REGULARITY OF LOEWNER CURVES

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Abstract. The Loewner equation encrypts a growing simple curve in the plane into a real-valued driving function. We show that if the driving function λ is in C^{β} with $\beta > 2$ (or real analytic) then the Loewner curve is in $C^{\beta+\frac{1}{2}}$ (resp. analytic). This is a converse of [EE01] and extends the result in [Won14].

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1. Introduction and results

The Loewner differential equation, a classical tool that has attracted recent attention due to Schramm-Loewner evolution, provides a unique way of encoding a simple 2-dimensional curve into a continuous 1-dimensional function. In particular, let $\gamma : [0,T] \to \mathbb{C}$ be a simple curve with $\gamma(0) = 0$ and $\gamma(0,T) \in \mathbb{H} = \{x + iy \,:\, y > 0\}$. For each $t \in [0,T]$, there is a unique conformal map $g_t : \mathbb{H} \setminus \gamma(0,t) \to \mathbb{H}$ with the so-called hydrodynamic normalization:

$$g_t(z) = z + \frac{a(t)}{z} + O(z^{-2}) \text{ for } z \text{ near infinity.} \tag{1}$$

Further, it is possible to reparametrize γ so that $a(t) = 2t$ in equation (1). In this case, we say that γ is parametrized by halfplane capacity (since $a(t)$ is called the halfplane capacity of $\gamma[0,T]$ and can be thought of as a measure of the size of $\gamma[0,T]$.) Unless stated otherwise, we will assume γ has this parametrization throughout the paper. The Loewner equation describes the time evolution of g_t:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}$$

where $\lambda(t) = g_t(\gamma(t))$ is a continuous real-valued function, called the driving function. (See [Law05] for further details.)

It is natural to ask how properties of the Loewner curve γ correspond to properties of the driving function λ. In this paper, we prove the following theorem relating the regularity of λ to the regularity of γ.

Theorem 1.1. *Let $\lambda \in C^{\beta}[0,T]$ for $\beta > 2$. Then the Loewner curve γ is $C^{\beta+\frac{1}{2}}(0,T]$, when $\beta + 1/2 \notin \mathbb{N}$, and γ is in $\Lambda_*^n(0,T]$ when $\beta = n + \frac{1}{2}$ for $n \in \mathbb{N}$.*

See Theorem 4.1 for quantitative version, and see Section 2.1 for precise definitions. This theorem extends the work in [Won14], where the result was proven for $\beta \in (1/2, 2]$. In Section 7, we discuss an example where $\lambda \in C^{3/2}$ but γ fails to be C^2, illustrating the fact that it is not possible to strengthen Theorem 1.1 to say that $\gamma \in C^{n+1}$ when $\lambda \in C^{n+1/2}$.

We also address the analytic case:

Theorem 1.2. *If λ is real analytic on $[0,T]$, then γ is also real analytic on $(0,T]$.*

Notice that in both of these theorems, the regularity of γ is on the time interval $(0,T]$. With the halfplane-capacity parametrization, it is not possible to extend these results to $t = 0$. To see this, consider the example when the driving function is $\lambda(t) \equiv 0$. Then the corresponding Loewner curve is $\gamma(t) = 2i\sqrt{t}$. Further, with the halfplane-capacity parametrization, $\gamma(t)$ can always be expanded at $t = 0$ in powers of $\sqrt{t}$, as we see in the following theorem.

Theorem 1.3. *Assume that $\lambda \in C^{n+\alpha}[0,T]$ for $n \in \mathbb{N}$ and $\alpha \in (0,1]$. Then near $t = 0$,*

$$\gamma(t) = \begin{cases} 2i\sqrt{t} + a_2 t + i\, a_3 t^{3/2} + a_4 t^2 + \cdots + a_{2n} t^n + O(t^{n+\alpha}) & \text{if } \alpha \le 1/2 \\ 2i\sqrt{t} + a_2 t + i\, a_3 t^{3/2} + a_4 t^2 + \cdots + a_{2n} t^n + i\, a_{2n+1} t^{n+1/2} + O(t^{n+\alpha}) & \text{if } \alpha > 1/2 \end{cases}$$

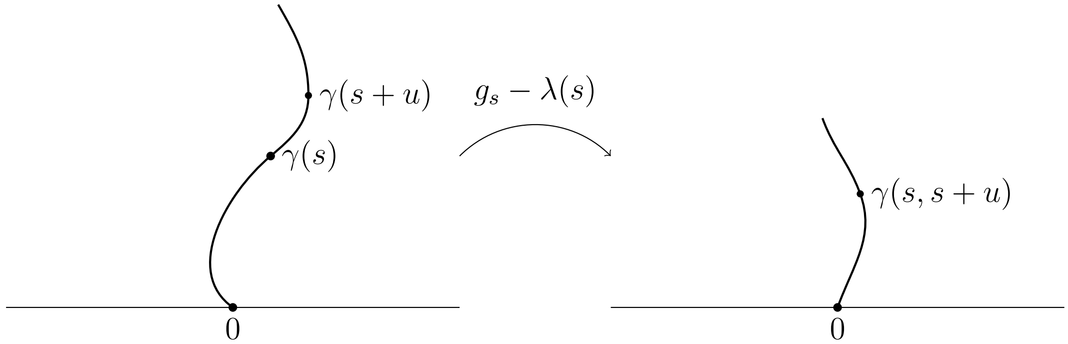

FIGURE 1. The curve $\gamma(s, s+u) = g_s(\gamma(s+u)) - \lambda(s)$.

where the real-valued coefficients a_m depend on $\lambda^{(k)}(0)$ for $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.

If we make the simple change of parametrization $t = s^2$, then the smoothness extends to $s = 0$.

Theorem 1.4. *Let $\Gamma(s) = \gamma(s^2)$ be the reparametrized Loewner curve with driving function λ. If λ is real analytic on $[0, T]$, then Γ is real analytic on $[0, \sqrt{T}]$. If $\lambda \in C^\beta[0, T]$, then $\Gamma \in C^{\beta+1/2}[0, \sqrt{T}]$ when $\beta + 1/2 \notin \mathbb{N}$.*

We wish to briefly describe the key tool used in this paper. For $s \in [0, T]$, consider the simple curve $g_s(\gamma(s+u)) - \lambda(s)$, which we denote by $\gamma(s, s+u), 0 \leq u \leq T - s$. See Figure 1. The curve $\gamma(s, s+u)$ corresponds to the time-shifted driving function $\lambda_s(u) = \lambda(u+s) - \lambda(s)$, $0 \leq u \leq T - s$. It follows from [Won14, Theorem 6.2] that under the assumption $\lambda \in C^2[0, T]$, the curve γ is in C^2 and

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s) \int_0^s \frac{\partial_s[\gamma(s-u, s)]}{\gamma(s-u, s)^3} du. \tag{2}$$

In order to understand the higher differentiability of γ, we need to understand $\gamma(s-u, s)$. Differentiating this function with respect to u, we obtain

$$\partial_u[\gamma(s-u, s)] = \partial_u[g_{s-u}(\gamma(s)) - \lambda(s-u)] = \frac{-2}{\gamma(s-u, s)} + \lambda'(s-u) \text{ for } 0 < u \leq s, \tag{3}$$

and $\gamma(s-u, s)|_{u=0} = \gamma(s, s) = 0$. We note that the above differential equation does not hold for $u = 0$. This is the reason for us to investigate the following ODE:

$$\begin{aligned} f'(u) &= \frac{-2}{f(u)} + \lambda'(s-u), \; 0 \leq u \leq s, \\ f(0) &= i\epsilon \in \mathbb{H}. \end{aligned} \tag{4}$$

The work in this paper depends on a deep understanding of the function $f(u) = f(u, s, \epsilon)$ which is the solution to (4). Once we show that $f(u, s, \epsilon)$ converges uniformly to $\gamma(s-u, s)$ as $\epsilon \to 0^+$, we can use (2) to translate information about f into information about the derivatives of γ.

Remark. Theorem 1.1 and Theorem 1.2 provide a converse to the results of Earle and Epstein in [EE01]. Their results (translated from the radial setting to the chordal setting using [Mar11]) state that if any parametrization of γ is C^n, then the halfplane-capacity parametrization of γ is in $C^{n-1}(0,T)$ and $\lambda \in C^{n-1}(0,T)$. They also prove that if γ is real analytic, then λ must be real analytic.

The paper is organized as follows: Section 2 includes initial properties of $f(u,s,\epsilon)$ and some lemmas regarding solutions to a particular class of ODEs. These lemmas will be useful in analyzing f and its partial derivatives, and this is the content of Section 3. In Section 4, we state and prove a quantitative version of Theorem 1.1. Theorem 1.2 is proved is Section 5. In Section 6, we analyze the behavior of the trace at its base, proving Theorem 1.4 and Theorem 1.3. The latter is proven by constructing a nice curve that well-approximates a given Loewner curve at its base. We conclude in Section 7 with two examples.

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2. Preliminaries

2.1. **Notation.** Let I be an interval on the real line. The space $C^0(I)$ consists of all continuous functions on I and $||\phi||_{\infty,I} = \sup_{t\in I}|\phi(t)|$ for $\phi \in C^0(I)$.

Let $\alpha \in (0,1)$. A function ϕ defined on I is in C^α if $||\phi||_{\infty,I} < \infty$ and

$$||\phi||_{C^\alpha} := \sup_{s,t\in I, s\neq t} \frac{|\phi(t)-\phi(s)|}{|t-s|^\alpha} < \infty.$$

Let $n \in \mathbb{N}_0$, $\alpha \in [0,1]$ and $M > 0$. A function ϕ is in $C^{n,\alpha}(I;M)$ if $\phi', \cdots, \phi^{(n)}$ exist and are continuous and the following two conditions hold:

$$||\phi^{(k)}||_{\infty,I} \leq M \text{ for all } 0 \leq k \leq n,$$
$$\text{and } ||\phi^{(n)}||_{C^\alpha} := \sup_{s,t\in I, s\neq t} \frac{|\phi^{(n)}(t)-\phi^{(n)}(s)|}{|t-s|^\alpha} \leq M.$$

In particular, the n^{th} derivative of functions in $C^{n,1}$ are Lipschitz. A function ϕ is in C^n if $\phi \in C^{n,0}(I;M)$ for some M. When $\alpha \in (0,1)$, we also write $C^{n+\alpha}$ for $C^{n,\alpha}$.

Zygmund introduced a generalization of $C^{0,1}$ called Λ_*. A continuous function ϕ is in $\Lambda_*(I)$ means that

$$||\phi||_{\Lambda_*} := \sup_{s-\delta, s+\delta \in I, \delta>0} \frac{|\phi(s+\delta)+\phi(s-\delta)-2\phi(s)|}{\delta} < \infty.$$

We say that $\phi \in \Lambda_*^n(I;M)$ if $\phi', \cdots, \phi^{(n)}$ exist and are continuous, $\phi^{(n)} \in \Lambda_*$, and the following two conditions hold:

$$||\phi^{(k)}||_{\infty,I} \leq M \text{ for all } 0 \leq k \leq n,$$
$$\text{and } ||\phi^{(n)}||_{\Lambda_*} \leq M.$$

The following proposition will be needed in Section 6.

Proposition 2.1. *If a function ϕ belongs to $C^{n,\alpha}(I;M)$ then there exists $c = c(n,M)$ such that for all $t_0, t+t_0 \in I$,*

$$|\phi(t+t_0) - \sum_{k=0}^{n} \frac{1}{k!} t^k \phi^{(k)}(t_0)| \leq ct^{n+\alpha}.$$

The proof follows from the integral form of the remainder of Taylor series.

We use C for a universal constant, and c for a constant depending on M, n, T. When constants depend on other factors, we will state this explicitly.

2.2. **Loewner equation.** In the introduction we described how the Loewner equation can be used to encode a simple curve into its driving function. This process can be reversed. Let λ be a real-valued continuous function on $[0,T]$ with $T > 0$. Then the forward chordal Loewner equation is the following initial value problem:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \ g_0(z) = z. \tag{5}$$

For each $z \in \mathbb{H}$, the solution $g_t(z)$ exists up to $T_z = \inf\{t > 0 : g_t(z) - \lambda(t) = 0\}$. Let $K_t = \{z \in \mathbb{H} : T_z \leq t\}$. It is known that g_t is the unique conformal map from $\mathbb{H}\backslash K_t$ to $\mathbb{H}$ that satisfies the hydrodynamic normalization at infinity:

$$g_t(z) = z + O(\frac{1}{z}), \quad \text{near } z = \infty.$$

We say that λ generates the curve $\gamma : [0,T] \to \overline{\mathbb{H}}$ if $\mathbb{H}\backslash K_t$ is the unbounded component of $\mathbb{H}\backslash\gamma(0,T]$. In this case $\lambda(t) = g_t(\gamma(t))$, $0 \leq t \leq T$. An important property of the chordal Loewner equation is the concatenation property, which says that for fixed s, the time-shifted driving function $\lambda(s+t)$ generates the mapped curve $g_s(\gamma(s,t))$. For more details, see [Law05].

