (see, e.g., [4, Section 3.8]) which can be used to show that there is a $c$ such that

$$\operatorname{diam}(A) \leq c\, r^{1/2} \operatorname{dist}(z, A).$$

A similar argument with more details is given in Section 8.

The asymptotic equivalence of the second and third distributions will be discussed in Section 8

# 6 Restriction property for two-sided radial $SLE_{8/3}$

As one might expect, two-sided radial $SLE_{8/3}$ satisfies the restriction property, which means that two-sided radial $SLE_{8/3}$ conditioned to stay in a subdomain of $\mathbb{D}$ is two-sided radial $SLE_{8/3}$ in that subdomain. More precisely, the restriction property states that the conditional distribution of $\Psi_A(\gamma, \hat{\gamma})$ given $\{(\gamma, \hat{\gamma}) \cap A = \emptyset\}$ is the same as two-sided $SLE_{8/3}$ from $(\Psi_A(e^{ix}), \Psi_A(e^{i\hat{x}}))$ to 0 in $\mathbb{D}$, where $(\gamma, \hat{\gamma})$ is a two-sided $SLE_{8/3}$ started at $(e^{ix}, e^{i\hat{x}})$ and $A \in \mathcal{A}(x) \cap \mathcal{A}(\hat{x})$. Here we have continued to use our notation from the second section, that is, $\mathcal{A}(x)$ denotes the set of $A \in \mathbb{D}$ such that $\mathbb{D} \setminus A$ is a simply connected domain containing the origin with $\operatorname{dist}(e^{ix}, A) > 0$, $D_A$ denotes $\mathbb{D} \setminus A$, and $\Psi_A : D_A \to \mathbb{D}$ is the unique conformal transformation with $\Psi_A(0) = 0$, and $\Psi'_A(0) > 0$.

In this section we prove the restriction property for two-sided radial $SLE_{8/3}$. The proof follows the same general outline as in the case of one-sided radial or chordal $SLE_{8/3}$. The main difference is that the martingale we will consider is more complicated, and hence our calculations using Itô calculus will be more involved. The following theorem contains the essential result for the restriction property.

**Theorem 1.** Suppose $x, \hat{x} \in \mathbb{R}$ and $A \in \mathcal{A}(x) \cap \mathcal{A}(\hat{x})$. If $(\gamma, \hat{\gamma})$ denotes two-sided radial $SLE_{8/3}$ from $(e^{ix}, e^{i\hat{x}})$ to 0, then

$$\mathbb{P}\{(\gamma, \hat{\gamma}) \cap A = \emptyset\} = \Psi_A'(0)^{2/3} \, |\Psi_A'(e^{ix}) \, \Psi_A'(e^{i\hat{x}})|^{5/8} \, \left| \frac{\Psi_A(e^{i\hat{x}}) - \Psi_A(e^{ix})}{e^{i\hat{x}} - e^{ix}} \right|^{3/4}.$$

If $x = \hat{x}$, this is to be interpreted as

$$\Psi_A'(0)^{2/3} \, |\Psi_A'(e^{ix})|^2.$$

We first discuss how this implies the restriction property. To specify the distribution of a pair of simple, non-intersecting paths $(\eta_1, \eta_2)$ from $(e^{x_1}, e^{x_2})$ to 0, it suffices to give $\mathbb{P}\{(\eta_1, \eta_2) \cap K = \emptyset\}$ for each $K \in \mathcal{A}(x_1) \cap \mathcal{A}(x_2)$. Thus, to prove the restriction property from the previous theorem, we need to show that for $A \in \mathcal{A}(x_1) \cap \mathcal{A}(x_2)$, then $P_0 := \mathbb{P}\{\Psi_A(\gamma, \hat{\gamma}) \cap K = \emptyset \mid (\gamma, \hat{\gamma}) \cap A = \emptyset\}$ is

$$\Psi_K'(0)^{2/3} \, |\Psi_K'(z) \, \Psi_K'(\hat{z})|^{5/8} \, \left| \frac{\Psi_K(\hat{z}) - \Psi_K(z)}{\hat{z} - z} \right|^{3/4}, \tag{13}$$

where $z = \Psi_A(e^{ix})$ and $\hat{z} = \Psi_A(e^{i\hat{x}})$. We first note that $P_0$ is equal to

$$\frac{\mathbb{P}\{(\gamma, \hat{\gamma}) \cap (A \cup \Psi_A^{-1}(K)) = \emptyset\}}{\mathbb{P}\{(\gamma, \hat{\gamma}) \cap A = \emptyset\}}.$$

Since $\Psi_{A \cup \Psi_A^{-1}(K)} = \Psi_K \circ \Psi_A$, we obtain (13) from another application of Theorem 1.

In order to prove Theorem 1, we start with a simple lemma.

**Lemma 6.1.** There is a $c < \infty$ such that if $A, x, \hat{x}$, and $\Psi_A$ are as in the theorem, then

$$\Psi_A'(0)^{2/3} \, |\Psi_A'(e^{ix})|^{5/8} \, |\Psi_A'(e^{i\hat{x}})|^{5/8} \, \left| \frac{\Psi_A(e^{ix}) - \Psi_A(e^{i\hat{x}})}{e^{ix} - e^{i\hat{x}}} \right|^{3/4} \leq c \, r^{1/3},$$

where $r = \mathrm{inrad}(\mathbb{D} \setminus A) = \mathrm{dist}(0, A)$.

*Proof.* If $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal transformation with $f(0) = 0$, the Koebe (1/4)-theorem and the Schwarz lemma imply that $1 \leq f'(0)/\mathrm{dist}[0, \partial f(\mathbb{D})] \leq 4$. Applying this to $f = \Psi_A^{-1}$ gives

$$\frac{1}{4r} \leq \Psi_A'(0) \leq \frac{1}{r}.$$

Suppose $I \subset \partial \mathbb{D}$ and let $h(A; I)$ denote the harmonic measure of $I$ in $\mathbb{D} \setminus A$ from 0; in other words, $h(A; I)$ is the probability that a Brownian motion starting at the origin leaves $\mathbb{D}$ at $I$. The Beurling estimate implies that there is a $c$ such that the probability that a Brownian motion starting at 0 reaches $\{|z| = 1/2\}$ without leaving $\mathbb{D}$ is at most $c \, r^{1/2}$. The probability that a Brownian motio starting at $\{|z| = 1/2\}$ leaves $\mathbb{D}$ at $I$ is bounded by $c\,l(I)$ where $l$ denotes length. Therefore

$$h(A; I) \leq c\, r^{1/2}\, l(I)$$

which implies

$$\left| \frac{\Psi_A(z) - \Psi_A(w)}{z - w} \right| \leq c\, r^{1/2}.$$

and $|\Psi'_A(z)| \leq c\, r^{1/2}$. □

The proof continues as in the one-sided case. The basic idea is to show the equality of two random variables: the first is a martingale $\tilde{M}_t$ that is equal to $\mathbb{P}\{(\gamma, \hat{\gamma}) \cap A = \emptyset\}$ when $t = 0$, and the second is our "martingale candidate" $M_t$, which has initial value

$$\Psi'_A(0)^{2/3}\, |\Psi'_A(e^{ix})\, \Psi'_A(e^{i\hat{x}})|^{5/8} \left| \frac{\Psi_A(e^{i\hat{x}}) - \Psi_A(e^{ix})}{e^{i\hat{x}} - e^{ix}} \right|^{3/4}. \tag{14}$$

We will think of generating $(\gamma, \hat{\gamma})$ in two steps. First we obtain $\gamma$ by solving the Loewner equation with the driving term $X_t$, where $X_t$ is described by (11). Then we take $\hat{\gamma}$ to be chordal $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma$. Let $\mathcal{F}_t$ denote the filtration generated by $X_t$, and set

$$\tilde{M}_t := \mathbb{E}\left[1_{\{\gamma \cap A = \emptyset\}}\, \mathbb{P}\{\hat{\gamma} \cap A = \emptyset \mid \gamma\} \mid \mathcal{F}_t\right].$$

Note that $\tilde{M}_0 = \mathbb{P}\{(\gamma, \hat{\gamma}) \cap A = \emptyset\}$ and $\tilde{M}_t$ is a continuous, bounded martingale. Additionally,

$$\lim_{t \to \infty} \tilde{M}_t = 1_{\{\gamma \cap A = \emptyset\}} \mathbb{P}\{\hat{\gamma} \cap A = \emptyset \mid \gamma\}.$$

In what follows, we will define our "martingale candidate" $M_t$, which will satisfy $M_0$ equal to (14), and we will show that $M_t$ is also a continuous, bounded martingale with the same limit at infinity as $\tilde{M}_t$. This will imply that $M_t = \tilde{M}_t$, completing the proof. The most tedious part of the work, which is contained in the subsection below, is the calculation to show that $M_t$ actually is a martingale.

