Lecture 1

Vector Spaces over $\mathbb{R}$

1.1 Definition

Definition 1. A **vector space** over $\mathbb{R}$ is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* $+$ and *multiplication by scalars* $\cdot$, satisfying the following properties:

A1 (Closure of addition)
For all $u, v \in V, u + v$ is defined and $u + v \in V$.

A2 (Commutativity for addition)
$u + v = v + u$ for all $u, v \in V$.

A3 (Associativity for addition)
$u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

A4 (Existence of additive identity)
There exists an element $\vec{0}$ such that $u + \vec{0} = u$ for all $u \in V$.

A5 (Existence of additive inverse)
For each $u \in V$, there exists an element -denoted by $-u$- such that $u + (-u) = \vec{0}$.

M1 (Closure for scalar multiplication)
For each number $r \in \mathbb{R}$ and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

M2 (Multiplication by 1)
$1 \cdot u = u$ for all $u \in V$.

M3 (Associativity for multiplication)
$r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

D1 (First distributive property)
$r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.

D2 (Second distributive property)
$(r + s) \cdot u = r \cdot u + s \cdot u$ for all $r, s \in \mathbb{R}$ and all $u \in V$.

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$.

Proof. We have $\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$ $\square$

Lemma. Let $u \in V$, then $0 \cdot u = \vec{0}$.

Proof.

$$\begin{aligned} u + 0 \cdot u &= 1 \cdot u + 0 \cdot u \\ &= (1 + 0) \cdot u \\ &= 1 \cdot u \\ &= u \end{aligned}$$

Thus
$$\begin{aligned} \vec{0} = u + (-u) &= (0 \cdot u + u) + (-u) \\ &= 0 \cdot u + (u + (-u)) \\ &= 0 \cdot u + \vec{0} \\ &= 0 \cdot u \end{aligned}$$
$\square$

Lemma. a) The element $-u$ is unique.

b) $-u = (-1) \cdot u$.

Proof of part (b).

$$\begin{aligned} u + (-1) \cdot u &= 1 \cdot u + (-1) \cdot u \\ &= (1 + (-1)) \cdot u \\ &= 0 \cdot u \\ &= \vec{0} \end{aligned}$$

$\square$

1.2 Examples

Before examining the axioms in more detail, let us discuss two examples.

Example. Let $V = \mathbb{R}^n$,considered as column vectors
$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ Then for

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad r \in \mathbb{R} :$$

Define

$$u + v = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad r \cdot u = \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix}$$

Note that the zero vector and the additive inverse of u are given by:
$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad -u = \begin{pmatrix} -x_1 \\ \vdots \\ -x_2 \end{pmatrix}$

Remark. $\mathbb{R}^n$ can also be considered as the space of all row vectors.

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

The addition and scalar multiplication is again given coordinate wise

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$$

Example. If $\vec{x} = (2, 1, 3), \vec{y} = (-1, 2, -2)$ and $r = -4$ find $\vec{x} + \vec{y}$ and $r \cdot \vec{x}$.

Solution.

$$\begin{aligned} \vec{x}+\vec{y} &= (2,1,3)+(-1,2,-2) \\ &= (2-1,1+2,3-2) \\ &= (1,3,1) \end{aligned}$$

$$r\cdot\vec{x} = -4\cdot(2,1,3) = (-8,-4,-12).$$

Remark.

$$\begin{aligned} (x_1,\dots,x_n)+(0,\dots,0) &= (x_1+0,\dots,x_n+0) \\ &= (x_1,\dots,x_n) \end{aligned}$$

So the additive identity is $\vec{0}=(0,\dots,0)$.

Note also that

$$\begin{aligned} 0\cdot(x_1,\dots,x_n) &= (0x_1,\dots,0x_n) \\ &= (0,\dots,0) \end{aligned}$$

for all $(x_1,\dots,x_n)\in\mathbb{R}^n$.

Example. Let A be the interval $[0,1)$ and V be the space of functions $f : A \longrightarrow \mathbb{R}$, i.e.,

$$V = \{f : [0,1) \longrightarrow \mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (r\cdot f)(x) &= rf(x) \end{aligned}$$

For instance, the function $f(x)=x^4$ is an element of V and so are

$$g(x)=x+2x^2, \qquad h(x)=\cos x, \qquad k(x)=e^x$$

We have $(f+g)(x)=x+2x^2+x^4$.

Remark. (a) The zero element is the function $\vec{0}$ which associates to each x the number 0:

$$\vec{0}(x) = 0 \text{ for all } x \in [0, 1)$$

Proof. $(f + \vec{0})(x) = f(x) + \vec{0}(x) = f(x) + 0 = f(x)$. $\square$

(b) The additive inverse is the function $-f : x \mapsto -f(x)$.

Proof. $(f + (-f))(x) = f(x) - f(x) = 0$ for all x. $\square$

Example. Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space V. We set

$$V^A = \{f : A \longrightarrow V\}$$

and set addition and scalar multiplication by

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (r \cdot f)(x) &= r \cdot f(x) \end{aligned}$$

Remark. (a) The zero element is the function which associates to each x the vector $\vec{0}$:

$0 : x \mapsto \vec{0}$

Proof

$$\begin{aligned} (f + 0)(x) &= f(x) + 0(x) \\ &= f(x) + \vec{0} = f(x) \quad \square \end{aligned}$$

Remark.

(b) Here we prove that $+$ is associative:

Proof. Let $f, g, h \in V^A$. Then

$$\begin{aligned} [(f + g) + h](x) &= (f + g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) \quad \text{associativity in } V \\ &= f(x) + (g + h)(x) \\ &= [f + (g + h)](x) \end{aligned}$$

$\square$

1.3 Exercises

Let $V = \mathbb{R}^4$. Evaluate the following:

a) $(2, -1, 3, 1) + (3, -1, 1, -1)$.

b) $(2, 1, 5, -1) - (3, 1, 2, -2)$.

c) $10 \cdot (2, 0, -1, 1)$.

d) $(1, -2, 3, 1) + 10 \cdot (1, -1, 0, 1) - 3 \cdot (0, 2, 1, -2)$.

e) $x_1 \cdot (1, 0, 0, 0) + x_2 \cdot (0, 1, 0, 0) + x_3 \cdot (0, 0, 1, 0) + x_4 \cdot (0, 0, 0, 1)$.