It was shown that if $\lambda \in C^{1/2}[0,T]$ with $||\lambda||_{C^{1/2}} < 4$ then λ generates a simple quasi-arc γ ([MR05], [Lin05]). Since we work with $\lambda \in C^\beta$ for $\beta > 2$, on small intervals $||\lambda||_{C^{1/2}} \leq 1$. Therefore we are guaranteed that the corresponding Loewner curve is a simple curve. We can prove Theorems 1.1 and 1.2 on small intervals, then use the concatenation property of the Loewner equation to derive the regularity of γ on $[0,T]$. Henceforth, we assume $||\lambda||_{C^{1/2}} \leq 1$.

Changing (5) by a negative sign gives the backwards chordal Loewner equation:

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \xi(t)}, \ h_0(z) = z \tag{6}$$

for a continuous real-valued function ξ defined on $[0,T]$. The solution $h_t(z)$ exists for all $z \in \mathbb{H}$ and $t \in [0,T]$, and h_t is a conformal map from $\mathbb{H}$ into $\mathbb{H}$. The forward and backward versions of the Loewner equation are related as follows: if g_t is the solution to (5) with driving function $\lambda \in C[0,T]$ and h_t is the solution to (6) with driving function $\xi(t) = \lambda(T-t)$, then $h_t = g_{T-t} \circ g_T^{-1}$, and in particular, $h_T = g_T^{-1}$.

We think of (4) as a variant of the backward Loewner equation (with $\xi(u) = \lambda(s-u)$ and $f(u) = h_u(i\epsilon) - \xi(u)$), and our first goal is to understand some basic properties of its solution $f(u) = f(u, s, \epsilon)$, when $(u, s) \in D := \{(u, s) : 0 \leq u \leq s \leq T\}$. Further properties of $f(u, s, \epsilon)$ are in Section 3.

Lemma 2.2. *Let $\lambda \in C^1([0,T]; M)$, and let $0 \leq s \leq T$ and $\epsilon > 0$. Then the ODE*

$$\begin{aligned} f'(u) &= \frac{-2}{f(u)} + \lambda'(s-u), \ 0 \leq u \leq s, \\ f(0) &= i\epsilon \in \mathbb{H}. \end{aligned}$$

has a unique solution $f(u) = f(u, s, \epsilon)$, with $0 \leq u \leq s$, satisfying the following properties:

(i) $Im\, f$ is increasing in u.

(ii) For all $(u, s) \in D = \{(u, s) : 0 \leq u \leq s \leq T\}$

$$\sqrt{3u + \epsilon^2} \leq Im\, f(u, s, \epsilon) \leq \sqrt{4u + \epsilon^2}$$

$$\text{and} \quad |Re\, f(u, s, \epsilon)| \leq \sqrt{u} \leq \frac{1}{\sqrt{3}} Im\, f(u, s, \epsilon).$$

(iii) For every $\delta > 0$, there is $\epsilon(\delta) > 0$ such that

$$|f(u, s, \epsilon_1) - f(u, s, \epsilon_2)| \leq \delta \text{ for all } (u, s) \in D \text{ and } \epsilon_1, \epsilon_2 \leq \epsilon(\delta).$$

In particular, $f(u, s, \epsilon)$ converges uniformly as $\epsilon \to 0+$ to a limit denoted by $f(u, s)$. This limit is the family of curves $\gamma(s-u, s)$ generated by λ_s, $0 \leq s \leq T$.

(iv) Suppose $\lambda \in C^n([0,T]; M)$, and let $l + k \leq n$ and $k \leq n - 1$. Then $\partial_u^l \partial_s^k f$ exists and is continuous in $(u, s) \in D$ for all $\epsilon > 0$.

(v) If $\lambda \in C^n([0,T]; M)$ and $1 \leq k \leq n-1$, then $\partial_s^k f(0, s, \epsilon) = 0$ for all $s \in [0,T]$ and $\epsilon > 0$.

Proof. The equation (4) is of the form:

$$f'(u) = G(f(u), u, s),$$

where $G(z, u, s) = \frac{-2}{z} + \lambda'(s-u)$ is jointly continuous in z, u, s, and Lipschitz in z variable whenever $\operatorname{Im} z \geq C > 0$. So the solution exists on some interval containing 0. To show that the solution to (4) exists on the whole interval $[0, s]$, it suffices to show that (i) always holds. The idea of $(i)-(iii)$ comes from [RTZ13], which contains a study of the Loewner equation when $||\lambda||_{C^{1/2}} < 4$. For the convenience of the reader, we will present the proof here.

Let $x = x(u), y = y(u)$ be real and imaginary parts of $f(u)$. It follows from (4) that

$$(x + \lambda(s - \cdot))' = \frac{-2x}{x^2 + y^2}, \tag{7}$$

$$y' = \frac{2y}{x^2 + y^2}. \tag{8}$$

In particular, y is increasing and $(y^2)' \leq 4$. The former shows (i), and the latter shows that $y \leq \sqrt{4u + \epsilon^2}$.

Now we will show that $|x(u)| \leq \sqrt{u}$, for $0 \leq u \leq s$. Suppose $0 \leq x(u)$ and let $u_0 = \sup\{v \in [0,u] : x(v) \leq 0\}$. So

$$\partial_v(x(v) + \lambda(s-v)) \leq 0 \text{ for } u_0 \leq v \leq u,$$

and

$$x(u) + \lambda(s-u) \leq x(u_0) + \lambda(s-u_0) = \lambda(s-u_0).$$

Hence

$$x(u) \leq \lambda(s-u_0) - \lambda(s-u) \leq \sqrt{|u_0 - u|} \leq \sqrt{u}.$$

where the very last inequality follows since $||\lambda||_{1/2} \leq 1$. The same argument applies when $x(u) \leq 0$, proving that $|x(u)| \leq \sqrt{u}$.

Next we will show $y(u) > \sqrt{3u}$ for $0 \leq u \leq s$. Suppose this is not the case. Then since $y(0) = \epsilon > 0$, there exists $u_0 \in (0,s]$ such that $y(u_0) = \sqrt{3u_0}$ and $y(u) \geq \sqrt{3u}$ for $u \in [0,u_0]$. It follows from (8) that

$$(y^2)' = \frac{4y^2}{x^2+y^2} \geq \frac{12u}{u+3u} = 3 \text{ for } 0 \leq u \leq u_0.$$

So $y(u_0) \geq \sqrt{3u_0 + \epsilon^2} > \sqrt{3u_0}$. This is a contradiction. Therefore $y(u) > \sqrt{3u}$ and $(y^2)' \geq 3$. These show (ii).

To show (iii), differentiate (4) with respect to ϵ to obtain

$$\partial_u(\partial_\epsilon f) = \partial_\epsilon \partial_u f = \frac{2\partial_\epsilon f}{f^2}.$$

Since $\partial_\epsilon f(0,s,\epsilon) = i$,

$$\partial_\epsilon f(u,s,\epsilon) = i \exp \int_0^u \frac{2}{f^2(v,s,\epsilon)}\, dv.$$

This implies

$$\begin{aligned} |\partial_\epsilon f(u,s,\epsilon)| &= \exp \int_0^u \operatorname{Re} \frac{2}{f^2(v,s,\epsilon)}\, dv \\ &= \exp \int_0^u \frac{2(x^2(v) - y^2(v))}{(x^2(v)+y^2(v))^2}\, dv \leq 1. \end{aligned}$$

The last inequality comes from (ii). It follows that

$$|f(u,s,\epsilon) - f(u,s,\epsilon')| \leq |\epsilon - \epsilon'|, \text{ for all } 0 \leq u \leq s \leq T,$$

and $f(u,s,\epsilon)$ converges uniformly in D to a limit, denoted by $f(u,s)$, as $\epsilon \to 0^+$.

Intuitively the limit $f(u,s)$ is equal to $\gamma(s-u,s)$ since $f(u,s,\epsilon)$ satisfies the same ODE as $\gamma(s-u,s)$ does, and $\lim_{\epsilon\to 0^+} f(0,s,\epsilon) = \gamma(s-u,s)|_{u=0} = 0$. Indeed, from (3) and (4) we can show that

$$(9) \quad |f(u,s,\epsilon) - \gamma(s-u,s)| = |f(u_0,s,\epsilon) - \gamma(s-u_0,s)| \exp \int_{u_0}^u \operatorname{Re} \frac{2\, dv}{f(v,s,\epsilon)\gamma(s-v,s)},$$

with $0 < u_0 \leq u \leq s \leq T$ and $\epsilon > 0$. Since $\gamma(s-v,s)$ is the tip of a Loewner curve generated by a driving function whose Hölder-1/2 norm is less than 1, then by [Won14, Lemma 3.1], it satisfies

$$|\operatorname{Re} \gamma(s-v,s)| \leq \operatorname{Im} \gamma(s-v,s).$$

This implies that

$$\operatorname{Re}\frac{2}{f(v,s,\epsilon)\gamma(s-v,s)} \le 0.$$

Let $u_0 \to 0^+$ and then $\epsilon \to 0^+$ in (9) we get $f(u,s) = \gamma(s-u,s)$.

Statement (iv) follows from the standard ODE theory (see [CL55], for instance) and the fact that G is C^{n-1} in (u,s).

We show (v) by induction. For the base case,

$$\partial_s f(0,s,\epsilon) = \lim_{\delta\to 0}\frac{f(0,s+\delta,\epsilon)-f(0,s,\epsilon)}{\delta} = \lim_{\delta\to 0}\frac{\epsilon-\epsilon}{\delta} = 0.$$

Now supppose $\partial_s^k f(0,s,\epsilon) = 0$ for all $s \in [0,T]$. Then

$$\partial_s^{k+1} f(0,s,\epsilon) = \lim_{\delta\to 0}\frac{\partial_s^k f(0,s+\delta,\epsilon)-\partial_s^k f(0,s,\epsilon)}{\delta} = 0.$$

□

Remark. For convenience, in this paper we only consider $\epsilon \in (0,1]$. In this case,

$$\sqrt{3u} \le |f(u,s,\epsilon)| \le \sqrt{Cu+\epsilon^2} \le C\sqrt{u} + C\epsilon \le c(T) \text{ for all } 0 \le u,s \le T.$$

Later in Lemma 3.2 we will show that $\partial_s^n f$ exists and is continuous in (u,s).

2.3. **ODE lemmas.** The next lemma is one of the key tools to investigate the regularity of $f(u,s,\epsilon)$.

Lemma 2.3. *Consider a complex-valued function X satisfying the initial value problem*

$$X'(u) = P(u)X(u) + Q(u), \ \ X(0) = 0.$$

Suppose $|P(u)| \le -C\,Re\,P(u)$ and $|Q(u)| \le M_1$ for $0 \le u \le u_0$. Then

$$|X(u)| \le (C+1)M_1 u \text{ for } 0 \le u \le u_0.$$

Proof. Solving the equation, one obtains

$$X(u) = R(u) + e^{-\mu(u)}\int_0^u e^{\mu(v)}P(v)R(v)\,dv,$$

where $\mu(u) = -\int_0^u P(v)\,dv$ and $R(u) = \int_0^u Q(v)\,dv$. Since $|R(u)| \le M_1 u$,

$$\begin{aligned}
|X(u)| &\le M_1 u + M_1 u|e^{-\mu(u)}|\int_0^u |e^{\mu(v)}|\cdot|P(v)|\,dv \\
&\le M_1 u + M_1 u e^{-\operatorname{Re}\mu(u)}\int_0^u e^{\int_0^v -\operatorname{Re}P(w)dw} C(-\operatorname{Re}P(v))\,dv \\
&= M_1 u + CM_1 u e^{-\operatorname{Re}\mu(u)}\left(e^{-\int_0^u \operatorname{Re}P(v)\,dv} - 1\right) \\
&= M_1 u + CM_1 u e^{-\operatorname{Re}\mu(u)}\left(e^{\operatorname{Re}\mu(u)} - 1\right) \\
&\le (C+1)M_1 u.
\end{aligned}$$

□

In some cases, we will need a more general version of Lemma 2.3.

Lemma 2.4. *Let Y be a solution to*

$$Y'(u) = P(u)Y(u) - P(u)Q(u) + R(u),\ Y(0) = Q(0)$$

with $|P| \leq -C\,\mathrm{Re}\,P$ and $|Q(v) - Q(0)| \leq \omega(v)$ on $[0, u_0]$, where ω is a non-decreasing function.

(i) If $|R| \leq M_2 u^{\beta-1}$, then

$$|Y(u) - Q(u)| \leq (C+1)\omega(u) + (C+1)\frac{M_2}{\beta}u^\beta.$$

(ii) If $Y(0) = Q(0) = 0$ and $|R| \leq M_2$, then

$$|Y(u)| \leq C\omega(u) + (C+1)M_2 u.$$

(iii) More generally,

$$|Y(u) - Q(u)| \leq (C+1)\omega(u) + (C+1)\int_0^u |R(v)|\,dv.$$

Proof. Let $\mu(u) = \int_0^u -P(v)\,dv$ and $S(u) = \int_0^u R(v)\,dv$. We have

$$\begin{aligned} Y(u) &= e^{-\mu(u)}Y(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-PQ + R)\,dv \\ &= Q(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)[Q - Q(0)]\,dv + e^{-\mu(u)}\int_0^u e^{\mu(v)}R\,dv \\ &= Q(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)[Q - Q(0)]\,dv + S(u) - e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)S\,dv, \end{aligned}$$

where the last equality follows from an integration by parts. Therefore under the first assumption, $|S(u)| \leq M_2 u^\beta/\beta$ and

$$\begin{aligned} |Y(u) - Q(u)| \leq{}& |Q(0) - Q(u)| + e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\omega(v)\,dv + |S(u)| \\ &+ e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\frac{M_2}{\beta}u^\beta\,dv \\ \leq{}& \omega(u) + C\omega(u) + \frac{M_2}{\beta}u^\beta + C\frac{M_2}{\beta}u^\beta. \end{aligned}$$

Under the second assumption,

$$\begin{aligned} |Y(u)| \leq{}& e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\omega(v)\,dv + |S(u)| \\ &+ e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)M_2 u\,dv \\ \leq{}& C\omega(u) + M_2 u + CM_2 u. \end{aligned}$$

□

3. Properties of $f(u, s, \epsilon)$

In this section, we will prove all important properties of $f(u, s, \epsilon)$, which are summarized in Proposition 3.7. Then we let $\epsilon \to 0^+$ to get properties of $f(u, s) = \gamma(s-u, s)$. The next two lemmas concern the s-derivatives of f.