## The martingale calculation

As usual, we begin by establishing the notation we will use. Set $X_0 = e^{ix}$ and $\hat{X}_0 = e^{i\hat{x}}$, and let $X_t$ and $\hat{X}_t$ satisfy

$$dX_t = -\frac{3}{8}\, K_t\, dt + dB_t, \qquad d\hat{X}_t = \frac{3}{8}\, K_t\, dt,$$

where we write

$$K_t = \cot\left[\frac{\hat{X}_t - X_t}{2}\right].$$

Recall that the Loewner equation with driving term $X_t$ generates one side of two-sided $SLE_{8/3}$, and we will refer to this curve as $\gamma$. Let $g_t$ be the conformal maps associated with this Loewner chain. That is,

$$\dot{g}_t(z) = \frac{3}{8}\, g_t(z)\, \frac{e^{iX_t} + g_t(z)}{e^{iX_t} - g_t(z)}.$$

As before, we let $h_t(z) = -i \log g_t(e^{iz})$. Then, as is discussed in Section 4.6 of [4], on the event $\{\gamma[0,t] \cap A = \emptyset\}$ we take $\Psi_t$ and $\Phi_t$ to be conformal perturbations of the Loewner chains with $\Psi_0 = \Psi_A$. More specifically, if $g_t^*$ is the the conformal map from $\mathbb{D} \setminus \Psi_A \circ \gamma[0,t]$ onto $\mathbb{D}$ with $g_t^*(0) = 0$ and $(g_t^*)'(0) > 0$, then $\Psi_t = g_t^* \circ \Psi_A \circ g_t^{-1}$ and $\Phi_t(z) = -i \log \Psi_t(e^{iz})$. Finally, let $\hat{X}_t^* = \Phi_t(\hat{X}_t)$ and $X_t^* = \Phi_t(X_t)$, and take

$$K_t^* = \cot\left[\frac{\hat{X}_t^* - X_t^*}{2}\right], \quad G_t = \csc^2\left[\frac{\hat{X}_t - X_t}{2}\right], \quad G_t^* = \csc^2\left[\frac{\hat{X}_t^* - X_t^*}{2}\right].$$

We will make use of the following five equations from Section 4.6 of [4]:

$$\dot{\Phi}_t(X_t) = -\frac{9}{8}\Phi_t''(X_t),$$

$$\dot{\Phi}_t(\hat{X}_t) = \frac{3}{8}\left[\Phi_t'(X_t)^2\, K_t^* - \Phi_t'(\hat{X}_t)\, K_t\right],$$

$$\dot{\Phi}_t'(X_t) = \frac{3}{8}\,\Phi_t'(X_t)\left[\frac{\Phi_t''(X_t)^2}{2\,\Phi_t'(X_t)^2} - \frac{4\,\Phi_t'''(X_t)}{3\,\Phi_t'(X_t)} + \frac{1 - \Phi_t'(X_t)^2}{6}\right],$$

$$\dot{\Phi}_t'(\hat{X}_t) = \frac{3}{8}\left[-\frac{1}{2}\,\Phi_t'(X_t)^2\,\Phi_t'(\hat{X}_t)\,G_t^* - \Phi_t''(\hat{X}_t)\,K_t + \frac{1}{2}\,\Phi_t'(\hat{X}_t)\,G_t\right],$$

$$\dot{\Psi}_t'(0) = \frac{3}{8}(\Phi_t'(X_t)^2 - 1)\,\Psi_t'(0).$$

Note that these differ from the results in [4] by a factor of 3/8 because of our choice of parametrization for $\gamma$.

We can now state our martingale candidate:

$$M_t := 1_{\{\gamma[0,t]\cap A = \emptyset\}}\,\Psi_t'(0)^{2/3}\,\Phi_t'(X_t)^{5/8}\,\Phi_t'(\hat{X}_t)^{5/8}\,\frac{F(\hat{X}_t^* - X_t^*)}{F(\hat{X}_t - X_t)},$$

where $F(x) = \sin^{3/4}(x/2)$. Equiped with the tools of Itô calculus, we wish to show that $M_t$ is a continuous martingale.

We begin by considering the three derivative terms in $M_t$. We first compute that

$$d[\Psi_t'(0)^{2/3}] = \frac{1}{4}\,\Psi_t'(0)^{2/3}\,(\Phi_t'(X_t)^2 - 1)\,dt.$$

Using the standard chain rule for functions of two variables, we obtain next that

$$\begin{aligned} d[\Phi_t'(\hat{X}_t)^b] &= b\,\Phi_t'(\hat{X}_t)^b\,\frac{\dot{\Phi}_t'(\hat{X}_t)\,dt + \Phi_t''(\hat{X}_t)\,d\hat{X}_t}{\Phi_t'(\hat{X}_t)} \\ &= \Phi_t'(\hat{X}_t)^b\,\left[-\frac{3b}{16}\,\Phi_t'(X_t)^2\,G_t^* + \frac{3b}{16}\,G_t\right]dt, \end{aligned}$$

and so,

$$d[\Phi_t'(\hat{X}_t)^{5/8}] = \Phi_t'(\hat{X}_t)^{5/8}\,\left[-\frac{15}{128}\,\Phi_t'(X_t)^2\,G_t^* + \frac{15}{128}\,G_t\right]dt.$$

For the $X_t$ term we need to use Itô's formula, which tells us that

$$d[\Phi_t'(X_t)] = \dot{\Phi}_t'(X_t)\;dt + \Phi_t''(X_t)\,dX_t + \frac{1}{2}\Phi_t'''(X_t)\,dt.$$

Hence,

$$\begin{aligned} d[\Phi_t'(X_t)] = &\left[\frac{3}{16}\frac{\Phi_t''(X_t)^2}{\Phi_t'(X_t)} + \frac{1}{16}\,\Phi_t'(X_t)\,[1-\Phi_t'(X_t)^2] - \frac{3}{8}K_t\,\Phi_t''(X_t)\right]\;dt \\ &+\Phi_t''(X_t)\;dB_t. \end{aligned}$$

From another use of Itô's formula, we obtain

$$\begin{aligned} d[\Phi_t'(X_t)^{5/8}] = \Phi_t'(X_t)^{5/8}\,\Bigg\{&\left[\frac{5}{128}\,[1-\Phi_t'(X_t)^2] - \frac{15}{64}\,K_t\,\frac{\Phi_t''(X_t)}{\Phi_t'(X_t)}\right]\;dt \\ &+\frac{5}{8}\,\frac{\Phi_t''(X_t)}{\Phi_t'(X_t)}\,dB_t\Bigg\}. \end{aligned}$$

Combining all of this gives that

$$d[\Psi_t'(0)^{2/3}\,\Phi_t'(\hat{X}_t)^{5/8}\,\Phi_t'(X_t)^{5/8}]$$

is $\Psi_t'(0)^{2/3}\,\Phi_t'(\hat{X}_t)^{5/8}\,\Phi_t'(X_t)^{5/8}$ times

$$\left[-\frac{27}{128}\,(1-\Phi_t'(X_t)^2)\; - \frac{15}{64}\,K_t\,\frac{\Phi_t''(X_t)}{\Phi_t'(X_t)} - \frac{15}{128}\,\Phi_t'(X_t)^2\,G_t^* + \frac{15}{128}\,G_t\right]\;dt$$

$$+\frac{5}{8}\,\frac{\Phi_t''(X_t)}{\Phi_t'(X_t)}\,dB_t.$$

We now turn our attention to the terms involving $F(x) = \sin^{3/4}(x/2)$. Note that

$$F'(x) = \frac{3}{8}\,\cot(x/2)\,F(x), \qquad F''(x) = \left[-\frac{3}{64}\,\csc^2(x/2) - \frac{9}{64}\right]\,F(x).$$

If $f(x) = 1/F(x) = \sin^{-3/4}(x/2)$, then

$$f'(x) = -\frac{3}{8}\,\cot(x/2)\,f(x), \qquad f''(x) = \left[\frac{21}{64}\csc^2(x/2) - \frac{9}{64}\right]\,f(x).$$

Let $Z_t = \hat{X}_t - X_t$, and $Z_t^* = \hat{X}_t^* - X_t^*$. Then,

$$dZ_t = \frac{3}{4}K_t\,dt - dB_t,$$

and we have

$$\begin{aligned} d[f(Z_t)] &= f'(Z_t)\,dZ_t + \frac{1}{2}f''(Z_t)\,dt \\ &= f(Z_t)\left\{\left[-\frac{15}{128}\,G_t + \frac{27}{128}\right]\,dt + \frac{3}{8}K_t dB_t\right\}, \end{aligned}$$

where we have made use of the trig identity $1 + K_t^2 = G_t$.

The last term we need to compute is $d[F(Z_t^*)]$, and to do this, we must first compute $d\hat{X}^*$ and $dX^*$:

$$d\hat{X}_t^* = \dot{\Phi}_t(\hat{X}_t)\,dt + \Phi_t'(\hat{X}_t)\,d\hat{X}_t = \frac{3}{8}\,\Phi_t'(U_t)^2\,K_t^*\,dt.$$

$$\begin{aligned} dX_t^* &= \dot{\Phi}_t(X_t)\,dt + \Phi_t'(X_t)\,dX_t + \frac{1}{2}\Phi_t''(X_t)\,dt \\ &= \left[-\frac{5}{8}\,\Phi_t''(X_t) - \frac{3}{8}\,K_t\,\Phi_t'(X_t)\right]\,dt + \Phi_t'(X_t)\,dB_t. \end{aligned}$$

Therefore,

$$dZ_t^* = \left[\frac{3}{8}\Phi_t'(X_t)^2\,K_t^* + \frac{5}{8}\,\Phi_t''(X_t) + \frac{3}{8}\,K_t\,\Phi_t'(X_t)\right]\,dt - \Phi_t'(X_t)\,dB_t,$$

and so,

$$d[F(Z_t^*)] = F'(Z_t^*)\,dZ_t^* + \frac{1}{2}F''(Z_t^*)\,\Phi_t'(X_t)^2\,dt.$$

Using that $1 + (K_t^*)^2 = G_t^*$, this simplifies to give

$$d[F(Z_t^*)] = F(Z_t^*) \left\{ \left[ \frac{15}{128} G_t^* \, \Phi_t'(X_t)^2 - \frac{27}{128} \Phi_t'(X_t)^2 + \frac{15}{64} K_t^* \, \Phi_t''(X_t) \right.\right.$$

$$\left.\left. + \frac{9}{64} K_t \, K_t^* \, \Phi_t'(X_t) \right] \, dt - \frac{3}{8} K_t^* \, \Phi_t'(X_t) \, dB_t \right\}.$$

Now that we have computed $d[\Psi_t'(0)^{2/3} \, \Phi_t'(\hat{X}_t)^{5/8} \, \Phi_t'(X_t)^{5/8}]$, $d[f(Z_t)]$, and $d[F(Z_t^*)]$, we are ready to compute the drift of $M_t$ on the event that $\{\gamma[0,t] \cap A = \emptyset\}$. We find that this is equal to zero, as desired.

