Lecture 2

Subspaces

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

The answer is NO.

Theorem. Let $W \neq \emptyset$ be a subset of a vector space V. Then W, with the same addition and scalar multiplication as V, is a vector space if and only if the following two conditions hold:

1. $u + v \in W$ for all $u, v \in W$ (or $W + W \subseteq W$)
2. $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ (or $\mathbb{R}W \subseteq W$).

In this case we say that W is a *subspace* of V.

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.
To show that W is a vector space we have to show that all the 10 axioms of Definition 1.1 hold for W. But that follows because the axioms hold for V and W is a subset of V:

A1 (Commutativity of addition)
For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.

A4 (Existence of additive identity)
Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

A5 (Existence of additive inverse)
If $u \in W$ then $-u = (-1) \cdot u \in W$.

One can check that the other axioms follow in the same way.

□

2.1 Examples

Usually the situation is that we are given a vector space V and a subset of vectors W satisfying some conditions and we need to see if W is a subspace of V.

$$W = \{v \in V : \underline{\text{some conditions}} \text{ on } v\}$$

We will then have to show that

$$\begin{array}{cc} u, v \in W & u + v \\ r \in \mathbb{R} & r \cdot u \end{array} \Bigg\} \text{Satisfy the } \underline{\text{same conditions}}.$$

2.2 Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x, y) | y = kx\} \qquad \text{for a given } k \\ &= \text{line through } (0,0) \text{ with slope } k. \end{aligned}$$

To see that W is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$

and

$$\begin{aligned} u + v &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, kx_1 + kx_2) \\ &= (x_1 + x_2, k(x_1 + x_2)) \in W \end{aligned}$$

Similarly, $r \cdot u = (rx_1, ry_1) = (rx_1, krx_1) \in W$

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$

2. Lines. But only those that contain $(0, 0)$. Why?

3. $\mathbb{R}^2$

Remark (First test). If W is a subspace, then $\vec{0} \in W$.
Thus: If $\vec{0} \notin W$, then W is not a subspace.

This is why a line not passing through $(0, 0)$ can not be a subspace of $\mathbb{R}^2$.

2.3 A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

Example. Lets consider the following subset of $\mathbb{R}^2$:

$$W = \{(x, y) | x^2 - y^2 = 0\}$$

Is W a subspace of $\mathbb{R}^2$? Why?

The answer is NO.

We have $(1, 1)$ and $(1, -1) \in W$ but $(1, 1) + (1, -1) = (2, 0) \notin W$. i.e., W is not closed under addition.

Notice that $(0, 0) \in W$ and W is closed under multiplication by scalars.

2.4 Subspaces of $\mathbb{R}^3$

What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector (a, b, c) and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z) | \underbrace{ax + by + cz = p}_{\text{condition on } (x,y,z)} \}$$

For W to be a subspace, $(0, 0, 0)$ must be in W by the *first test*. Thus

$$p = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$

or

$$p = 0$$

2.4.1 Planes containing the origin

A plane containing $(0, 0, 0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W$. Then

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}$$

Then we have

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) &= \underbrace{(ax_1 + by_1 + cz_1)}_{0} + \underbrace{(ax_2 + by_2 + cz_2)}_{0} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a(rx_1) + b(ry_1) + c(rz_1) &= r(ax_1 + by_1 + cz_1) \\ &= 0 \quad \square \end{aligned}$$

2.5 Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes containing $(0,0,0)$.
3. Lines containing $(0,0,0)$.
(Intersection of two planes containing $(0,0,0)$)

2.6 Exercises

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

a) $W = \{(x,y) \in \mathbb{R}^2 |\ xy = 0\}$.

b) $W = \{(x,y,z) \in \mathbb{R}^3 |\ 3x + 2y^2 + z = 0\}$.

c) $W = \{(x,y,z) \in \mathbb{R}^3 |\ 2x + 3y - z = 0\}$.

d) The set of all vectors (x_1, x_2, x_3) satisfying

$$2x_3 = x_1 - 10x_2.$$

e) The set of all vectors in $\mathbb{R}^4$ satisfying the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 5x_4 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

f) The set of all points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying

$$x_1 + 2x_2 + 3x_3 + x_4 = -1.$$