Lemma 3.1. *Suppose $\lambda \in C^n([0,T]; M)$ with $n \geq 2$. For every $1 \leq k \leq n-1$, there exists a function $Q_k = Q_k(u, s, \epsilon)$ such that*

$$\partial_u(\partial_s^k f) = \frac{2}{f^2}\partial_s^k f + Q_k.$$

with $(u, s) \in D$, and $\epsilon \in (0, 1]$. Moreover there exists constant $c = c(M, n, T) > 0$ so that

$$|\partial_s^k f(u, s, \epsilon)| \leq cu.$$

Proof. We will prove the lemma by induction. Let $k = 1$ and $n \geq 2$. Fix $s \in [0, T]$ and $\epsilon \in (0, 1)$, and let $X(u) = \partial_s f(u, s, \epsilon)$. Then

$$\begin{aligned} X'(u) = \partial_u \partial_s f(u, s, \epsilon) = \partial_s \partial_u f(u, s, \epsilon) &= \frac{2}{f^2(u, s, \epsilon)}\partial_s f(u, s, \epsilon) + \lambda''(s-u) \\ &= \frac{2}{f^2(u, s, \epsilon)}X(u) + \lambda''(s-u), \end{aligned}$$

and $X(0) = \partial_s f(0, s, \epsilon) = 0$. Let $P_s = P_s(u, \epsilon) = \frac{2}{f^2(u,s,\epsilon)}$ and $Q_1(u, s, \epsilon) = \lambda''(s-u)$. Clearly, $|Q_1| \leq M$. We will show that $P_s(\cdot, \epsilon)$ satisfies the property of P in Lemma 2.3. Indeed, let $f(u, s, \epsilon) = x + iy$. It follows from Lemma 2.2(ii) that there exists a constant $C > 0$ such that

$$|P_s(u, \epsilon)| = \frac{2}{x^2 + y^2} \leq -C\frac{2(x^2 - y^2)}{(x^2+y^2)^2} = -C \operatorname{Re} P_s(u, \epsilon).$$

Applying Lemma 2.3, we obtain

$$|\partial_s f(u, s, \epsilon)| \leq cu$$

completing the base case.

Now suppose the lemma holds for $1 \leq k-1 \leq n-2$ and $\partial_u(\partial_s^{k-1} f) = P_s \partial_s^{k-1} f + Q_{k-1}$. Then

$$\partial_u \partial_s^k f = \partial_s(\partial_u \partial_s^k f) = P_s \partial_s^k f + Q_k,$$

with $Q_k = \partial_s Q_{k-1} - \dfrac{4}{f^3}(\partial_s f)(\partial_s^{k-1} f)$. One can show by induction that

$$Q_k = \lambda^{(k+1)}(s-u) + R_k$$

with $R_k(u, s, \epsilon) = \sum$ terms, where the number of terms is no more than $k-1$ and each term has the form

$$\frac{c}{f^m}\prod_{j=1}^{m-1} \partial_s^{m_j} f,$$

for some $3 \leq m \leq k+1$, and $1 \leq m_j \leq k-1$. This term is by induction no bigger than

$$\frac{c}{u^{m/2}}u^{m-1} = cu^{m/2-1} \leq c(M, k, T)\sqrt{u}.$$

So $|Q_k|$ is bounded by a constant $c = c(M, k, T)$, and hence Lemma 2.3 implies that $|\partial_s^k f| \le cu$. □

Remark. $R_1 = 0$ and R_k satisfies a recursive formula:

$$R_{k+1}(u, s, \epsilon) = \partial_s R_k(u, s, \epsilon) - \frac{4}{f(u, s, \epsilon)^3}(\partial_s f(u, s, \epsilon))(\partial_s^k f(u, s, \epsilon)).$$

We have shown that for $1 \le k \le n-1$,

$$|R_k| \le c(M, k, T)\sqrt{u}.$$

Since R_n is only related to $\partial_s^k f$ for $0 \le k \le n-1$, we have the same inequality:

$$|R_n| \le c(M, n, T)\sqrt{u}.$$

Lemma 3.2. *Suppose $\lambda \in C^n([0, T]; M)$ then $\partial_s^n f(u, s, \epsilon)$ exists and if $\lambda \in C^{n,\alpha}([0, T]; M)$ then*

$$|\partial_s^n f(u, s, \epsilon)| \le cu^\alpha,$$

where $c = c(M, n, T)$.

Remark. If $\lambda \in C^n([0, T]; M)$, then $|\partial_s^n f(u, s, \epsilon)| \le c\,\mathrm{osc}(\lambda^{(n)}, u, [0, s]) \le cM$.

Proof. It follows from the proof of the previous lemma that

$$\begin{aligned} \partial_u(\partial_s^{n-1} f) &= P_s \partial_s^{n-1} f + Q_{n-1} \\ &= P_s \partial_s^{n-1} f + \lambda^{(n)}(s-u) + R_{n-1}. \end{aligned}$$

So

$$\partial_u X = P_s X + Q,\ \ X|_{u=0} = \lambda^{(n-1)}(s),$$

where $X = \partial_s^{n-1} f + \lambda^{(n-1)}(s-u)$ and $Q = -P_s \lambda^{(n-1)}(s-u) + R_{n-1}$. Since Q is C^1 jointly in (u, s), $\partial_s X$ exists and satisfies

$$\partial_u(\partial_s X) = P_s \partial_s X - P_s \lambda^{(n)}(s-u) + R_n.$$

and $\partial_s X|_{u=0} = \lambda^{(n)}(s)$. Hence $\partial_s^n f$ exists and is continuous in (u, s). Since $|R_n| \le c(M, n, T)$, apply Lemma 2.4 (i) with $\omega \equiv Mu^\alpha$, $M_2 = c$, and $\beta = 1$ to obtain

$$|\partial_s^n f| = |\partial_s X - \lambda^{(n)}(s-u)| \le (C+1)Mu^\alpha + cu \le cu^\alpha.$$

□

The next three lemmas concern the oscillation of $\partial_s^k f$ in the variable s. In the proofs, we omit ϵ from the formulas at times (for ease of reading), but we remind the reader that the functions f, P_s, Q_k, R_k do depend on the three variables u, s, ϵ.

Lemma 3.3. *Suppose $\lambda \in C^{1,\alpha}([0, T]; M)$ with $\alpha \in (0, 1]$. Then*

$$|f(u, s+\delta, \epsilon) - f(u, s, \epsilon)| \le c\min(u\delta^\alpha, \delta u^\alpha),$$

$$|\partial_s f(u, s+\delta, \epsilon) - \partial_s f(u, s, \epsilon)| \le c(1 + \frac{\epsilon}{\alpha})\min(u^\alpha, \delta^\alpha)$$

for $0 \le u \le s \le s+\delta \le T$ and $\epsilon > 0$.

Proof. Since $|\partial_s f(u,s,\epsilon)| \le cu^\alpha$ (by Lemma 3.2),

$$|f(u,s+\delta,\epsilon) - f(u,s,\epsilon)| \le c\delta u^\alpha.$$

Omitting the parameter ϵ for convenience, we have

$$\partial_u[f(u,s+\delta) - f(u,s)] = \frac{2}{f(u,s)f(u,s+\delta)}[f(u,s+\delta) - f(u,s)] + \lambda'(s+\delta-u) - \lambda'(s-u),$$

and $f(0,s+\delta) - f(0,s) = 0$. We see that $P := \dfrac{2}{f(u,s)f(u,s+\delta)}$ satisfies

$$|P(u)| \le -C\mathrm{Re}\, P(u)$$

and that $Q = \lambda'(s+\delta-u) - \lambda'(s-u)$ is bounded by $M\delta^\alpha$. Therefore, Lemma 2.3 implies

$$|f(u,s+\delta) - f(u,s)| \le CMu\delta^\alpha.$$

It remains to prove the last inequality. We have

$$\partial_u[\partial_s f(u,s+\delta) + \lambda'(s+\delta-u)] = P_{s+\delta}\partial_s f(u,s+\delta),$$

and

$$\partial_u[\partial_s f(u,s) + \lambda'(s-u)] = P_s \partial_s f(u,s).$$

So

$$\begin{aligned}
&\partial_u[\partial_s f(u,s+\delta) + \lambda'(s+\delta-u) - \partial_s f(u,s) - \lambda'(s-u)] \\
&\qquad = P_{s+\delta}[\partial_s f(u,s+\delta) + \lambda'(s+\delta-u) - \partial_s f(u,s) - \lambda'(s-u)] \\
&\qquad\qquad - P_{s+\delta}\left(\lambda'(s+\delta-u) - \lambda'(s-u)\right) + (P_{s+\delta} - P_s)\partial_s f(u,s).
\end{aligned}$$

We will apply Lemma 2.4 with $Q(u) = \lambda'(s+\delta-u) - \lambda'(s-u)$ and $R(u) = (P_{s+\delta} - P_s)\partial_s f(u,s)$. Note

$$|\lambda'(s+\delta-u) - \lambda'(s-u) - \lambda'(s+\delta) + \lambda'(s)| \le 2M\min(u^\alpha, \delta^\alpha).$$

Further

$$\begin{aligned}
|P_{s+\delta} - P_s| \cdot |\partial_s f(u,s)| &\le \frac{c|f(u,s+\delta) - f(u,s)| \cdot |f(u,s) + f(u,s+\delta)|}{u^2} u^\alpha \\
&\le \frac{cu\delta^\alpha\sqrt{Cu+\epsilon^2}}{u^2} u^\alpha \\
&\le c\delta^\alpha u^{\alpha-1/2} + c\delta^\alpha \epsilon u^{\alpha-1},
\end{aligned}$$

and so

$$\int_0^u |R(v)|\, dv \le \int_0^u \left(c\delta^\alpha v^{\alpha-1/2} + c\delta^\alpha \epsilon v^{\alpha-1}\right)\, dv \le c\delta^\alpha u^{\alpha+1/2} + c\delta^\alpha \frac{\epsilon}{\alpha} u^\alpha.$$

Therefore, by Lemma 2.4 (iii) with $\omega \equiv 2M\min(u^\alpha, \delta^\alpha)$,

$$\begin{aligned}
|\partial_s f(u,s+\delta) - \partial_s f(u,s)| &\le CM\min(u^\alpha,\delta^\alpha) + c\delta^\alpha u^{\alpha+1/2} + c\delta^\alpha\frac{\epsilon}{\alpha}u^\alpha \\
&\le c(1+\frac{\epsilon}{\alpha})\min(u^\alpha, \delta^\alpha).
\end{aligned}$$

□

Lemma 3.4. *Suppose $\lambda \in C^{n,\alpha}([0,T];M)$ with $n \geq 2$ and $\alpha \in (0,1]$. Then*

$$|R_k(u,s+\delta,\epsilon) - R_k(u,s,\epsilon)| \leq c\delta\sqrt{u} \quad \textit{when} \quad 1 \leq k \leq n-1,$$

and

$$|\partial_s^k f(u,s+\delta,\epsilon) - \partial_s^k f(u,s,\epsilon)| \leq cu\delta \quad \textit{when} \quad 1 \leq k \leq n-2,$$

and

$$|\partial_s^{n-1} f(u,s+\delta,\epsilon) - \partial_s^{n-1} f(u,s,\epsilon)| \leq c\min(u^\alpha\delta, u\delta^\alpha).$$

Proof. From the Remark following Lemma 3.1, we know that $R_1 = 0$, R_k satisfies the recursive formula:

$$R_{k+1} = \partial_s R_k - \frac{4}{f^3}(\partial_s f)(\partial_s^k f),$$

and $|R_k| \leq c\sqrt{u}$ for $1 \leq k \leq n$. Therefore, for $k+1 \leq n$, Lemma 3.1 implies that

$$\begin{aligned} |\partial_s R_k| &\leq |R_{k+1}| + \frac{4}{|f|^3}|\partial_s f| \cdot |\partial_s^k f| \\ &\leq c\sqrt{u}. \end{aligned}$$

Thus

$$|R_k(u,s+\delta,\epsilon) - R_k(u,s,\epsilon)| \leq \int_s^{s+\delta} |\partial_s R_k(u,r,\epsilon)| dr \leq c\delta\sqrt{u},$$

proving the first statement.