To show that $M_t$ is a continuous martingale, one must also check that the limit of $M_t$ as $t$ approaches $t_A$ from below is zero, where $t_A$ is the first time that $\gamma[0,t] \cap A \neq \emptyset$. We leave this to the reader. Note that Lemma 6.1 implies that $M_t$ is bounded.

## The behavior of the martingale at infinity

To finish our proof of the restriction property for two-sided radial $SLE_{8/3}$, we must show that $M_t$ approaches $1_{\{\gamma \cap A = \emptyset\}} \mathbb{P}\{\hat{\gamma} \cap A = \emptyset \mid \gamma\}$ as $t \to \infty$. We first note that when $t$ is large, $A$ has small harmonic measure as viewed from zero in the domain $\mathbb{D} \setminus (\gamma[0,t] \cup A)$, and this implies that the harmonic measure of $g_t(A)$ in $\mathbb{D} \setminus g_t(A)$ is also small. Therefore, away from $g_t(A)$, the maps $\Psi_t$ and $\Phi_t$ will be close to the identity, and so $\Psi_t'(0)^{2/3}$ and $\Phi_t'(X_t)^{5/8}$ will be close to 1. Since $e^{i\hat{X}_t}$ will be near to $g_t(A)$, we do not have immediate control over the derivative of $\Phi_t$ at $\hat{X}_t$. However, the small harmonic measure of $g_t(A)$ does imply that $\Phi_t$ cannot move $\hat{X}_t$ much, which gives that $F(Z_t^*) \, f(Z_t) = \frac{F(\hat{X}_t^* - X_t^*)}{F(\hat{X}_t - X_t)}$ is also close to 1.

The last step is to show that the remaining term, $\Phi_t'(\hat{X}_t)^{5/8}$, approaches $\mathbb{P}\{\hat{\gamma} \cap A = \emptyset \mid \gamma\}$ as $t \to \infty$. The underlying idea here is that if we take $t$ to be large and just look at the part of the boundary of the disk near $g_t(A)$ and $e^{i\hat{X}_t}$, then our picture will look roughly like the upper halfplane, and in this setting $\Phi_t'(\hat{X}_t)^{5/8}$ gives the probability that a chordal $SLE_{8/3}$ started at $\hat{X}_t$ avoids $g_t(A)$. To make things more precise, we will use our result from Section 8 that tells us that the limit as $t \to \infty$ of radial $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma[0,t]$ from $e^{i\hat{X}_0}$ to 0 is a chordal $SLE_{8/3}$ from $e^{i\hat{X}_0}$ to 0 in $\mathbb{D} \setminus \gamma$. Note that

$$\begin{aligned} &\mathbb{P}\{\text{a radial } SLE_{8/3} \text{ in } \mathbb{D} \setminus \gamma[0,t] \text{ from } e^{i\hat{X}_0} \text{ to 0 avoids } A \mid \gamma[0,t]\} \\ &\quad = \mathbb{P}\{\text{a radial } SLE_{8/3} \text{ in } \mathbb{D} \text{ from } e^{i\hat{X}_t} \text{ to 0 avoids } g_t(A)\} \\ &\quad = \Psi_t'(0)^{5/48} \Phi_t(\hat{X}_t)^{5/8}. \end{aligned}$$

Since $\Psi_t'(0)$ is approaching 1 and $\hat{\gamma}$ is precisely a chordal $SLE_{8/3}$ in $\mathbb{D} \setminus \gamma$ from $e^{i\hat{X}_0}$ to 0, we have that

$$\lim_{t \to \infty} \Phi_t(\hat{X}_t)^{5/8} = \mathbb{P}\{\hat{\gamma} \cap A = \emptyset \mid \gamma\},$$

as desired.

## A connection to Brownian motion

We end our discussion by mentioning an alternate way to view two-sided radial $SLE_{8/3}$ started at $(1,1)$. We can obtain this process by taking the outer boundary of two independent Brownian motions from 0 to 1 in $\mathbb{D}$ that are conditioned not to disconnect 0 from $\partial\mathbb{D}$.

# 7 Two-sided chordal $SLE_{8/3}$

Although we have previously concerned ourselves only with the radial case, one can also define two-sided chordal $SLE_{8/3}$ and two-sided whole plane $SLE_{8/3}$. The chordal version of this process is actually one of the $SLE(8/3,\rho)$ processes, which were introduced in [6]. We will describe this connection after constructing the process and discussing the restriction property that it satisfies. Here we follow the same general outline as our discussion of the radial case: Given $\gamma$ and $\hat{\gamma}$, two independent chordal $SLE_{8/3}$ processes, we first wish to understand $\mathbb{P}\{\gamma(0,t]\cap\gamma(0,\infty)=\emptyset\}$ as $t\to\infty$. From this we obtain a martingale, $M_s$, and weighting an $SLE_{8/3}$ by $M_s/M_0$ gives one side of the two-sided chordal process. Girsanov's Theorem allows us to describe this process via Loewner's equation. We finish with the definition of the two-sided process.

Let $B_t$ and $\hat{B}_t$ be two independent standard Brownian motions with $B_0<\hat{B}_0$. Let $\gamma,\hat{\gamma},g_t$, and $\hat{g}_t$ be the corresponding $SLE_{8/3}$. In other words, $g_t$ is the conformal transformation of $\mathbb{H}\setminus\gamma(0,t]$ onto $\mathbb{H}$ such that $g_t(z)-z\to 0$ as $z\to\infty$. It satisfies the Loewner equation

$$\dot{g}_t(z)=\frac{3/4}{g_t(z)-B_t},\qquad g_0(z)=z. \tag{15}$$

Here $\gamma$ has been parametrized so that $\mathrm{hcap}(\gamma(0,t])=3t/4$, instead of $2t$. All the same holds of $\hat{\gamma}$ and $\hat{g}_t$.

By the restriction property for chordal $SLE_{8/3}$, we know that

$$\mathbb{P}\{\gamma(0,t]\cap\hat{\gamma}(0,\infty)=\emptyset\mid\gamma[0,t]\}=g_t'(\hat{B}_0)^{5/8},$$

and therefore,

$$\mathbb{P}\{\gamma(0,t]\cap\hat{\gamma}(0,\infty)=\emptyset\}=\mathbb{E}[g_t'(\hat{x})^{5/8}].$$

By differentiating (15), we see that

$$g_t'(\hat{B}_0)^{5/8}=\exp\left\{-\frac{15}{32}\int_0^t\frac{ds}{Y_s^2}\right\},$$

where $Y_t=g_t(\hat{B}_0)-B_t$ satisfies

$$dY_t=\frac{3/4}{Y_t}\,dt-dB_t.$$

We assume for ease that $B_0 = 0$, and we let $\phi(t,x) = \mathbb{E}[g_t'(x)^{5/8}]$ for $x > 0$. Since

$$\phi(T-t, Y_t) \exp\left\{-\frac{15}{32}\int_0^t \frac{ds}{Y_s^2}\right\}, \qquad 0 \le t < T$$

is a martingale, Itô's formula shows that $\phi$ must satisfy

$$\dot\phi(t,x) = \frac{1}{2}\phi''(t,x) + \frac{3}{4x}\,\phi'(t,x) - \frac{15}{32\,x^2}\,\phi(t,x).$$

One could also obtain this differential equation from the Feynman-Kac formula. If $\psi(x) = \phi(1,x)$, then scaling implies that $\phi(t,x) = \psi(x/\sqrt{t})$. Letting $y = x/\sqrt{t}$, we see that

$$\psi''(y) + \left(y + \frac{3}{2y}\right)\,\psi'(y) - \frac{15}{16\,y^2}\,\psi(y) = 0.$$

We must have boundary conditions $\psi(\infty) = 1$ and $\psi(0) = 0$. The solution to this initial value problem, discussed in Appendix B.2 of [4], is $\psi(x) = x^{3/4} f(x)$, where

$$f(x) = e^{-x^2/2}\frac{\Gamma(13/8)}{2^{3/8}\,\Gamma(2)}\,\Phi(13/8, 2; x^2/2)$$

and $\Phi$ denotes the confluent hypergeometric function (of the first kind). The actual expression for $f$ is unneeded, as all we will use is that $c_0 = f(0)$ is well-defined and non-zero. We have now established that

$$\mathbb{E}[g_t'(\hat B_0)^{5/8}] = \hat B_0^{3/4}\, t^{-3/8}\, f(\hat B_0/\sqrt{t}),$$

and therefore

$$\lim_{t\to\infty} t^{3/8}\,\mathbb{P}\{\gamma(0,t] \cap \hat\gamma(0,\infty) = \emptyset\} = c_0\,(\hat B_0 - B_0)^{3/4}. \tag{16}$$