When $1 \leq k \leq n-2$, Lemma 3.1 implies that

$$|\partial_s^k f(u,s+\delta,\epsilon) - \partial_s^k f(u,s,\epsilon)| \leq \int_s^{s+\delta} |\partial_s^{k+1} f(u,r,\epsilon)| dr \leq cu\delta,$$

proving the second statement. From Lemma 3.2

$$|\partial_s^{n-1} f(u,s+\delta,\epsilon) - \partial_s^{n-1} f(u,s,\epsilon)| \leq \int_s^{s+\delta} |\partial_s^n f(u,r,\epsilon)| dr \leq cu^\alpha\delta.$$

To prove the third statement, it remains to show

$$|\partial_s^{n-1} f(u,s+\delta,\epsilon) - \partial_s^{n-1} f(u,s,\epsilon)| \leq c\delta^\alpha u. \tag{10}$$

Omitting the parameter ϵ, we have

$$\begin{aligned} \partial_u[\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)] = P_{s+\delta}&[\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)] \\ &+ (\lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u)) \\ &+ (P_{s+\delta} - P_s)\partial_s^{n-1} f(u,s) \\ &+ R_{n-1}(u,s+\delta) - R_{n-1}(u,s). \end{aligned}$$

Since

$$|\lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u)| \leq M\delta^\alpha,$$

and

$$|P_{s+\delta} - P_s| \cdot |\partial_s^{n-1} f(u,s)| \leq \frac{c\delta uC}{u^2} u \leq c\delta \leq c\delta^\alpha,$$

and

$$|R_{n-1}(u,s+\delta) - R_{n-1}(u,s)| \leq c\delta\sqrt{u} \leq c\delta^\alpha,$$

we apply Lemma 2.3 with $M_1 = c\delta^\alpha$ to prove (10). $\square$

Lemma 3.5. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$ *with* $n \geq 2$ *and* $\alpha \in (0,1]$. *There exists* $c = c(M,n,T)$ *so that*

$$\begin{aligned} |R_{n+1}(u,s,\epsilon)| &\leq cu^{\alpha-1/2}, \\ |R_n(u,s+\delta,\epsilon) - R_n(u,s,\epsilon)| &\leq cu^{\alpha-1/2}\delta, \\ |\partial_s^n f(u,s+\delta,\epsilon) - \partial_s^n f(u,s,\epsilon)| &\leq c(1+\frac{\epsilon}{\alpha})\min(u^\alpha,\delta^\alpha). \end{aligned}$$

Proof. Let's note that

$$R_n = \sum \frac{c}{f^m} \prod_{j=1}^{m-1} \partial_s^{m_j} f$$

with $3 \leq m \leq n+1$, $1 \leq m_j \leq n-1$, and the number of terms in the sum is no more than $n-1$. Since $\partial_s^n f$ exists, so does R_{n+1}:

$$R_{n+1} = \sum \frac{c}{f^m} \prod_{j=1}^{m-1} \partial_s^{m_j} f,$$

with $3 \leq m \leq n+2$ and $1 \leq m_j \leq n$. We can check that in each product, there is at most one $m_j = n$. Hence

$$|R_{n+1}| \leq cn\frac{u^{m-2}u^\alpha}{u^{m/2}} \leq cu^{\alpha+m/2-2} \leq c(M,n,T)u^{\alpha-1/2},$$

and

$$|\partial_s R_n| \leq |R_{n+1}| + \frac{4}{|f|^3}|\partial_s f| \cdot |\partial_s^n f| \leq cu^{\alpha-1/2}.$$

This implies that

$$|R_n(u,s+\delta) - R_n(u,s)| \leq cu^{\alpha-1/2}\delta.$$

It remains to prove the last statement. Now we have

$$\partial_u(\partial_s^n f(u,s+\delta) + \lambda^{(n)}(s+\delta-u)) = P_{s+\delta}\partial_s^n f(u,s+\delta) + R_n(u,s+\delta),$$

and

$$\partial_u(\partial_s^n f(u,s) + \lambda^{(n)}(s-u)) = P_s\partial_s^n f(u,s) + R_n(u,s).$$

Let

$$\begin{aligned} &Y(u) = \partial_s^n f(u,s+\delta) + \lambda^{(n)}(s+\delta-u) - \partial_s^n f(u,s) - \lambda^{(n)}(s-u) \text{ and} \\ &Q(u) = \lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u). \end{aligned}$$

Then

$$\partial_u Y = P_{s+\delta}Y - P_{s+\delta}Q + (P_{s+\delta} - P_s)\partial_s^n f(u,s) + R_n(u,s+\delta) - R_n(u,s).$$

We see that

$$|Q(u) - Q(0)| \leq c\min(u^\alpha,\delta^\alpha),$$

and

$$|(P_{s+\delta} - P_s)\partial_s^n f(u,s)| \leq \frac{cu\delta\sqrt{Cu+\epsilon^2}}{u^2}u^\alpha \leq c\delta u^{\alpha-1/2} + c\epsilon\delta u^{\alpha-1}.$$

By Lemma 2.4 (iii) with $|R(u)| \leq c\delta u^{\alpha-1/2} + c\epsilon\delta u^{\alpha-1}$,

$$|\partial_s^n f(u,s+\delta,\epsilon) - \partial_s^n f(u,s,\epsilon)| = |Y - Q| \leq c\min(u^\alpha,\delta^\alpha) + c\delta u^{\alpha+1/2} + \frac{c\epsilon\delta}{\alpha}u^\alpha.$$

□

Lemma 3.6. *(Boundedness of mixed u and s derivatives.) Suppose $\lambda \in C^n([0,T];M)$. Let $s_0 \in (0,T)$ and $D_0 = \{(u,s) \in D : s_0 \le u\}$. There exists $L_0 = L_0(M,n,T,s_0)$ such that for all $l + k \le n$,*

$$|\partial_u^l \partial_s^k f(u,s,\epsilon)| \le L_0.$$

In other words, $f \in C^n(D_0; L_0)$ for every $\epsilon \in (0,1]$.

Proof. The case $l = 0$ and $k \le n$ is proven by Lemmas 3.1 and 3.2. Consider $k = 0$ and $1 \le l \le n$. We have

$$\partial_u f = \frac{-2}{f} + \lambda'(s-u).$$

This implies that when $u_0 \le u$,

$$|\partial_u f| \le \frac{2}{C\sqrt{u}} + M \le L_0.$$

We can show by induction in l that

$$\partial_u^l f = \frac{2}{f^2}\partial_u^{l-1} f + (-1)^{l-1}\lambda^{(l)}(s-u) + \hat{R}_l,$$

where $\hat{R}_l$ is the sum of a finite number (depending on l) of terms of the form

$$\frac{c}{f^m}\prod_{j=1}^{m-1}\partial_u^{m_j} f$$

with $3 \le m \le l-1$ and $1 \le m_j \le l-2$. Hence by induction $|\partial_u^l f| \le L_0$ for $s_0 \le u \le T$. The other cases $1 \le k \le n-1$ are proved similarly.

□

In summary, we have proved the following results about $f(u,s,\epsilon)$:

Proposition 3.7. *If λ is in $C^{n,\alpha}[0,T]$, then $f(u,s,\epsilon)$ satisfies the following properties:*

- $C\sqrt{u+\epsilon^2} \le |f(u,s,\epsilon)| \le C'\sqrt{u} + C'\epsilon$.
- $|\partial_s^k f(u,s,\epsilon)| \le cu$ *for* $1 \le k \le n-1$.
- $|\partial_s^n f(u,s,\epsilon)| \le cu^\alpha$.
- $|\partial_s^k f(u,s+\delta,\epsilon) - \partial_s^k f(u,s,\epsilon)| \le cu\delta$ *for* $1 \le k \le n-2$.
- $|\partial_s^{n-1} f(u,s+\delta,\epsilon) - \partial_s^{n-1} f(u,s,\epsilon)| \le c\min(u\delta^\alpha, u^\alpha\delta)$ *if* $0 \le n-1$.
- $|\partial_s^n f(u,s+\delta,\epsilon) - \partial_s^n f(u,s,\epsilon)| \le c(1+\frac{\epsilon}{\alpha})\min(u^\alpha, \delta^\alpha)$ *for* $1 \le n$.
- *For every* $0 < s_0 < T$, *there exists* $L_0 = L_0(M,n,T,s_0)$ *such that for all* $l+k \le n$, $|\partial_u^l \partial_s^k f(u,s,\epsilon)| \le L_0$.

We emphasize that c depends only on M, n, T, not on α and ϵ. We know from Lemma 2.2 that $f(u,s,\epsilon)$ converges uniformly in D to $f(u,s)$ as $\epsilon \to 0^+$. For all $l+k = n$, it follows from the proof of previous lemmas that $\partial_u^l \partial_s^k f(u,s,\epsilon)$ can be expressed in terms of lower derivatives in u and s of $f(u,s,\epsilon)$. Therefore in $D_0 = \{(u,s) \in D : 0 < s_0 \le u \le s \le T\}$, $\partial_u^l \partial_s^k f(u,s,\epsilon)$ converges uniformly. This implies the following:

Corollary 3.8. *If λ is in $C^{n,\alpha}[0,T]$, then $f(u,s)$ is in $C^n(D_0)$ and satisfies*

- $C\sqrt{u} \leq |f(u,s)| \leq C'\sqrt{u}$.
- $|\partial_s^k f(u,s)| \leq cu$ *for* $1 \leq k \leq n-1$.
- $|\partial_s^n f(u,s)| \leq cu^\alpha$.
- $|\partial_s^k f(u,s+\delta) - \partial_s^k f(u,s)| \leq cu\delta$ *for* $1 \leq k \leq n-2$.
- $|\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)| \leq c\min(u\delta^\alpha, u^\alpha\delta)$ *if* $0 \leq n-1$.
- $|\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)| \leq c\min(u^\alpha, \delta^\alpha)$ *for* $1 \leq n$.
- *For every* $0 < s_0 < T$, *there exists* $L_0 = L_0(M,n,T,s_0)$ *such that for all* $l+k \leq n$, $|\partial_u^l \partial_s^k f(u,s)| \leq L_0$.

Corollary 3.9. *If λ is in $C^{n,\alpha}[0,T]$ with $n \geq 2$ and $\alpha \in (0,1]$, then γ is in $C^n(0,T]$.*

Proof. The previous arguments imply that $\gamma(s-u,s) \in C^n(D_0)$ for every $s_0 \in (0,T)$. Hence $s \mapsto \gamma(0,s) \in C^n(0,T]$. Since $\gamma(s) = \gamma(0,s) + \lambda(0)$, the curve γ is in $C^n(0,T]$. □

4. Smoothness of γ

The goal of this section is to prove the following:

Theorem 4.1. *Suppose $\lambda \in C^{n,\alpha}([0,T];M)$ with $n \geq 2$ and $\alpha \in (0,1]$.*

(i) If $\alpha < 1/2$, then $\gamma \in C^{n,\alpha+1/2}(0,T]$. For every $0 < s_0 < T$, there exists $c_0 = c_0(M,n,T,s_0)$ such that $\gamma \in C^n([s_0,T];c_0)$ and

$$|\gamma^{(n)}(s+\delta) - \gamma^{(n)}(s)| \leq \frac{c_0}{1-2\alpha}\delta^{\alpha+1/2},$$

(ii) If $\alpha = 1/2$, then $\gamma \in \Lambda_*^n(0,T]$. For every $0 < s_0 < T$, there exists $c_0 = c_0(M,n,T,s_0)$ such that $\gamma \in C^n([s_0,T];c_0)$ and

$$|\gamma^{(n)}(s+\delta) + \gamma^{(n)}(s-\delta) - 2\gamma^{(n)}(s)| \leq c_0\delta.$$

(iii) If $\alpha \in (\frac{1}{2},1]$, then $\gamma \in C^{n+1,\alpha-1/2}(0,T]$. For every $0 < s_0 < T$, there exists $c_0 = c_0(M,n,T,s_0)$ such that $\gamma \in C^{n+1}([s_0,T];c_0)$ and

$$|\gamma^{(n+1)}(s+\delta) - \gamma^{(n+1)}(s)| \leq \frac{c_0}{2\alpha-1}\delta^{\alpha-1/2}.$$

Proof. Assume that $\lambda \in C^{n,\alpha}([0,T];M)$ with $n \geq 2$ and $\alpha \in (0,1]$. Fix $s_0 \in (0,T)$ and let $D_0 = \{(u,s) \in D : 0 < s_0 \leq u \leq s \leq T\}$. Recall from [Won14] that

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)\int_0^s \frac{\partial_s[f(u,s)]}{f(u,s)^3}\,du.$$

We need to show

$$F(s) := \int_0^s \frac{\partial_s f(u,s)}{f(u,s)^3}\,du \text{ is } \begin{cases} \text{in } C^{n-2} & \text{and} \quad F^{(n-2)} \in C^{\alpha+1/2} \text{ when } \alpha \in (0,1/2) \\ \text{in } C^{n-2} & \text{and} \quad F^{(n-2)} \in \Lambda_* \text{ when } \alpha = 1/2 \\ \text{in } C^{n-1} & \text{and} \quad F^{(n-1)} \in C^{\alpha-1/2} \text{ when } \alpha \in (1/2,1] \end{cases}.$$

Let $F_1(u,s) = \dfrac{\partial_s f(u,s)}{f(u,s)^3}$ and $\hat{R}_1(u,s) = 0$. We define F_k and $\hat{R}_k$ recursively as follows:

$$\begin{aligned} \hat{R}_k &= \partial_s \hat{R}_{k-1} - \frac{3(\partial_s f)(\partial_s^{k-1} f)}{f^4}, \\ F_k &= \partial_s F_{k-1} = \frac{\partial_s^k f}{f^3} + \hat{R}_k. \end{aligned}$$

Let $\hat{F}_k(s) = F_k(s,s)$. Then formally

$$F^{(n-2)}(s) = \hat{F}_1^{(n-3)}(s) + \hat{F}_2^{(n-4)}(s) + \cdots \hat{F}_{n-2}(s) + \int_0^s \left[\frac{\partial_s^{n-1} f(u,s)}{f^3(u,s)} + \hat{R}_{n-1}(u,s) \right] du, \tag{11}$$

and

$$F^{(n-1)}(s) = \hat{F}_1^{(n-2)}(s) + \hat{F}_2^{(n-3)}(s) + \cdots \hat{F}_{n-1}(s) + \int_0^s \left[\frac{\partial_s^{n} f(u,s)}{f^3(u,s)} + \hat{R}_{n}(u,s) \right] du. \tag{12}$$

We notice that

$$\hat{R}_k = \sum \frac{c}{f^m} \prod_{j=1}^{m-2} (\partial_s^{m_j} f), \tag{13}$$

where there are at most $k-1$ terms for the sum, $4 \le m \le k+2$, and $1 \le m_j \le k-1$. Further, when $k \ge 3$ each product contains at most one $m_j = k-1$. Therefore, $\hat{R}_k \in C^{n-(k-1)}(D_0)$, $F_k \in C^{n-k}(D_0)$ and $\hat{F}_k \in C^{n-k}[s_0, T]$. The representation of $\hat{R}_k$ in (13) also implies that

$$|\hat{R}_k(u,s)| \le c \text{ for } 1 \le k \le n, \tag{14}$$

$$\text{and } |\hat{R}_{n+1}(u,s)| \le \frac{c}{u^{1/2}} \text{ if } \alpha \ge \frac{1}{2}. \tag{15}$$

Hence equation (11) holds for all $\alpha \in (0,1]$ and equation (12) holds when $\alpha \in (1/2, 1]$.

Let

$$I_k(s) := \int_0^s \frac{\partial_s^k f(u,s)}{f(u,s)^3}\, du \quad \text{and} \quad IR_k(s) = \int_0^s \hat{R}_k(u,s)\, du.$$

Theorem 4.1 will be proven once we show that

- $I_{n-1} + IR_{n-1} \in C^{\alpha+1/2}[s_0, T]$ for $\alpha \in (0, 1/2)$,
- $I_{n-1} + IR_{n-1} \in \Lambda_*[s_0, T]$ for $\alpha = 1/2$, and
- $I_n + IR_n \in C^{\alpha-1/2}[s_0, T]$ for $\alpha \in (1/2, 1]$,

along with the needed bounds on $|I_k(s+\delta) - I_k(s)|$ and $|IR_k(s+\delta) - IR_k(s)|$ (and the appropriate estimates for the $\alpha = 1/2$ case.) This is the content of the next three lemmas.

□

Lemma 4.2. *Suppose $\lambda \in C^{n,\alpha}([0,T]; M)$, with $n \ge 2$ and $\alpha \in (0,1]$. Then there exists $c = c(M, n, T)$ such that for all $0 < s_0 \le s \le s+\delta \le T$,*

$$\begin{aligned} |IR_k(s+\delta) - IR_k(s)| &\le c\delta \textit{ for all } 1 \le k \le n-1 \textit{ and} \\ |IR_n(s+\delta) - IR_n(s)| &\le c\delta \textit{ if } \alpha \ge \frac{1}{2}. \end{aligned}$$

Proof. It follows from the definition of $\hat{R}_k$ and formula (14) that for $1 \leq k \leq n-1$,

$$|\hat{R}_k(u,s+\delta) - \hat{R}_k(u,s)| \leq \int_s^{s+\delta} |\partial_v \hat{R}_k(u,v)|\, dv \leq c\delta.$$

Similarly if $\alpha \geq \frac{1}{2}$ equation (15) implies

$$|\hat{R}_n(u,s+\delta) - \hat{R}_n(u,s)| \leq \frac{c\delta}{u^{1/2}}.$$

Integrating completes the lemma. □

Lemma 4.3. *Suppose $\lambda \in C^{n,\alpha}([0,T];M)$, with $n \geq 2$ and $\alpha \in (0,\frac{1}{2}]$. Then $I_{n-1} \in C^{\alpha+1/2}[s_0,T]$ when $\alpha \in (0,1/2)$ and $I_{n-1} \in \Lambda_*[s_0,T]$ when $\alpha = 1/2$. In particular, there exists $c = c(M,n,T)$ such that for all $0 < s_0 \leq s \leq s+\delta \leq T$,*

$$|I_{n-1}(s+\delta) - I_{n-1}(s)| \leq \begin{cases} c(\frac{1}{1-2\alpha}+1)\delta^{\alpha+1/2} + c(1+\frac{1}{\sqrt{s_0}})\delta & \text{when } 0<\alpha<\frac{1}{2} \\ c(1+\log^+\frac{s}{\delta}+\frac{1}{\sqrt{s_0}})\delta & \text{when } \alpha=\frac{1}{2} \end{cases}$$

and when $\alpha = 1/2$,

$$|I_{n-1}(s+\delta) + I_{n-1}(s-\delta) - 2I_{n-1}(s)| \leq c\left(1+\frac{1}{\sqrt{s_0}}\right)\delta \tag{16}$$

for all $0 < s_0 \leq s-\delta \leq s+\delta \leq T$.

Proof. We decompose $I_{n-1}(s+\delta) - I_{n-1}(s)$ into the sum of four integrals and bound each integral.

$$\begin{aligned} I_{n-1}(s+\delta) - I_{n-1}(s) &= \int_0^{\delta\wedge s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_{\delta\wedge s}^{s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_0^{s} \frac{\partial_s^{n-1} f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\, du \\ &+ \int_s^{s+\delta} \frac{\partial_s^{n-1} f(u,s+\delta)}{f(u,s+\delta)^3}\, du. \end{aligned}$$

The first integral:

$$\begin{aligned} \left|\int_0^{\delta\wedge s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du\right| &\leq \int_0^{\delta\wedge s} \frac{cu\delta^\alpha}{u^{3/2}}\, du \\ &= c\delta^\alpha\sqrt{\delta\wedge s} \leq c\delta^{\alpha+1/2}. \end{aligned}$$

The second integral, when $0 < \alpha < 1/2$:

$$\begin{aligned} \left|\int_{\delta\wedge s}^{s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du\right| &\leq \int_{\delta\wedge s}^{s} \frac{cu^\alpha\delta}{u^{3/2}}\, du \\ &\leq \frac{c\delta}{1-2\alpha}(\delta^{\alpha-1/2} - s^{\alpha-1/2}) \\ &\leq \frac{c}{1-2\alpha}\delta^{\alpha+1/2}. \end{aligned}$$

In the case $\alpha = 1/2$, the second integral is bounded by

$$\int_{\delta \wedge s}^{s} c\delta u^{-1}\, du = c\delta \log \frac{s}{s \wedge \delta} = c\delta \log^{+} \frac{s}{\delta}.$$

The third integral:

$$\begin{aligned}\left| \int_0^s \frac{\partial_s^{n-1} f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3)}\, du \right| &\leq \int_0^s \frac{cu(u\delta u)}{u^3}\, du \\ &= c\delta s \leq c\delta.\end{aligned}$$

The last integral:

$$\begin{aligned}\left| \int_s^{s+\delta} \frac{\partial_s^{n-1} f(u,s+\delta)}{f(u,s+\delta)^3}\, du \right| \leq \int_s^{s+\delta} \frac{cu}{u^{3/2}}\, du &= c(\sqrt{s+\delta} - \sqrt{s}) \\ &= \frac{c\delta}{\sqrt{s+\delta} + \sqrt{s}} \leq \frac{c}{\sqrt{s_0}}\delta.\end{aligned}$$

To finish the proof, it remains to show (16). Set $\alpha = 1/2$ and write

$$I_{n-1}(s+\delta) + I_{n-1}(s-\delta) - 2I_{n-1}(s) = [I_{n-1}(s+\delta) - I_{n-1}(s)] - [I_{n-1}(s) - I_{n-1}(s-\delta)].$$

As with $I_{n-1}(s+\delta) - I_{n-1}(s)$ above, we can decompose $I_{n-1}(s) - I_{n-1}(s-\delta)$ into the sum of four integrals. In both cases, the first, third and fourth integrals yield adequate bounds. When $\delta \geq s - \delta$, the second integral is also adequately controlled. Thus, we assume $\delta < s - \delta$ and we only need to control the difference of the second integrals:

$$\int_\delta^s \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du - \int_\delta^{s-\delta} \frac{\partial_s^{n-1} f(u,s) - \partial_s^{n-1} f(u,s-\delta)}{f(u,s)^3}\, du.$$

We can decompose this into the sum $J_1 + J_2 + J_3$ where

$$\begin{aligned} J_1 &= \int_\delta^{s-\delta} \frac{(f(u,s)^3 - f(u,s+\delta)^3)\,(\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s))}{f(u,s+\delta)^3 f(u,s)^3}\, du \\ J_2 &= \int_\delta^{s-\delta} \frac{\partial_s^{n-1} f(u,s+\delta) + \partial_s^{n-1} f(u,s-\delta) - 2\partial_s^{n-1} f(u,s)}{f(u,s)^3}\, du \\ J_3 &= \int_{s-\delta}^{s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du. \end{aligned}$$

Then

$$|J_1| \leq \int_\delta^{s-\delta} \frac{c(u\delta u)(u\sqrt{\delta})}{u^3}\, du \leq c\delta^{3/2},$$

and

$$|J_3| \leq \int_{s-\delta}^{s} c\delta u^{-1}\, du = c\delta \log \frac{s}{s-\delta} \leq c\delta \log \frac{T}{s_0}.$$

Since

$$\begin{aligned}&\Big| \left[\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)\right] - \left[\partial_s^{n-1} f(u,s) - \partial_s^{n-1} f(u,s-\delta)\right] \Big| \\ &\quad = \left| \int_s^{s+\delta} \partial_s^n f(u,r) - \partial_s^n f(u,r-\delta)\, dr \right| \\ &\quad \le \int_s^{s+\delta} c\sqrt{\delta}\, dr \le c\delta^{3/2},\end{aligned}$$

then

$$|J_2| \le \int_\delta^{s-\delta} \frac{c\delta^{3/2}}{u^{3/2}}\, du \le c\delta.$$

This establishes (16) and completes the lemma. □

Lemma 4.4. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$ *with* $n \ge 2$ *and* $\alpha \in (\frac{1}{2}, 1]$. *Then* $I_n \in C^{\alpha-1/2}[s_0,T]$, *and there exists* $c = c(M,T,n)$ *such that for all* $0 \le s \le s+\delta \le T$

$$|I_n(s+\delta) - I_n(s)| \le \frac{c}{2\alpha-1}\delta^{\alpha-1/2}.$$

Proof. We proceed in a manner similar to the previous proof.

$$\begin{aligned} I_n(s+\delta) - I_n(s) &= \int_0^{\delta\wedge s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_{\delta\wedge s}^s \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_0^s \frac{\partial_s^n f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\, du \\ &+ \int_s^{s+\delta} \frac{\partial_s^n f(u,s+\delta)}{f(u,s+\delta)^3}\, du. \end{aligned}$$

The first integral:

$$\begin{aligned} \left| \int_0^{\delta\wedge s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du \right| &\le \int_0^{\delta\wedge s} \frac{c\min(u^\alpha,\delta^\alpha)}{u^{3/2}}\, du \\ &\le c\int_0^{\delta\wedge s} u^{\alpha-3/2}\, du \le \frac{c}{2\alpha-1}\delta^{\alpha-1/2}. \end{aligned}$$

The second integral:

$$\begin{aligned} \left| \int_{\delta\wedge s}^s \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du \right| &\le \int_{s\wedge\delta}^s \frac{c\min(u^\alpha,\delta^\alpha)}{u^{3/2}}\, du \\ &\le \int_{s\wedge\delta}^s \frac{c\delta^\alpha}{u^{3/2}}\, du \le c\delta^\alpha(\delta^{-1/2} - s^{-1/2}) \le c\delta^{\alpha-1/2}. \end{aligned}$$

The third integral:

$$\begin{aligned} \left| \int_0^s \frac{\partial_s^n f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\, du \right| &\le \int_0^s cu^\alpha \frac{u^2\delta}{u^3}\, du \\ &= \int_0^s c\delta u^{\alpha-1}\, du = \frac{c\delta}{\alpha}s^\alpha \le c\delta^{\alpha-1/2}. \end{aligned}$$

The last integral:

$$\left|\int_s^{s+\delta} \frac{\partial_s^n f(u,s+\delta)}{f(u,s+\delta)^3}\,du\right| \le \int_s^{s+\delta} \frac{cu^\alpha}{u^{3/2}}\,du = \frac{c}{2\alpha-1}((s+\delta)^{\alpha-1/2} - s^{\alpha-1/2}) \le \frac{c}{2\alpha-1}\delta^{\alpha-1/2}.$$

□

5. Real analyticity of γ

In this section we prove Theorem 1.2. There exists $\delta > 0$ such that λ can be extended (complex) analytically to $E = \{z \in \mathbb{C} : d(z,[0,T]) \le \delta\}$. Notice that $f(s,s) = \gamma(0,s) = \gamma(s) - \lambda(0)$ and $f(u,s,\epsilon)$ converges uniformly to $f(u,s)$ on $D = \{(u,s) : 0 < u \le s, 0 < s \le T\}$. So it suffices to show that $f(u,s,\epsilon)$ can be extended analytically in the same neighborhood of D (in $\mathbb{C}\times\mathbb{C}$) for all ϵ. Recall that $G(z,u,s) = \frac{-2}{z} + \lambda'(s-u)$ is analytic in (z,u,s), hence by the dependence of solutions of ODE on parameters (see [CL55, Theorem 8.1]) the function $f(\cdot,s,\epsilon)$ in (4) exists and is analytic in a neighborhood of $u = 0$ for each $\epsilon \in (0,1]$ and $s \in E$. The main difficulty is to show this neighborhood is the same for all ϵ and s.