Now we define

$$M_s := c_0\, g_s'(\hat B_0)^{5/8}\, Y_s^{3/4},$$

and a simple calculation show us that

$$dM_s = -\frac{3/4}{Y_s}\, M_s\, dB_s.$$

Equation (16) allows us to conclude that

$$M_s = \lim_{t\to\infty} t^{3/8}\,\mathbb{P}\{\gamma(0,t] \cap \hat\gamma(0,\infty) = \emptyset \mid \mathcal{F}_s\},$$

since

$$\begin{aligned}\lim_{t\to\infty} t^{3/8}\,\mathbb{P}\{\gamma(0,t]\cap\hat\gamma(0,\infty)=\emptyset \mid \mathcal{F}_s\} \\ &= g_s'(\hat B_0)^{5/8}\lim_{t\to\infty}(t-s)^{3/8}\,\mathbb{P}\{g_s\circ\gamma(s,t]\cap g_s\circ\hat\gamma(0,\infty)=\emptyset\} \\ &= g_s'(\hat B_0)^{5/8}\,c_0\,|g_s(\hat B_0)-B_s|^{3/4}.\end{aligned}$$

It is this latter view of $M_s$ that leads us to define one side of two-sided chordal $SLE_{8/3}$ as $SLE_{8/3}$ weighted by $M_s/M_0$. By making use of Girsanov's Theorem, this is the same as saying that one side of two-sided chordal $SLE_{8/3}$ is the process obtained from the chordal Loewner equation with driving term $X_s$, where $W_s$ is a standard Brownian motion (with respect to the probability measure induced by $M_s/M_0$) and

$$dX_s = -\frac{3/4}{\hat X_s - X_s}\,ds + dW_s, \qquad d\hat X_s = \frac{3/4}{\hat X_s - X_s}\,ds.$$

Next we would like to define the general two-sided chordal process. As in the radial case, we will do so by weighting two independent chordal $SLE_{8/3}$ processes by a two-parameter martingale $N_{s,r}/N_{0,0}$. We define

$$N_{s,r} = \lim_{t\to\infty} t^{3/8}\,\mathbb{P}\{\gamma(0,t]\cap\hat\gamma(0,\infty)=\emptyset \mid \mathcal{F}_{s,r}\}.$$

In the lemma below, we will show that this limit exits and that $N_{s,r}$ is symmetric in $s$ and $r$.

First we introduce some notation. On the event $E_{s,r} := \{\gamma(0,s]\cap\hat\gamma(0,r]=\emptyset\}$, let $v_{s,r}$ be the unique conformal transformation of $\mathbb{H}\setminus(\gamma(0,s]\cup\hat\gamma(0,r])$ such that $v_{s,r}(z)-z\to 0$ as $z\to\infty$. Let $U_{s,r}=v_{s,r}(\gamma(s))$ and $\hat U_{s,r}=v_{s,r}(\hat\gamma(r))$, and define $g_{s,r}$ and $\hat g_{s,r}$ by the relations $v_{s,r}=g_{s,r}\circ\hat g_r=\hat g_{s,r}\circ g_s$.

**Lemma 7.1.**

$$N_{s,r} = c_0\,1_{E_{s,r}}\,g_{s,r}'(\hat U_{0,r})^{5/8}\hat g_{s,r}'(U_{s,0})^{5/8}\,|\hat U_{s,r}-U_{s,r}|^{3/4}.$$

*Proof.* We first write $N_{s,r}$ as

$$\lim_{t\to\infty} t^{3/8}\,\mathbb{E}[\mathbb{P}\{\gamma(0,t]\cap\hat\gamma(0,\infty)=\emptyset \mid \mathcal{F}_{t,r}\} \mid \mathcal{F}_{s,r}].$$

Then the restriction property implies that

$$\mathbb{P}\{\gamma(0,t]\cap\hat\gamma(0,\infty)=\emptyset \mid \mathcal{F}_{t,r}\} = 1_{E_{t,r}}\,g_{t,r}'(\hat U_{0,r})^{5/8},$$

and

$$\lim_{t\to\infty}\mathbb{E}[E_{t,r} \mid \mathcal{F}_{s,r}] = 1_{E_{s,r}}\,\hat g_{s,r}'(U_{s,o})^{5/8}.$$

Therefore,

$$N_{s,r} = 1_{E_{s,r}}\, \hat{g}'_{s,r}(U_{s,o})^{5/8} \lim_{t\to\infty} t^{3/8}\, \mathbb{E}[g'_{t,r}(\hat{U}_{0,r})^{5/8} \mid \mathcal{F}_{s,r},\, E_{t,r}].$$

On the event $E_{t,r}$, we define $u_{t,s,r}$ by $g_{t,r} = u_{t,s,r} \circ g_{s,r}$, or equivalently by $v_{t,r} = u_{t,s,r} \circ v_{s,r}$, and we set $\eta(\tau) = v_{s,r} \circ \gamma(\tau + s)$ for $\tau \geq 0$. Notice that

$$\begin{aligned} &\mathbb{E}[u'_{t,s,r}(U_{s,r})^{5/8} \mid \mathcal{F}_{s,r},\, E_{t,r}] \\ &\qquad = \mathbb{P}\{ \text{ a } SLE_{8/3} \text{ started at } \hat{U}_{s,r} \text{ avoids } \eta(0, t-s] \ \} \end{aligned}$$

by a third use of the restriction property. In order to use (16) to conclude that

$$\lim_{t\to\infty} t^{3/8}\, \mathbb{E}[u'_{t,s,r}(U_{s,r})^{5/8} \mid \mathcal{F}_{s,r}, E_{t,r}] = c_0\, |\hat{U}_{s,r} - U_{s,r}|^{3/4},$$

we must have that $\lim_{t\to\infty} \frac{\mathrm{hcap}(\eta(0,t-s])}{3t/4} = 1$. This, however, follows from the fact that $\mathrm{hcap}(\eta(0,t-s]) = \mathrm{hcap}(\gamma(0,t] \cup \hat{\gamma}(0,r]) - \mathrm{hcap}(\gamma(0,s] \cup \hat{\gamma}(0,r])$ and $\lim_{t\to\infty} \frac{\mathrm{hcap}(\gamma(0,t]\cup\hat{\gamma}(0,r])}{3t/4} = 1$. Thus we have established that

$$N_{s,r} = c_0\, 1_{E_{s,r}}\, g'_{s,r}(\hat{U}_{0,r})^{5/8} \hat{g}'_{s,r}(U_{s,0})^{5/8}\, |\hat{U}_{s,r} - U_{s,r}|^{3/4}.$$

$\square$

**Definition**. If $x < \hat{x}$, then two-sided chordal $SLE_{8/3}$ in $\mathbb{H}$ starting at $(x, \hat{x})$ is the measure on ordered pairs of paths $(\gamma, \hat{\gamma})$ such that for each $s, r < \infty$, the distribution of

$$\gamma(s'), 0 \leq s \leq s', \qquad \hat{\gamma}(r'), 0 \leq r \leq r'$$

is given by saying that the Radon-Nikodym derivative of this distribution with respect to that of *independent* chordal $SLE$'s starting at $x, \hat{x}$ is

$$\frac{N_{s,r}}{N_{0,0}} = 1_{E_{s,r}}\, g'_{s,r}(\hat{U}_{0,r})^{5/8} \hat{g}'_{s,r}(U_{s,0})^{5/8} \left( \frac{\hat{U}_{s,r} - U_{s,r}}{\hat{x} - x} \right)^{3/4}.$$

Notice that as in the radial case,

$$\frac{N_{s,r}}{N_{0,0}} = \frac{N_{s,0}}{N_{0,0}} \frac{N_{s,r}}{N_{s,0}} = \frac{N_{0,r}}{N_{0,0}} \frac{N_{s,r}}{N_{0.r}},$$

which implies that we can grow some of the first curve and then some of the second, or vice versa. Again we can make sense of what this means using Girsanov's Theorem. Since $M_s/M_0 = N_{s,0}/N_{0,0}$, we can obtain part of the first curve by running the Loewner equation with driving term $X_t$ until time $s$, where $X_0 = x, \hat{X}_0 = \hat{x}$, and

$$dX_t = -\frac{3/4}{\hat{X}_t - X_t}\, dt + dW_t, \qquad d\hat{X}_t = \frac{3/4}{\hat{X}_t - X_t}\, dt. \tag{17}$$

To obtain a piece of the second curve, we map the first curve down by $g_s$ and then proceed as before, switching the roles of $X_t$ and $\hat{X}_t$

Alternately, we could create the two-sided chordal $SLE_{8/3}$ process in two steps. First grow one complete curve $\gamma$ as above by using the Loewner equation with driving term $X_t$ described by (17). Then the second curve $\hat{\gamma}$ is chordal $SLE_{8/3}$ from $\hat{x}$ to infinity in the smaller domain $D_\gamma$, where $D_\gamma$ is the simply connected component of $\mathbb{H} \setminus \gamma(0,\infty)$ that has $\hat{x}$ on the boundary. This is a consequence of the fact that

$$\lim_{s\to\infty} \frac{N_{s,r}}{N_{0.r}} = 1_{E_{\infty,r}}\, \tilde{\Phi}_r'(\hat{X}_r)^{5/8},$$

where $\tilde{\Phi}_r$ is the conformal perturbation of a $SLE_{8/3}$ Loewner chain with $\tilde{\Phi}_0$ a conformal map from $D_\gamma$ onto $\mathbb{H}$. Weighting a chordal $SLE_{8/3}$ by $1_{E_{\infty,r}}\, \tilde{\Phi}_r'(\hat{X}_r)^{5/8}$ gives $SLE_{8/3}$ in the domain $D_\gamma$.

## Restriction property for two-sided chordal $SLE_{8/3}$

Two-sided chordal $SLE_{8/3}$ satisfies the restriction property: if $(\gamma,\hat{\gamma})$ is two-sided chordal $SLE_{8/3}$ starting at $(x,\hat{x})$, then the conditional distribution of $\Phi_A(\gamma,\hat{\gamma})$ given $\{(\gamma,\hat{\gamma})\cap A = \emptyset\}$ is the same as two-sided chordal $SLE_{8/3}$ starting at $(\Psi_A(x),\Psi_A(\hat{x}))$. Here $A$ is a compact set in $\mathbb{H}$ such that $\mathbb{H}\setminus A$ is simply connected and $\mathrm{dist}(\{x,\hat{x}\},A)>0$, and $\Phi_A$ denotes a conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ with $\Phi_A(z)\sim z$ for $z$ near infinity. The restriction property follows from the following theorem.

**Theorem 2.** If $(\gamma,\hat{\gamma})$ denotes two-sided chordal $SLE_{8/3}$ starting at $(x,\hat{x})$, then

$$\mathbb{P}\{(\gamma,\hat{\gamma})\cap A = \emptyset\} = \Phi_A'(x)^{5/8}\,\Phi_A'(\hat{x})^{5/8}\left(\frac{\Phi_A(\hat{x})-\Phi_A(x)}{\hat{x}-x}\right)^{3/4}.$$

If $x=\hat{x}$, this is to be interpreted as $\Phi_A(x)^2$.

This theorem is proved in the same manner as Theorem 1. We use our third method of obtaining a two-sided chordal $SLE_{8/3}$: $\gamma$ is generated by the Loewner equation with driving term $X_t$, where $X_t$ satisfies (17), and $\hat{\gamma}$ is chordal $SLE_{8/3}$ in $D_\gamma$. Let $\mathcal{F}_t$ denote the filtration generated by $X_t$, and let $\Phi_t$ be the conformal perturbation of this Loewner chain with $\Phi_0 = \Phi_A$. Then, one must show that the martingale

$$\tilde{M}_t := \mathbb{E}[1_{\{\gamma\cap A=\emptyset\}}\mathbb{P}\{\hat{\gamma}\cap A=\emptyset \mid \gamma\} \mid \mathcal{F}_t]$$

is equal to the "martingale candidate"

$$M_t := 1_{\{\gamma(0,t]\cap A=\emptyset\}}\Phi_t'(X_t)^{5/8}\,\Phi_t'(\hat{X}_t)^{5/8}\left(\frac{\Phi_t(\hat{X}_t)-\Phi_t(X_t)}{\hat{X}_t-X_t}\right)^{3/4}.$$

This is done by showing that $M_t$ is a bounded martingale and that $\tilde{M}_t$ and $M_t$ have the same limit at infinity. We omit the details.

We end by noting a connection between two-sided chordal $SLE_{8/3}$ and the $SLE(\kappa, \rho)$ processes. The latter processes can be defined as solutions to the Loewner equation driven by a random function having the appropriate drift. In particular, if $(O_t, U_t)$ are a pair of processes satisfying

$$dO_t = \frac{a}{O_t - U_t}\,dt, \qquad dU_t = \frac{-v}{O_t - U_t} + dB_t, \qquad O_0 = U_0 = 0,$$

then the solution to the Loewner equation with driving term $U_t$ is the $SLE(2/a, 2v/a)$ process. See Section 9.3 of [4] for a brief introduction. Therefore, one side of two-sided chordal $SLE$ started from $(0,0)$ is the same as $SLE(8/3, 2)$. This also follows from the restriction exponent, since both processes satisfy the restriction property with exponent 2.

# 8 Chordal $SLE$ as the limit of radial $SLE$

In the construction of two-sided $SLE_{8/3}$ we used the fact that chordal $SLE$ can be obtained as a limit of radial $SLE$. We will be more precise about this here. Since it is no more difficult, we will discuss $\kappa \leq 4$ and as before we let $a = 2/\kappa$.

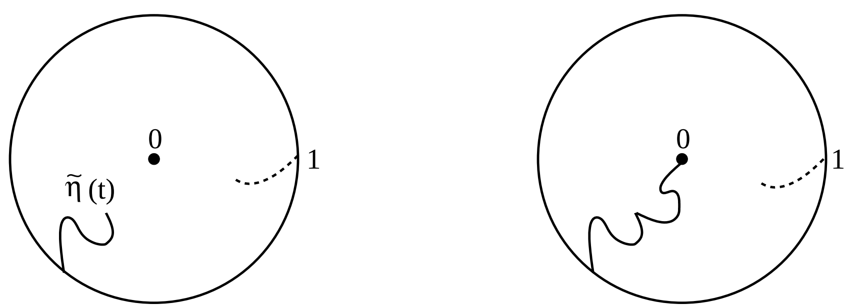



Figure 2: A comparison of radial and chordal $SLE$.

Suppose $\tilde{\eta} : (0, \infty) \to \mathbb{D} \setminus \{0\}$ is a simple curve with $\tilde{\eta}(0+) \in \partial\mathbb{D} \setminus \{1\}$ and $\tilde{\eta}(t) \to 0$ as $t \to \infty$. Define the following measures on paths (modulo reparametrization) $\tilde{\gamma} : [0, \infty) \to \mathbb{D}$:

- $\nu_t$: Radial $SLE_\kappa$ in $\mathbb{D} \setminus \tilde{\eta}(0, t]$ from 1 to 0;
- $\nu_\infty$: Chordal $SLE_\kappa$ in $\mathbb{D} \setminus \tilde{\eta}(0, \infty)$ from 1 to 0.

In this section, we will give a precise version of the result that as $t \to \infty$, $\nu_t$ approaches $\nu_\infty$. By considering $\gamma(t) = -i \log \tilde{\gamma}(t)$, we can can consider $\nu_t, \nu_\infty$ as measures on paths (modulo reparametrization) $\gamma : (0, \infty) \to \mathbb{H}$ with $\gamma(0+) = 0$. We choose the parametrization to be the half-plane parametrization. To be more precise, if $g_t$ denotes the conformal transformation of $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$ with $g_t(z) - z = o(1)$ as $z \to \infty$, then $g_t$ has expansion

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \qquad z \to \infty.$$

In this case, $g_t$ satisfies the chordal Loewner equation

$$\dot{g}_s(z) = \frac{a}{g_s(z) - U_s}, \quad g_0(z) = z, \tag{18}$$

with "driving function" $U_s = g_s(\gamma(s))$. For fixed $r < \infty$, let $\nu_{t,r}, \nu_{\infty,r}$ denote these measures on paths stopped at time $r$. We write $\eta(t) = -i \log \tilde{\eta}(t)$ where the branch of the logarithm is chosen so that $-2\pi < \eta(0+) < 0$.

**Proposition 8.1.** Suppose $\kappa \leq 4$, $\eta$ is a curve as in the previous paragraph, and $0 < r < \infty$. Let $\nu_{t,r}, \nu_{\infty,r}$ be $\nu_t, \nu_\infty$ restricted to curves up to time $r$, $\gamma(s), 0 \leq s \leq r$. There exists a $T = T(\eta, r)$ such that for $t \geq T$, $\nu_{t,r}$ and $\nu_{\infty,r}$ are mutually absolutely continuous with respect to each other. Moreover, with probability one with respect to $\nu_{\infty,r}$, the Radon-Nikodym derivative has a limit of 1 as $t$ approaches infinity, i.e.

$$\lim_{t\to\infty} \frac{d\nu_{t,r}}{d\nu_{\infty,r}} = 1.$$

We start by giving the basic idea for the proof. Without loss of generality we will assume that $r = 1$; other values of $r$ can be handled by scaling. Let $g_s$ denote the conformal transformation of $\mathbb{H} \setminus \gamma(0,s]$ onto $\mathbb{H}$ satisfying $g_s(z) - z = o(1)$ as $z \to \infty$. To give a measure on the maps $g_s$ (or, equivalently, on the curve $\gamma$) we give a measure on the driving function $U_s$. As we will see, this measure $\mu_{\infty,r}$ can be obtained by solving (18) where the driving function $U_s$ satisfies a stochastic differential equation

$$dU_s = R_s\, ds + dB_s. \tag{19}$$

The drift term $R_s$ depends on $\eta$ and is adapted to the Brownian motion. Similarly, the measure $\nu_t$ can be obtained from the Loewner equation using the driving function $U_{s,t}$ satisfying

$$dU_{s,t} = R_{s,t}\, ds + dB_s = [R_{s,t} - R_s]\, ds + dU_s. \tag{20}$$

Let $\mathcal{W}$ denote the standard Wiener measure, i.e., the measure on paths $B_s, 0 \leq s \leq 1$ that gives the standard Brownian motion. Then the Girsanov transformation tells us that the measure on paths whose Radon-Nikodym deriviative with respect to $\mathcal{W}$ is

$$\exp\left\{ \int_0^1 R_s\, dB_s - \frac{1}{2}\int_0^1 R_s^2\, ds \right\},$$

is the same as paths satisfying the differential equation (19). Similarly, if we choose Radon-Nikodym derivative

$$\exp\left\{ \int_0^1 R_{s,t}\, dB_s - \frac{1}{2}\int_0^1 R_{s,t}^2\, ds \right\},$$

the paths satisfy (20). In other words, we can define the paths on the same probability space so that

$$v_t = \exp\left\{ \int_0^1 (R_{s,t} - R_s)\, dB_s - \frac{1}{2}\int_0^1 [R_{s,t}^2 - R_s^2]\, ds \right\}.$$