The outline of this section is as follows: First we show in Lemma 5.1 that the equation (4) still has solution when s is in the domain

$$E_1 = \{t : 0 < \operatorname{Re} t < T + \delta_1, |\operatorname{Im} t| < \delta_1\}$$

with δ_1 small enough and not depending on ϵ. Then in Lemma 5.2 we show that one can take complex u-derivatives in (4), which means the solutions are extended analytically. Finally by [CL55, Theorem 8.3] the solutions are analytic in (u,s) on the same domain for all ϵ.

Let M be an upper bound for the sup-norms of λ' and λ'' on E. As a first step, we will show the following:

Lemma 5.1. *There exists $\delta_1 \in (0,\delta)$ depending on δ, M and T such that for every $s \in E_1$ and $\epsilon \in (0,1]$, the solution to the equation*

$$\begin{aligned} \partial_u f(u,s,\epsilon) &= \frac{-2}{f(u,s,\epsilon)} + \lambda'(s-u),\ u \ge 0, \\ f(0,s,\epsilon) &= i\epsilon, \end{aligned}$$

exists uniquely for $u \in [0, Re\, s + \delta_1]$. Moreover,

$$\max(\sqrt{2u}, \frac{\epsilon}{2}) \le Im\, f(u,s,\epsilon) \text{ for } 0 \le u \le Re\, s + \delta_1.$$

Proof. The solution $f(u,s,\epsilon)$ exists on a neighborhood of $u = 0$, and it continues to exists as long as it stays above the real line. The uniqueness of this solution comes from standard ODE techniques. To establish the results of the lemma, we will compare

$f(u,s,\epsilon)$ to $f(u,s_0,\epsilon)$ where $s_0 = \operatorname{Re} s$ and

$$\begin{aligned} \partial_u f(u,s_0,\epsilon) &= \frac{-2}{f(u,s_0,\epsilon)} + \lambda'(s_0 - u), \ u \geq 0, \\ f(0,s_0,\epsilon) &= i\epsilon. \end{aligned}$$

It follows from Lemma 2.2 (i, ii) that

$$\begin{aligned} \sqrt{3u+\epsilon^2} &\leq \operatorname{Im} f(u,s_0,\epsilon) \\ \text{and} \quad |\operatorname{Re} f(u,s_0,\epsilon)| &\leq \sqrt{u} \ \text{ for } 0 \leq u \leq s_0 + \delta_1, \end{aligned}$$

where $\delta_1 < \delta$ will be specified momentarily. By following the same argument in Lemma 3.3, we get a bound for the difference of $f(u,s,\epsilon)$ and $f(u,s_0,\epsilon)$:

$$|f(u,s,\epsilon) - f(u,s_0,\epsilon)| \leq CMu|s - s_0| \leq CMu\delta_1$$

whenever $0 \leq u \leq S$ with

$$S = \inf\{0 \leq v \leq u_0 + \delta_1 : \operatorname{Im} f(v,s,\epsilon) < \frac{\epsilon}{3} \text{ or } \frac{|\operatorname{Re} f(v,s,\epsilon)|}{\operatorname{Im} f(v,s,\epsilon)} > C_1\},$$

where C_1 is a constant in $(0,1)$ and close to 1. It follows that

$$\operatorname{Im} f(u,s,\epsilon) \geq \operatorname{Im} f(u,s_0,\epsilon) - CMu\delta_1 \geq \sqrt{3u+\epsilon^2} - CMu\delta_1,$$

and

$$|\operatorname{Re} f(u,s,\epsilon)| \leq |\operatorname{Re} f(u,s_0,\epsilon)| + CMu\delta_1 \leq \sqrt{u} + CMu\delta_1.$$

By choosing δ_1 small enough, $\operatorname{Im} f(u,s,\epsilon) \geq \max(\sqrt{2u}, \epsilon/2)$ and

$$\frac{|\operatorname{Re} f(u,s,\epsilon)|}{\operatorname{Im} f(u,s,\epsilon)} < C_1$$

for all $0 \leq u \leq S$. It follows that $S = u_0 + \delta_1$ and the lemma follows. □

Now we will show that

Lemma 5.2. *For every $\epsilon \in (0,1]$, $s \in E_1$ and $0 < \tilde{u} < \operatorname{Re} s + \delta_1$, there exist $r = r(\tilde{u}, M, \delta, T) \in (0, \delta - \delta_1)$ and an analytic extension of $f(\cdot, s, \epsilon)$ on $B_{\tilde{u}} = \{z \in \mathbb{C} : |z - \tilde{u}| < r\}$ such that*

$$\partial_u f(u,s,\epsilon) = \frac{-2}{f(u,s,\epsilon)} + \lambda'(s-u).$$

Proof. We will use the Picard iteration to show that the equation

$$\begin{aligned} g'(u) &= -\frac{2}{g(u)} + \lambda'(s-u), \\ g(\tilde{u}) &= f(\tilde{u},s,\epsilon) \end{aligned} \tag{17}$$

has a solution on $B_{\tilde{u}} = \{z \in \mathbb{C} : |z - \tilde{u}| < r\}$, where r will be specified later. Indeed for $|u - \tilde{u}| < r$ define $g_0(u) = f(\tilde{u},s,\epsilon)$ and

$$g_{n+1}(u) = f(\tilde{u},s,\epsilon) + \int_{\tilde{u}}^{u} \frac{-2}{g_n(v)} + \lambda'(s-v)\, dv.$$

We will show by induction on n that g_n is well-defined and analytic in $B_{\tilde{u}}$ and

$$\operatorname{Im} g_n(u) \geq \sqrt{\tilde{u}}.$$

The base case $n = 0$ is clear because of Lemma 5.1. Suppose the claim holds for n. The function g_{n+1} is well-defined and analytic in $B_{\tilde{u}}$ since $\frac{1}{g_n}$ is analytic in a simply connected domain. Now

$$\begin{aligned}\operatorname{Im} g_{n+1}(u) &\geq \operatorname{Im} f(\tilde{u}, s, \epsilon) - |u - \tilde{u}| \max_{v \in B_{\tilde{u}}} \left(\frac{2}{|g_n(v)|} + |\lambda'(s - v)| \right) \\ &\geq \sqrt{2\tilde{u}} - r(\frac{2}{\sqrt{\tilde{u}}} + M).\end{aligned}$$

The claim holds for $n + 1$ by choosing r small enough depending on $\tilde{u}, M$ and T. We also require that r is small enough so that $2r/\tilde{u} < 1$. Then the sequence g_n converges uniformly in $B_{\tilde{u}}$ since

$$\begin{aligned}|g_{n+1}(u) - g_n(u)| &\leq |u - \tilde{u}| \max_{v \in B_{\tilde{u}}} \frac{2|g_n(v) - g_{n-1}(v)|}{|g_n(v) g_{n-1}(v)|} \\ &\leq \frac{2r}{\tilde{u}} ||g_n - g_{n-1}||_{B_{\tilde{u}}, \infty}.\end{aligned}$$

Let g be the limit. Then this function is analytic and satisfies the differential equation (17). In particular $g(u)$ and $f(u, \tilde{u}, \epsilon)$ solve same initial value problem. Hence they are equal when u is real. In order words, $f(\cdot, s, \epsilon)$ is extended analytically on $B_{\tilde{u}}$. □

Proof of Theorem 1.2. By [CL55, Theorem 8.3], for every $\epsilon \in (0, 1]$ the function $f(u, s, \epsilon)$ is analytic in the domain $\{(u, s) : s \in E_1, u \in B_{\tilde{u}}$ for some $\tilde{u} \in (0, \operatorname{Re} s + \delta_1)\}$. It follows that $f(u, s)$ is also analytic in the same domain which contains $\{(s, s) : 0 < s \leq T\}$. Hence $f(s, s)$ and $\gamma(s)$ is real analytic on $(0, T]$. □

6. Behavior of γ at $s = 0$

In this section we analyze the behavior of γ at its base, proving Theorem 1.4 and Theorem 1.3.

6.1. **Smoothness of $\gamma(s^2)$ at $s = 0$.** We may extend λ smoothly on $(-\delta, T)$ by the concatenation property of the Loewner equation. Thus, it suffices to show that for fixed $t_0 \in (0, T)$, the curve $\gamma_0(s^2) = g_{t_0}(\gamma(s^2 + t_0))$ is smooth at $s = 0$ provided γ is smooth on $(0, T)$. The idea, illustrated in Figure 2, is as follows. Let U be the intersection of $\mathbb{H}$ and a small disk centered at $\lambda(0)$ and let $V = g_{t_0}^{-1}(U)$. Define an analytic branch ϕ of $\sqrt{z - \gamma(t_0)}$ in a neighborhood of $\gamma(t_0)$ such that the branch cut is $\gamma(0, t_0]$. Let $W = \phi(V)$. All we need to check is that for small $\epsilon > 0$ the images under ϕ of $\gamma(t_0 - \epsilon, t_0]$ and $\gamma(t_0 + s^2), 0 \leq s^2 \leq \epsilon$, are smooth. Finally the smoothness of $\gamma_0(s^2)$ follows immediately from the Schwarz reflection principle through $E = \phi(\gamma(t_0 - \epsilon, t])$ (in the case γ is analytic) or Kellogg-Warschawski theorem (in the case γ is $C^{n,\alpha}$) for the map $\phi \circ g_{t_0}^{-1}$ from U to W.

Proof of Theorem 1.4 when λ is analytic. It follows from (2) that $\gamma'(t) \neq 0$ for all t. Thus, there exists an (real) analytic function h on $(-\sqrt{\epsilon}, \sqrt{\epsilon})$ such that

$$\frac{\gamma(t_0 + s) - \gamma(t_0)}{s} = h(s)^2 \text{ for all } s \in (-\sqrt{\epsilon}, \sqrt{\epsilon}) \backslash \{0\}.$$

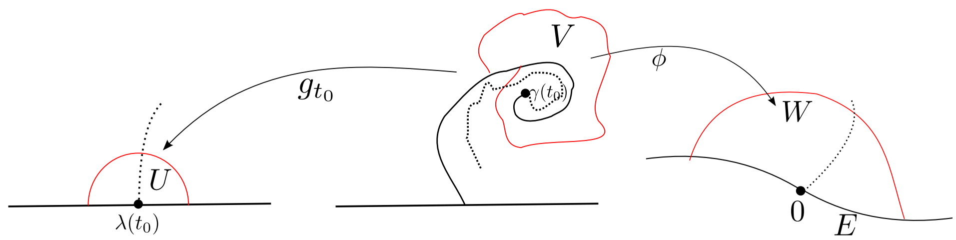

FIGURE 2. Illustration for the proof of Theorem 1.4

Let $\phi_1(s) = ish(-s^2)$ and $\phi_2(s) = sh(s^2)$. We see that these two functions are analytic and one-to-one. Moreover,

$$\phi_1(s)^2 = \gamma(t_0 - s^2) - \gamma(t_0) \text{ and}$$
$$\phi_2(s)^2 = \gamma(t_0 + s^2) - \gamma(t_0).$$

Therefore the boundary E of W, which is parametrized by $\phi_1(s)$ near 0, and $\phi(\gamma(t_0+s^2))$ are analytic. Since the latter map is the image of $\gamma_0(s^2)$ under $\phi\circ g_{t_0}^{-1}$, it follows from the Schwarz reflection principle that $\gamma_0(s^2)$ is analytic at 0. □

Proof of Theorem 1.4 when λ is C^β. By Theorem 1.4, $\gamma \in C^{n,\alpha}(0,T]$ for appropriate $\alpha \in (0,1)$. It is not obvious that the function h in the previous case is $C^{n,\alpha}$. Indeed one can find an example of function $\gamma \in C^{n,\alpha}$ but h is not. Now let

$$H(s) = \frac{\gamma(t_0+s)-\gamma(t_0)}{s} \text{ for } s \in (-\sqrt{\epsilon},\sqrt{\epsilon})\backslash\{0\}, \text{ and } H(0) = \gamma'(t_0).$$

We claim that $H \in C^{n-1,\alpha}(-\sqrt{\epsilon},\sqrt{\epsilon})$. Indeed

$$H^{(n)}(s) = \frac{n!}{s^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k}{k!}s^k\gamma^{(k)}(t_0+s) - \frac{(-1)^n n!}{s^{n+1}}\gamma(t_0) \text{ for } s \neq 0.$$

Apply Proposition 2.1 for functions $\gamma, \gamma', \cdots, \gamma^{(n)}$ to get $|H^{(n)}(s)| \le cs^{\alpha-1}$ which implies the claim.

Since $\inf_{s\in(-\sqrt{\epsilon},\sqrt{\epsilon})}|H(s)| > 0$, it follows from the claim that the function $s \mapsto \sqrt{H(-s^2)}$ is $C^{n-1,\alpha}(-\sqrt{\epsilon},\sqrt{\epsilon})$ for any well-defined square-root function. Let $\phi_1(s)$ be a parametrization near 0 of E such that $\phi_1(s)^2 = \gamma(t_0 - s^2) - \gamma(t_0)$ and $\phi_1(s) = s\sqrt{H(-s^2)}$ for $s \in (-\sqrt{\epsilon},\sqrt{\epsilon})$. Since $\phi_1'(s) = \dfrac{\gamma'(t_0-s^2)}{\sqrt{H(-s^2)}}$, the function ϕ_1 is $C^{n,\alpha}(-\sqrt{\epsilon},\sqrt{\epsilon})$. The same argument shows that the function $\phi(\gamma(t_0+s^2))$ is $C^{n,\alpha}[0,\sqrt{\epsilon})$. Combined with the last two statements, the Kellogg-Warschawski theorem [Pom92, Theorem 3.6] implies that the function $\gamma_0(s^2)$ is $C^{n,\alpha}[0,\sqrt{\epsilon})$. □

Remark. The proof also shows that if $\lambda \in C^{n,\alpha}([0,T];M)$ then $\Gamma \in C^{n,\alpha+1/2}([0,T];c)$ with $c = c(T,M,n,\alpha)$.

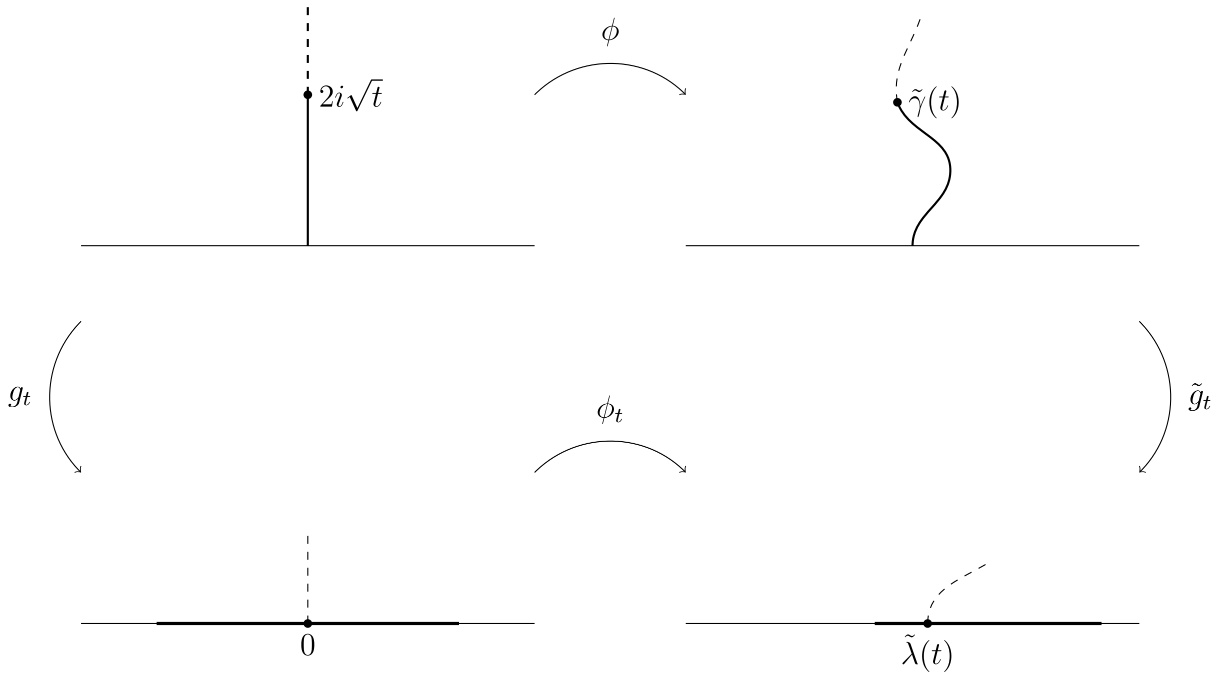

FIGURE 3. The conformal maps $\phi, g_t, \tilde{g}_t, \phi_t$, the comparison curve $\tilde{\gamma}$, and $\tilde{\lambda}$.

6.2. **Expansion of γ at $s = 0$.** The goal of this section is to prove Theorem 1.3, which illuminates why the s^2 parametrization is a natural parametrization at the base of a Loewner curve γ. To accomplish this, we create a comparison curve $\tilde{\gamma}$ that closely approximates γ near its base and is "nice" at $s = 0$ (that is, $\tilde{\Gamma}(s) = \tilde{\gamma}(s^2)$ is smooth at $t = 0$.) The properties of the comparison curve are summarized in Proposition 6.2 below.

Assume γ is generated by $\lambda \in C^{n,\alpha}[0,T]$. We define $\tilde{\gamma}$ as a perturbation of a vertical slit, as done in Section 4.6 of [Law05]. Set

$$\phi(z) = z + \sum_{m=2}^{4n+1} \frac{b_m}{2^m} z^m,$$

which is conformal on a neighborhood of the origin. The real-valued coefficients b_m will depend on $\lambda^{(k)}(0)$ as we will describe later. Then define

$$\begin{aligned}\tilde{\gamma}(t) = \phi(2i\sqrt{t}) &= 2i\sqrt{t} + \sum_{m=2}^{4n+1} i^m b_m t^{m/2} \\ &= 2i\sqrt{t} - b_2 t - i\, b_3 t^{3/2} + b_4 t^2 + \cdots + i\, b_{4n+1} t^{2n+1/2}.\end{aligned}$$

Let $g_t : \mathbb{H} \setminus [0, 2i\sqrt{t}] \to \mathbb{H}$ and $\tilde{g}_t : \mathbb{H} \setminus \tilde{\gamma}[0,t] \to \mathbb{H}$ be conformal maps with the hydrodynamic normalization at infinity. Then we set $\phi_t = \tilde{g}_t \circ \phi \circ g_t^{-1}$ and $\tilde{\lambda}(t) = \phi_t(0)$, as illustrated in Figure 3. In this form, $\tilde{\gamma}$ and $\tilde{\lambda}$ are not parametrized by halfplane capacity. We will need to reparametrize by $t = t(s)$, which satisfies $t(0) = 0$ and $\frac{dt}{ds} = \phi_t'(0)^{-2}$. Note in particular that $\frac{dt}{ds}\Big|_{s=0} = 1$.

Lemma 6.1. *Assume ϕ_t, $\tilde{\lambda}$ and $t = t(s)$ are defined as above, and let $k \in \mathbb{N}$. Then there exists $\tilde{T} > 0$, there exist polynomials $p_k(x_1, x_2, \cdots, x_{k+2})$, $q_k(x_1, x_2, \cdots, x_{2k})$ and*

$r_k(x_1, x_2, \cdots, x_{2k-1})$, *and there exist nonzero constants* c_k, d_k, e_k *so that for* $t \in [0, \tilde{T}]$,

$$\partial_t \phi_t^{(k)}(0) = c_k\, \phi_t^{(k+2)}(0) + p_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(k+1)}(0), \phi_t'(0)^{-1}\right), \tag{18}$$

$$\partial_s^k \tilde{\lambda}(t) = d_k\, \phi_t^{(2k)}(0) \cdot \phi_t'(0)^{-2k} + q_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-1)}(0), \phi_t'(0)^{-1}\right), \text{ and} \tag{19}$$

$$\partial_s^k t = e_k\, \phi_t^{(2k-1)}(0) \cdot \phi_t'(0)^{-(2k+1)} + r_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-2)}(0), \phi_t'(0)^{-1}\right). \tag{20}$$

Further $\tilde{\lambda} \in C^\infty[0, s(\tilde{T})]$ *under the halfplane-capacity parametrization.*

Proof. Write $\phi_t(z) = \sum_{k=0}^\infty a_k z^k$, keeping in mind that a_k depends on t. Then from Proposition 4.40 in [Law05],

$$\begin{aligned}\partial_t \phi_t(z) &= 2\left(\frac{\phi_t'(0)^2}{\phi_t(z) - \phi_t(0)} - \frac{\phi_t'(z)}{z}\right)\\ &= -2\,\frac{\sum_{k=0}^\infty (a_1 a_{k+2} + 2a_2 a_{k+1} + \cdots + (k+2)a_{k+2}a_1) z^k}{\sum_{k=0}^\infty a_{k+1} z^k}.\end{aligned} \tag{21}$$

Since $a_1 = 1$ when $t = 0$, there exists a neighborhood U of 0 and $\tilde{T} > 0$ so that the denominator is nonzero for $z \in U$ and $t \le \tilde{T}$. Therefore $\partial_t \phi_t^{(k)}(z)$ is defined for $(z, t) \in U \times [0, \tilde{T}]$. Equation (18) follows from (21) (with $c_k = -\frac{2(k+3)}{(k+2)(k+1)}$.)

We verify (19) inductively. For the base case,

$$\partial_s \tilde{\lambda}(t) = \partial_t \phi_t(0) \cdot \frac{dt}{ds} = -3\, \phi_t''(0) \cdot \phi_t'(0)^{-2}.$$

Assume (19) holds for a fixed k. Then

$$\begin{aligned}\partial_s^{k+1}\tilde{\lambda}(t) &= \partial_t\left(d_k\, \phi_t^{(2k)}(0) \cdot \phi_t'(0)^{-2k} + q_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-1)}(0), \phi_t'(0)^{-1}\right)\right) \cdot \phi_t'(0)^{-2}\\ &= d_k\, c_{2k}\, \phi_t^{(2k+2)}(0) \cdot \phi_t'(0)^{-2k-2} + q_{k+1}\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k+1)}(0), \phi_t'(0)^{-1}\right).\end{aligned}$$

We also prove (20) inductively. When $k = 1$,

$$\frac{dt}{ds} = \phi_t'(0) \cdot \phi_t'(0)^{-3}.$$

If (20) holds for fixed k, then

$$\begin{aligned}\partial_s^{k+1} t &= \frac{d}{dt}\left(e_k\, \phi_t^{(2k-1)}(0) \cdot \phi_t'(0)^{-(2k+1)} + r_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-2)}(0), \phi_t'(0)^{-1}\right)\right) \cdot \phi_t'(0)^{-2}\\ &= e_k\, c_{2k-1}\, \phi_t^{(2k+1)}(0) \cdot \phi_t'(0)^{-(2k+3)} + r_{k+1}\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k)}(0), \phi_t'(0)^{-1}\right).\end{aligned}$$

The last assertion follows from (19). □

We are now ready to recursively define the coefficients of ϕ. The coefficient b_m will depend on $\lambda^{(k)}(0)$ for $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor \wedge n$. For even values of m, our choice of b_m will ensure that $\partial_s^k \tilde{\lambda}(0) = \lambda^{(k)}(0)$ for $k \le n$. For odd values of m, we choose b_m so that the t-parametrization of $\tilde{\gamma}$ is close to the halfplane-capacity parametrization.

- Set $b_2 = -\frac{2}{3}\lambda'(0)$. Since $\partial_s\tilde{\lambda}(0) = -\frac{3}{2}b_2$, this implies that $\partial_s\tilde{\lambda}(0) = \lambda'(0)$.
- Set $b_3 = \dfrac{b_2^2}{8}$. This implies that $\frac{d^2t}{ds^2}\Big|_{s=0} = 2b_3 - b_2^2/4 = 0$.
- Assume that $b_2, b_3, \cdots, b_{2k-1}$ have been defined. Then by Lemma 6.1,
$$\partial_s^k\tilde{\lambda}(0) = d_k\,\frac{(2k)!}{2^{2k}}\,b_{2k} + q_k\left(1, \frac{1}{2}b_2, \cdots, \frac{(2k-1)!}{2^{2k-1}}b_{2k-1}, 1\right).$$
If $k \le n$, define b_{2k} so that $\partial_s^k\tilde{\lambda}(0) = \lambda^{(k)}(0)$. If $k > n$, we may define b_{2k} however we like; for instance, we choose b_{2k} so that $\partial_s^k\tilde{\lambda}(0) = 0$.
- Assume that $b_2, b_3, \cdots, b_{2k}$ have been defined. Then by Lemma 6.1,
$$\frac{d^{k+1}t}{ds^{k+1}}\bigg|_{s=0} = e_{k+1}\,\frac{(2k+1)!}{2^{2k+1}}\,b_{2k+1} + r_{k+1}\left(1, \frac{1}{2}b_2, \cdots, \frac{(2k)!}{2^{2k}}b_{2k}, 1\right).$$
Define b_{2k+1} so that this quantity is zero.