We will let $T = T(\eta, 1) = \sigma_{4\sqrt{a}}$ where $\sigma_r = \sup\{t : \mathrm{Im}[\eta(t)] \leq r\}$. By properties of half-plane capacity, $\gamma(0,1]$ is contained in $\{z : \mathrm{Im}(z) \leq 2\sqrt{a}\}$. For $T(\eta, 1) \leq t \leq \infty$, $\nu_{1,t}$ is supported on those paths $\gamma$ with $\gamma(0,1] \cap \eta(0,\infty) = \emptyset$ and $(\gamma(0,1] \cap \eta(0,\infty) + 2\pi) = \emptyset$; this shows the mutual absolute continuity. Therefore, to prove the proposition it suffices to show that with $\nu_{\infty,1}$ probability one,

$$\lim_{t\to\infty} \sup_{0\leq s\leq 1} |R_{s,t} - R_s| = 0. \tag{21}$$

## Chordal and radial SLE in subdomains of $\mathbb{H}$

Chordal $SLE_\kappa$ from 0 to $\infty$ in $\mathbb{H}$ is defined by solving the Loewner equation (18) where the driving function is Brownian motion. Radial $SLE_\kappa$ in $\mathbb{D}$ is defined by solving the radial equation. Chordal and radial $SLE_\kappa$ in simply connected subdomains is defined by conformal transformation. In this section we describe a different way of obtaining radial $SLE_\kappa$ in $\mathbb{D}$ and chordal and radial $SLE_\kappa$ in subdomains by solving the Loewner equation (18) with a driving function with appropriate drift.

For this section we let $\kappa \leq 4$ and set $a = 2/\kappa$, $b = (3a-1)/2$. Suppose $D \subset \mathbb{H}$ is a domain contaning $\{z \in \mathbb{H} : |z| < \epsilon\}$ for some $\epsilon$ and suppose $w \in \partial D \setminus \{0\}$. Chordal $SLE_\kappa$ from 0 to $w$ in $D$ is defined (modulo time reparametrization) to be the image of $SLE_\kappa$ in $\mathbb{H}$ from 0 to infinity under a conformal map taking 0 to 0 and $\infty$ to $w$. We can construct this measure in a different way.

If $\gamma : (0,t] \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$, let $g_t$ denote the conformal transformation of $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$ satisfying $g_t(z) - z = o(1)$ as $z \to \infty$. If $\gamma(0,t] \subset D$, let $D_t = g_t(D \setminus \gamma(0,t])$, $U_t = g_t(\gamma(t))$, $w_t = g_t(w)$. Let $F_t$ denote a conformal transformation of $\mathbb{H}$ onto $D_t$ with $F_t(\infty) = w_t$, $F_t(0) = U_t$. Let $\Phi_t = F_t^{-1}$, which is a conformal transformation of $D_t$ onto $\mathbb{H}$. Then (see, e.g., [5]) chordal $SLE_\kappa$ in $D$ can be given by solving the chordal Loewner equation (18) with driving function $U_t$ satisfying the SDE

$$dU_t = b\, \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dt + dB_t,$$

where $B_t$ is a standard Brownian motion.

A similar construction can be given for radial $SLE$. Suppose $\tilde{\gamma}(t)$ denotes radial $SLE_\kappa$ in $\mathbb{D}$ and $\tilde{g}_t$ denotes the unique conformal transformation of $\mathbb{D} \setminus \tilde{\gamma}(0,t]$ onto $\mathbb{D}$ with $\tilde{g}_t(0) = 0, \tilde{g}_t'(0) > 0$. Then (under a suitable parametrization), the maps $h_t := -i \log \tilde{g}_t$ satisfy

$$\dot{h}_t(z) = \frac{a}{2} \cot\left( \frac{h_t(z) - B_t}{2} \right),$$

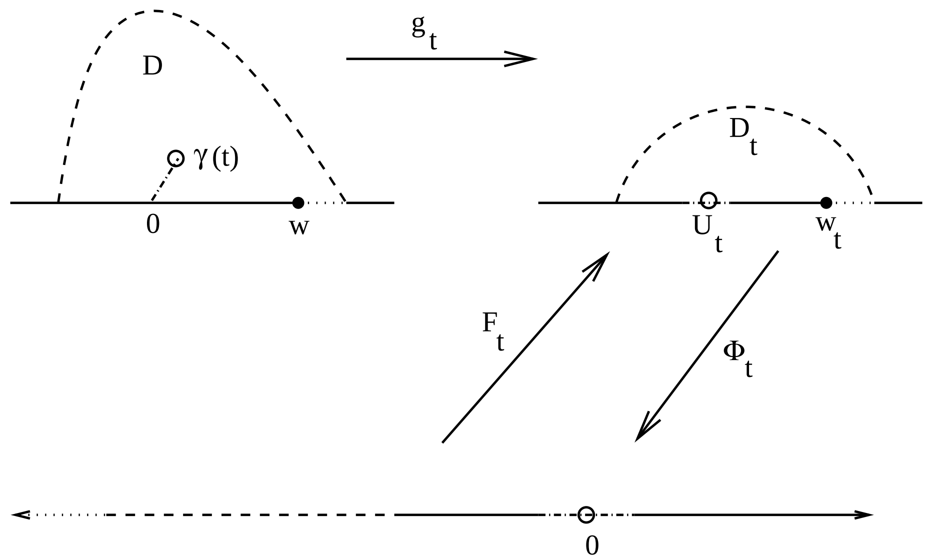



Figure 3: The domain $D_t$ and the maps $F_t, \Phi_t$.

where $B_t$ is a standard Brownian motion. Let $\gamma = -i \log \tilde{\gamma}(t)$ where a branch of the logarithm is chosen with $\log 1 = 0$. Note that for $t$ very small, $\gamma$ grows almost like chordal $SLE_\kappa$ (at time 0 it is growing exactly like this).

To see the difference between radial and chordal, suppose that the path has produced $\gamma(0,t]$. For radial $SLE$, the path has also produced all the $2\pi$ translates of $\gamma(0,t]$. Therefore, locally the path is now growing like chordal $SLE$ from $\gamma(t)$ to $\infty$ in the domain

$$\hat{D}_t := \mathbb{H} \setminus \left[ \bigcup_{k=-\infty}^{\infty} (2\pi k + \gamma(0,t]) \right].$$

Let $D_t = g_t(\hat{D}_t), U_t = g_t(\gamma(t))$. Although $\hat{D}_t$ is periodic, the domain $D_t$ is not periodic. By conformal invariance, radial $SLE_\kappa$ is the process that acts locally like chordal $SLE_\kappa$ from $U_t$ to $\infty$ in the domain $D_t$. Let $\Psi_t$ denote a conformal transformation of $D_t$ onto $\mathbb{H}$ with $\Psi_t(\infty) = \infty$. This transformation is not unique, but if $\tilde{\Psi}_t$ is another such transformation, then $\tilde{\Psi}_t = c\,\Psi_t + x$ for some $c > 0, x \in \mathbb{R}$. If we parametrize the curve $\gamma$ so that $g_t(z) = z + (at/z) + O(|z|^{-2})$ as $z \to \infty$, then the maps $g_t$ satisfy (18) where $U_t$ satisfies the SDE

$$dU_t = b\,\frac{\Psi_t''(U_t)}{\Psi_t'(U_t)}\,dt + dB_t,$$

and $B_t$ is a standard Brownian motion. Note that $\Psi''/\Psi'$ is independent of the choice of $\Psi$.

Now suppose that $A \subset \overline{\mathbb{D}}$ is a closed set not containing 0 or 1 such that $\mathbb{D} \setminus A$ is simply connected. Let $D_A = -i \log(\mathbb{D} \setminus A)$ which is the upper half plane with a peridoic set removed. Suppose $\Theta = \Theta_A$ is a conformal transformation of $\mathbb{D}$ onto $\mathbb{D} \setminus A$ with $\Theta(0) = 0, \Theta(1) = 1$

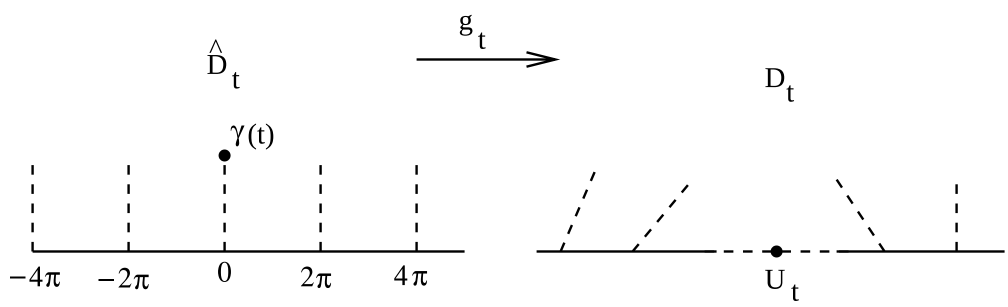



Figure 4: The domains $\hat{D}_t$ and $D_t$ for radial $SLE$.

and let $\theta(z) = -i \log \Theta(e^{iz})$ which is a conformal transformation of $\mathbb{H}$ onto $D_A$. The image of radial $SLE_\kappa$ from 1 to 0 under $\Theta$ is radial $SLE_\kappa$ in $\mathbb{D} \setminus A$ from 1 to 0. Therefore radial $SLE_\kappa$ in $D_A$ can be obtained as the image under $\theta$ of the measure described in the previous paragraph. By combining, we see that (the image under the logarithm map of) radial $SLE_\kappa$ in $\mathbb{D} \setminus A$ looks locally like chordal $SLE$ in $D_A$. Let

$$\hat{D}_{t,A} = D_A \setminus \left[ \bigcup_{k=-\infty}^{\infty} (2\pi k + \gamma(0,t]) \right],$$

and $D_{t,A} = g_t(\hat{D}_{t,A})$. Then locally (the image under the logarithm map of) radial $SLE_\kappa$ in $D_A$ looks like chordal $SLE_\kappa$ from 0 to infinity in the domain $D_{t,A}$. In particular, it satisfies (18) with a driving process $U_t$ satisfying

$$dU_t = b \, \frac{\hat{\Psi}_t''(U_t)}{\hat{\Psi}_t'(U_t)} \, dt + dB_t,$$

where $\hat{\Psi}_t$ is a conformal transformation of $D_{t,A}$ onto $\mathbb{H}$ fixing infinity.