This construction ensures that $\partial_s^k\tilde{\lambda}(0) = \lambda^{(k)}(0)$ for $k \le n$ and that $t = s + O(s^{2n+2})$. The first fact, together with by Theorem 3.3 in [Won14], implies that $|\gamma(s) - \tilde{\gamma}(t(s))| = O(s^{n+\alpha})$ for s near 0. The second fact implies that under the halfplane-capacity parametrization $\tilde{\gamma}(t(s))$ will have the same coefficients as $\tilde{\gamma}(t)$ for the terms with exponents at most $n + 1/2$. Together, this provides precise information about the expansion of $\gamma(s)$ near $s = 0$. In summary, we have proved the following, which establishes Theorem 1.3.

Proposition 6.2. *Assume that $\lambda \in C^{n,\alpha}[0,T]$ generates the curve γ. Then there exists $\tilde{\lambda} \in C^\infty[0,S]$ that generates a (halfplane-capacity-parametrized) curve $\tilde{\gamma} \in C^\infty(0,S]$ with the following properties:*

- $\lambda^{(k)}(0) = \tilde{\lambda}^{(k)}(0)$ *for* $1 \le k \le n$.
- $\tilde{\Gamma}(s) = \tilde{\gamma}(s^2)$ *is in* $C^\infty[0, \sqrt{S}]$.
- $\tilde{\Gamma}^{(m)}(0)$ *depends on* $\lambda^{(k)}(0)$ *for* $m \le 2n+1$ *and* $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.
- $|\gamma(s) - \tilde{\gamma}(s)| = O(s^{n+\alpha})$.

In particular near $s = 0$, the curve γ has the form
$$\gamma(s) = \begin{cases} 2i\sqrt{s} + a_2 s + i\,a_3 s^{3/2} + a_4 s^2 + \cdots + a_{2n}s^n + O(s^{n+\alpha}) & \text{if } \alpha \le 1/2 \\ 2i\sqrt{s} + a_2 s + i\,a_3 s^{3/2} + a_4 s^2 + \cdots + a_{2n}s^n + i\,a_{2n+1}s^{n+1/2} + O(s^{n+\alpha}) & \text{if } \alpha > 1/2 \end{cases}$$
where the real-valued coefficients a_m depend on $\lambda^{(k)}(0)$ for $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.

We note the equations for the first few coefficients:
$$a_2 = \frac{2}{3}\lambda'(0)$$
$$a_3 = -\frac{1}{18}\lambda'(0)^2$$
$$a_4 = \frac{4}{15}\lambda''(0) + \frac{1}{135}\lambda'(0)^3$$
$$a_5 = -\frac{1}{15}\lambda''(0)\lambda'(0) + \frac{1}{2160}\lambda'(0)^4$$

Coefficients a_2, a_3, a_4 were discovered in [LR13] by comparison with specific example curves (such as those generated by $c\sqrt{\tau - t}$.)

Along with the tools developed in Sections 3 and 4, Proposition 6.2 could be used to show that if $\Gamma(s) = \gamma(s^2)$, then $\Gamma^{(k)}(0)$ exists and equals $\tilde{\Gamma}^{(k)}(0)$ for $k = 1, \cdots, n+1$.

7. Examples

In this section we discuss two examples that illustrate the two special cases of Theorem 4.1. The first special case is when the driving function is $C^{n+1/2}$. Here the conclusion is weaker than we might initially expect: it is not necessarily true that $\gamma \in C^{n+1}$, but rather γ is in the larger space Λ_*^n (which contains both C^{n+1} and $C^{n,1}$.) This case is illustrated in the first example where the driving function is $C^{3/2}$ and the associated curve is $C^{1,1}$ but not C^2. The second special case of Theorem 4.1 is when the driving function is $C^{n,1}$. Here the conclusion is slightly stronger than might be initially expected: $\gamma \in C^{n+1,1/2}$. This is illustrated in the second example, where the driving function is $C^{0,1}$ but not C^1 and the associated curve is $C^{3/2}$. We describe the needed computational steps to verify these examples, but leave details for the reader.

7.1. Example 1: $\lambda \in C^{3/2}$ and $\gamma \in C^{1,1} \backslash C^2$.

This example was communicated to us from Don Marshall.

We will create γ via a sequence of conformal maps, as pictured in Figure 4. Let $f_1(z) = z + \frac{1}{z} + c \ln z$, and let $r_{1,2} = \frac{-c \pm \sqrt{c^2+4}}{2}$ be the finite critical points of f_1. Define

$$g(z) = \frac{c\pi}{f_1(z) - f_1(r_1)},$$

which is a conformal map from $\mathbb{H}$ onto the $C^{1,1}$ domain $\mathbb{C} \backslash ((-\infty, 0] \cup$ a circle arc). Finally, set

$$F(z) = i\sqrt{g(z)+1}.$$

The image of $\mathbb{H}$ under F is a slit half-plane, and we let γ be the resulting slit.

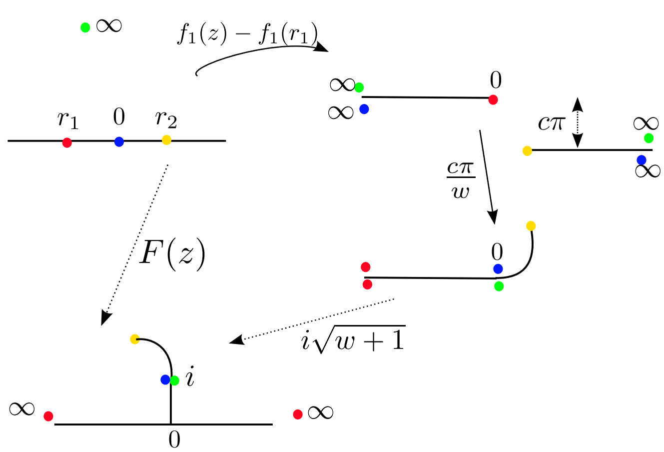

Figure 4. Conformal maps used in the construction of γ for Example 1.

For $t \in [0,1/4]$, $\gamma(t) = 2i\sqrt{t}$ and $\lambda(t) \equiv 0$. To compute λ and γ for $t > 1/4$, we will need to use the conformal maps, since $\gamma(t) = F(r_2)$ and $\lambda(t) = L^{-1}(r_2)$ for the automorphism L of $\mathbb{H}$ with

$$F(L(z)) = z + 0 + \frac{-2t}{z} + \cdots \quad \text{near infinity.} \tag{22}$$

Since L must send ∞ to r_1,

$$L(z) = r_1 + \frac{a}{z-b} = r_1 + \frac{a}{z} + \frac{ab}{z^2} + \frac{ab^2}{z^3} + \frac{ab^3}{z^4} + O(|z|^{-5}) \text{ near infinity,}$$

where $a < 0$ and $b \in \mathbb{R}$. Using this and the Taylor series expansion of $f_1 - f_1(r_1)$ at $z = r_1$, one can compute that

$$f_1(L(z)) - f_1(r_1) = \frac{A}{z^2} + \frac{B}{z^3} + \frac{D}{z^4} + O(1/|z|^5) \text{ near infinity,}$$

with

$$A = \frac{a^2 f_1^{(2)}(r_1)}{2}, \quad B = a^2 b f^{(2)}(r_1) + \frac{a^3 f^{(3)}(r_1)}{6},$$
$$\text{and } D = \frac{3a^2b^2 f^{(2)}(r_1)}{2} + \frac{a^3 b f^{(3)}(r_1)}{2} + \frac{a^4 f^{(4)}(r_1)}{24}.$$

Thus near infinity,

$$\begin{aligned} F(L(z)) &= i\sqrt{\frac{c\pi}{A}z^2 - \frac{c\pi B}{A^2}z - \frac{c\pi D}{A^2} + \frac{c\pi B^2}{A^3} + 1 + O(1/|z|)} \\ &= i\left(-i\sqrt{\frac{c\pi}{|A|}}z - iB\frac{\sqrt{c\pi}}{2|A|^{3/2}} + O(1/|z|)\right) \end{aligned}$$

Note that in choosing the appropriate branch for the square root, we used the fact that $A < 0$. In order to satisfy (22), we must have

- $A = -c\pi$, or equivalently, $a = \dfrac{r_1\sqrt{-2\pi c r_1}}{\sqrt{2 - cr_1}}$, and
- $B = 0$, or equivalently, $b = \dfrac{(cr_1 - 3)\sqrt{-2\pi c r_1}}{3(2 - cr_1)^{3/2}}$.

Using these two facts, we expand further and find that at infinity,

$$F(L(z)) = z + 0 - \frac{1}{2}\left(\frac{D}{A} + 1\right)\frac{1}{z} + O(1/|z|^2),$$

which implies that

$$4t = \frac{D}{A} + 1 = \frac{-\pi c r_1 (c^2 r_1^2 - 6cr_1 + 6)}{3(2 - cr_1)^3} + 1.$$

Next we compute $\lambda(t)$ for $t > 1/4$:

$$\lambda(t) = L^{-1}(r_2) = b + \frac{a}{r_2 - r_1} = \frac{-2\sqrt{2\pi}(-cr_1)^{3/2}}{3(2 - cr_1)^{3/2}}.$$

Thus with $y = -cr_1$, we have

$$t = \frac{1}{4} + \frac{\pi y(y^2+6y+6)}{12(2+y)^3} \quad \text{and} \quad \lambda(t) = \frac{-2\sqrt{2\pi}y^{3/2}}{3(2+y)^{3/2}}.$$

So for $t > 1/4$,

$$\lambda'(t) = \frac{\frac{d\lambda}{dy}}{\frac{dt}{dy}} = \frac{-2\sqrt{2}\sqrt{y}(2+y)^{3/2}}{\sqrt{\pi}(y+1)}.$$

Using this, one can show that for $s > t \geq 1/4$,

$$|\lambda'(s) - \lambda'(t)| \leq c\sqrt{y_s - y_t} \leq c'\sqrt{s-t},$$

proving that $\lambda \in C^{3/2}[0,T]$. We also note that away from $t = 1/4$, one can check that $\lambda(t)$ is C^2.

Lastly, for $t \geq 1/4$, $\gamma(t) = F(r_2)$. Using this, one can determine computationally that with the halfplane-capacity parametrization, γ' and γ'' exist on $[1/4, T]$ (by computing, for instance, $\gamma'(t) = \frac{dF(r_2)}{dc}/\frac{dt}{dc}$ and $\gamma'' = \frac{d\gamma'(t)}{dc}/\frac{dt}{dc}$). Further,

$$\lim_{t \searrow 1/4} \gamma'(t) = 2i = \lim_{t \nearrow 1/4} \gamma'(t),$$

but

$$\lim_{t \searrow 1/4} \gamma''(t) = -4i - 16 \neq \lim_{t \nearrow 1/4} \gamma''(t) = -4i.$$

Therefore on the full interval $(0,T]$, γ is $C^{1,1}$ but not C^2.

7.2. **Example 2: $\lambda \in C^{0,1}$ and $\gamma \in C^{3/2}$.** Consider the driving function

$$\lambda(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq \frac{1}{4} \\ \frac{3}{2} - \frac{3}{2}\sqrt{1 - 8(t-1/4)} & \text{for} \quad \frac{1}{4} \leq t < \frac{1}{4} + \frac{1}{10} \end{cases}.$$

There exists $c > 0$ so that

$$|\lambda(t) - \lambda(s)| \leq c|t-s|$$

for all $s, t \in [0, 0.35]$, implying that $\lambda \in C^{0,1}$. However, λ is not in C^1 since λ' is not continuous.

The driving function $\frac{3}{2} - \frac{3}{2}\sqrt{1-8s}$, defined on $[0, \frac{1}{8}]$, generates the upper half-circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$. Let $\hat{\gamma}$ be the portion of this circle generated on the time interval $[0, \frac{1}{10}]$. Then the curve γ generated by λ is the image of $[-1,1] \cup \hat{\gamma}$ by the map $S(z) = \sqrt{z^2-1}$. See Figure 5. By Proposition 3.12 in [MR07], $\gamma \in C^{3/2}$ (and no better) under the arclength parametrization. This is also true under the halfplane-capacity parametrization. Note that $\hat{\gamma}$ is smooth on $(0, \frac{1}{10}]$ (because its driving function is smooth), and near $s = 0$

$$\hat{\gamma}(s) = 2i\sqrt{s} + 4s - 2is^{3/2} + O(s^2)$$

by Theorem 1.3. Thus γ is piecewise smooth, and for $t \geq 1/4$

$$\gamma(t) = S(\hat{\gamma}(t-1/4)) = i + 2i(t-1/4) + 8(t-1/4)^{3/2} + O((t-1/4)^2).$$

From this we can determine that $\gamma \in C^{3/2}(0, 0.35]$ (and no better) under the halfplane-capacity parametrization.

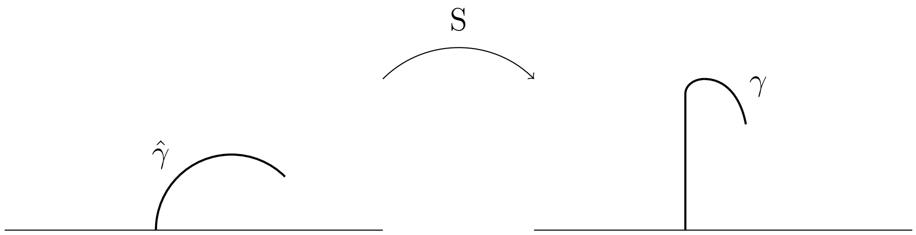

FIGURE 5. The curve γ for Example 2.

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