Finally, suppose that $\eta : (0, \infty) \to \mathbb{H}$ is a simple curve with $-2\pi < \eta(0-) < 0$ and $\mathrm{Im}[\eta(s)] \to \infty$. Assume also that

$$\eta(0,\infty) \cap [2\pi + \eta(0,\infty)] = \emptyset.$$

For each $r > 0$, let $\sigma_r$ denote the largest $s$ with $\mathrm{Im}(\eta_s) \leq r$. Let $D$ denote the domain bounded by $[\eta(0+), \eta(0+) + 2\pi], \eta(0,\infty)$, and $[2\pi + \eta(0,\infty)]$. For each $s < \infty$, let $D^{(s)}$ denote the domain

$$D^{(s)} = \mathbb{H} \setminus \left[ \bigcup_{k=-\infty}^{\infty} (2\pi k + \eta(0,s]) \right].$$

We need to compare chordal $SLE_\kappa$ in $D$ from 0 to $\infty$ to radial $SLE_\kappa$ in $D_s$ from 0 to $\infty$. Both processes can be considered as measures on paths $\gamma : (0,\infty) \to \mathbb{H}$ with $\gamma(0+) = 0$.

In both cases, the measures can be obtained by solving (18) with a driving function $U_t$; the difference comes in the SDE that $U_t$ satisfies. Let $D_t = g_t(D)$ and $D_t^{(s)} = g_t(D^{(s)})$. Let $\Phi_t$ be a conformal transformation of $D_t$ onto $\mathbb{H}$ fixing infinity and let $\Phi_{t,s}$ be a conformal transformation of $D_t^{(s)}$ onto $\mathbb{H}$ fixing infinity. Then the driving processes, $U_t, U_{t,s}$ satisfy

$$dU_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\,dt + dB_t, \qquad dU_{t,s} = b\,\frac{\Phi_{t,s}''(U_{t,s})}{\Phi_{t,s}'(U_{t,s})}\,dt + dB_t,$$

Let $I_r$ be the open interval $(b_1+ri, b_2+2\pi+r_i)$ where $b_1 = \max\{x : x+ri \in \eta(0,\sigma_r]\}, b_2 = \min\{x : x+ri \in \eta(0,t_r]\}$. Let $D_r$ denote the Jordan domain bounded by $I_r, (\eta(0+), \eta(0+)+2\pi), \eta(0,t_1], 2\pi+\eta(0,t_2)$ where $\eta(t_1) = b_1 + ri, \eta(t_2) = b_2 + ri$. Note that if $s \geq \sigma_r$, then any curve from 0 to $D^{(s)} \setminus D$ in $D^{(s)}$ must go through $I_r$.

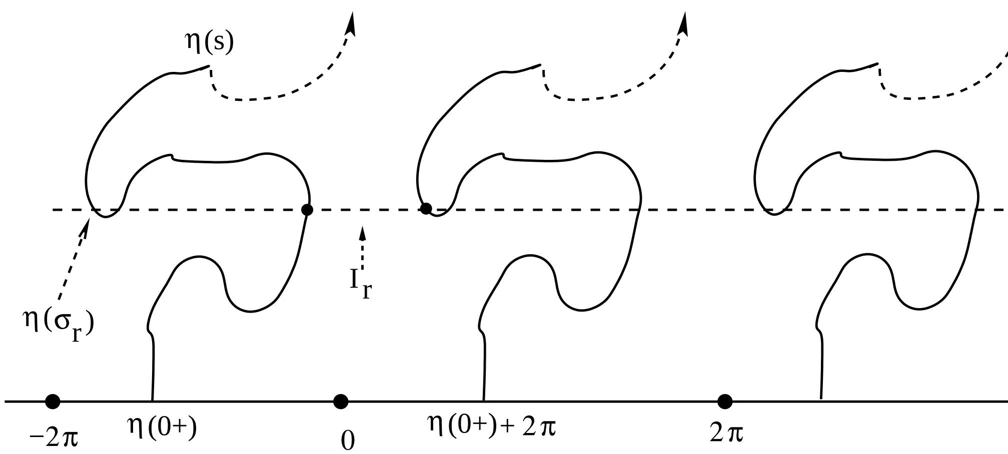



Figure 5: The domain $D^{(s)}$ is $\mathbb{H}$ with the solid curves removed. The domain $D$ is the connected component of $\mathbb{H} \setminus (\eta(0,\infty) \cup [2\pi + \eta(0,\infty)])$ with $0 \in \partial D$.

We will restrict to $0 \leq t \leq 1$. From (18) we can see that if $|z| \geq 2\sqrt{a}$ and $t \leq 1$, then $|g_t(z) - z| \leq 2a/|z|$. In particular, if $r \geq 2\sqrt{a}$, then $g_t(I_r) \subset \{\mathrm{Im}(z) \geq r - \sqrt{a}\}$. Note that any curve from $U_t$ to $g_t(D^{(s)})$ in $g(D^{(s)})$ must go through $g_t(I_r)$. We list some other properties here.

- There exists an $\epsilon > 0$ such that for all $0 \leq t \leq 1$,

$$\{z \in \mathbb{H} : |z - U_t| < \epsilon\} \subset g_t(D_r).$$

- There exists a $c < \infty$ such that the probability that a Brownian motion starting at $U_t$ reflected off the real axis into $\mathbb{H}$ reaches $g_t(I_r)$ before leaving $g_t(D_r)$ is bounded above by $c\,e^{-r/2}$. This follows from the Beurling estimate (see, e.g., [4, Section 3.8]) and the fact that $g_t(I_r) \subset \{\mathrm{Im}(z) \geq r - \sqrt{a}\}$.

We point out that the constant $c$ in the second statement depends only on $a$ while the $\epsilon$ in the first statement depends on $\gamma$ and $\eta$.

To prove (21) it suffices to establish an estimate on conformal maps. We do this in the next subsection. From (22), we can conclude that if

$$s \geq \sup\{s' : \mathrm{Im}(\eta(s')) \leq r\},$$

then

$$|R_{s,t} - R_t| \leq c\,\epsilon^{-1}\,e^{-r/2}.$$

This implies (21).

## Lemmas about conformal maps

Here we will discuss some of the necessary estimates about conformal maps. We start with some setup.

Suppose $\gamma^1, \gamma^2 : (0,1] \to \mathbb{H}$ are simple curves satisfy

- $x_1 = \gamma^1(0+) < 0 < \gamma^2(0+) = x_2$
- $\gamma^1(0,1] \cap \gamma^2(0,1] = \emptyset$.
- If $I = (\gamma^1(1), \gamma^2(1))$ denote the open line segment connecting the endpoints, then

$$I \cap (\gamma^1(0,1] \cup \gamma^2(0,1]) = \emptyset.$$

Let $\hat{D}$ denote the Jordan domain bounded by the curves $\gamma^1(0,1], \gamma^2(0,1], I$, and $[x_1, x_2]$. Let

$$\hat{D}^* = D \cup (x_1, x_2) \cup \{z : \overline{z} \in D\},$$

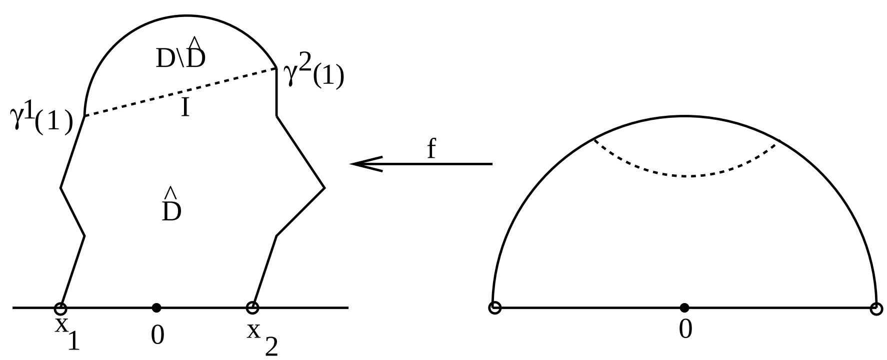
be the extension of $\hat{D}$ by Schwarz reflection. Finally, let $q = q(\gamma^1, \gamma^2)$ denote the harmonic measure of $I \cup I^*$ in $\hat{D}^*$ from 0. Equivalently, $q$ is the probability that a Brownian motion starting at 0 leaves $\hat{D}^*$ at $I$ or $I^*$. By symmetry, the probability of leaving at $I$ is $q/2$.

**Lemma 8.2.** There is a $c < \infty$ such that the following holds. Assume $\gamma^1, \gamma^2$ are given as above. Suppose $D$ is a simply connected domain with

$$\hat{D} \cup I \subset D \subset \mathbb{H} \setminus (\gamma^1(0,1] \cup \gamma^2(0,1]).$$

Let $F : D \to \hat{D}$ be the unique conformal transformation with $F(x_1) = x_1, F(0) = 0, F(x_2) = x_2$. Suppose $\hat{D}^*$ contains the open ball of radius $\epsilon$ about 0. Then,

$$|F'(0) - 1| \leq cq, \qquad |F''(0)| \leq c\,\epsilon^{-1}\,q.$$




*Proof.* By scaling we may assume that $\epsilon = 1$. The Koebe-1/4 and the Bieberbach estimate give $1/4 \leq F'(0) \leq 4, |F''(0)| \leq 2\,F'(0)$, so it suffices to prove the result for $q$ sufficiently small.

The Riemann mapping theorem states that there is a unique conformal transformation $f : \mathbb{D} \cap \mathbb{H} \to D$ with $f(-1) = x_1, f(0) = 0, f(1) = x_2$. By Schwarz reflection, this can be extended to a conformal transformation $f : \mathbb{D} \to D^*$ where $D^* = D \cup (-x_1, x_2) \cup \{z : \overline{z} \in D\}$.

Let $U = f^{-1}(\hat{D}^*)$. Then $U$ is a simply connected subdomain of $\mathbb{D}$ with the property that the probability that a Brownian motion starting at the origin leaves $U$ before leaving $\mathbb{D}$ equals $q$. Since $U$ is simply connected (and, hence, $\partial U$ is connected), we can see that there is a $c$ such that

$$(1 - \,c\,q)\,\mathbb{D} \subset U.$$

We will assume that $q$ is sufficient small so that $cq < 1/2$ and write $\delta = cq$.

Let $h : \mathbb{D} \to U$ be the unique conformal transformation with $h(0) = 0, h'(0) > 0$. The Schwarz lemma tells us that $(1 - \delta) \leq h'(0) \leq 1$. We will show that $|h''(0)| \leq c\,\delta$. Let $g(z) = \log(h(z)/z)$ which is a well-defined analytic function since $h'(0) > 0$ and $h(z) \neq 0$ for $z \neq 0$. The maximum principle implies that

$$|\mathrm{Re}\, g(z)| \leq \sup\{|\mathrm{Re}\, g(z)| : |z| = 1\} \leq |\log(1 - \delta)| \leq c\,\delta.$$

Since $\mathrm{Re}\, g$ is a harmonic function, this implies that the partial derivatives of $\mathrm{Re}\, g(z)$ are $O(\delta)$ for $|z| \leq 1/4$. Hence, by the Cauchy-Riemann equations, $|g'(z)| = O(\delta)$ for $|z| \leq 1/4$. Since $g(0) = \log h'(0) = O(\delta)$, we conclude that $|g(z)| \leq c\,\delta$ for $|z| \leq 1/4$, and hence

$$|h(z) - z| \leq c\delta.$$

From this we conclude that $|h''(0)| = |(h - z)''(0)| \leq c\,\delta$.

Since $h$ is unique, we can see that $h = f^{-1} \circ F \circ f$. The chain rule gives

$$F'(0) = h'(0),$$

$$F''(0) = \frac{1}{f'(0)} \left[ h''(0) - h'(0)\,[h'(0) - 1]\,\frac{f''(0)}{f'(0)} \right].$$

The Koebe-1/4 and the Bieberbach estimate give $|f'(0)| \geq 1/4, |f''(0)| \leq 2\, f'(0)$. Therefore,

$$F'(0) - 1 = h'(0) - 1, \qquad |F''(0)| \leq 8\, [|h''(0)| + |h'(0)|\, |h'(0) - 1|].$$

However we have seen that

$$|h'(0) - 1|, |h''(0)| \leq c\, q,$$

at least if $q$ is sufficiently small.

□

**Lemma 8.3.** Suppose $\hat{D}$ is as above and $D, \tilde{D}$ are two domains satisfying the conditions (on $D$) of the previous lemma and let $q, \epsilon$ be as in that lemma. Let $\Phi : D \longrightarrow \mathbb{H}$ denote the unique conformal transformation with $\Phi(0) = 0, \Phi(\infty) = \infty, \Phi'(0) = 1$; and let $\tilde{\Phi}$ denote the corresponding transformation for $\tilde{D}$. Then

$$|\Phi''(0) - \tilde{\Phi}''(0)| \leq c\, q\, \epsilon^{-1}. \tag{22}$$

*Proof.* Let $F$ be the unique conformal transformation of $D$ onto $\tilde{D}$ with $F(x_1) = x_1, F(0) = 0, F(x_2) = x_2$. By applying the previous lemma we cam see that

$$|F'(0) - 1| \leq c\, q, \qquad |F''(0)| \leq c\, \epsilon^{-1}\, q.$$

Let

$$\Psi(z) = \frac{\tilde{\Phi} \circ F(z)}{F'(0)}.$$

Then $\Psi$ is a conformal transformation of $D$ onto $\mathbb{H}$ with $\Psi(0) = 0, \Psi'(0) = 1$. Also,

$$\Psi''(0) = \tilde{\Phi}''(0)\, F'(0) + \frac{F''(0)}{F'(0)}.$$

The transformation $\Psi$ might not equal $\Phi$ since $\Psi(\infty)$ might not equal $\infty$. However, it is easy to check that

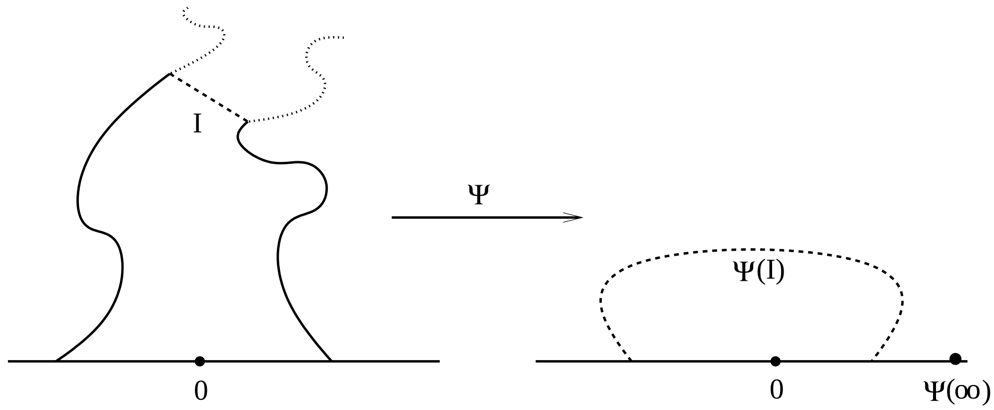
$$\Phi(z) = \frac{\Psi(\infty)\, \Psi(z)}{\Psi(\infty) - \Psi(z)},$$

where $\Psi(\infty)/[\Psi(\infty) - \Psi(z)]$ is interpreted to equal 1 if $\Psi(\infty) = \infty$. Note that $\Phi'(0) = 1$ and

$$\Phi''(0) = \Psi''(0) + \frac{2}{\Psi(\infty)} = \tilde{\Phi}''(0)\, F'(0) + \frac{F''(0)}{F'(0)} + \frac{2}{\Psi(\infty)}.$$

Therefore,

$$|\Phi''(0) - \tilde{\Phi}''(0)] \leq |\tilde{\Phi}''(0)|\, |F'(0) - 1| + \left|\frac{F''(0)}{F'(0)}\right| + \frac{2}{|\Psi(\infty)|}.$$




Applying the Bieberbach estimate to (the Schwarz reflection extension of) $z \mapsto \tilde{\Phi}(\epsilon z)/\epsilon$ gives $|\tilde{\Phi}''(0)| \leq 2/\epsilon$. We have already bounded $|F'(0) - 1|$ and $|F''(0)|$. We now need to estimate $|\Psi(\infty)|$. Note that $\Psi(I)$ is a curve in $\mathbb{H}$ connecting the negative real axis to the positive real axis. Let $d$ be the distance of this curve from the origin. Using the gambler's ruin estimate, it is not difficult to show that the probability that a Brownian motion starting at $\delta i$ hits this image before leaving $\mathbb{H}$ is bounded below by $c\,\delta/d$. [In fact, if $\delta < d/2$, and $z \in \mathbb{H}$ with $|z| \leq 2d$; $B_t$ is a Brownian motion starting at $i\delta$; and $T$ denotes the first $t$ with $B_t \in \mathbb{R}$, then with probability at least $O(\delta/d)$ the point $z$ will be in a bounded component of $\mathbb{H} \setminus (B[0,T] \cup [0, \delta i])$. In this case we must have $B(0,T)$ intersecting the image curve.] Note that $\Psi(\infty)$ lies outside this curve so $|\Psi(\infty)| \geq d$. The probability starting at $\delta i$ that a Brownian motion leaves $D$ at $I$ is bounded above by $c\,(\delta/\epsilon)\,q$. [Here, $c\,\delta/\epsilon$ bounds the probability to reach the sphere of radius $\epsilon/2$ and the Harnack inequality implies that the probability of reaching $I$ given this is bounded by $c\,q$.] Hence, since $\Psi'(0) = 1$, we get that

$$|\Psi(\infty)| \geq c\,\epsilon\,q^{-1}.$$

This establishes (22). □

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