# Lecture 4

# Linear Maps

We have all seen linear maps before. In fact, most of the maps we have been using in Calculus are linear.

## 4.1 Two Important Examples

### 4.1.1 The Integral

To integrate the function $f(x) = x^2 + 3x - \cos x$ over the interval $[a, b]$, we first find the antiderivative of $x^2$, that is $\frac{1}{3}x^3$, then the antiderivative of $x$, which is $\frac{1}{2}x^2$, and then multiply that by 3 to get $\frac{3}{2}x^2$. Finally, we find the antiderivative of $\cos x$, which is $\sin x$, and then multiply that by $-1$ to get $-\sin x$. To finish the problem we insert the endpoints. Thus,

$$
\begin{aligned}
\int_{-1}^{1} x^2 + 3x - \cos x \, dx &= \int_{-1}^{1} x^2 \, dx + 3 \int_{-1}^{1} x \, dx \\
&\quad - \int_{-1}^{1} \cos x \, dx \\
&= \left[\frac{1}{3}x^3\right]_{-1}^{1} + \left[\frac{3}{2}x^2\right]_{-1}^{1} - [\sin x]_{-1}^{1} \\
&= \frac{2}{3} - \sin 1 + \sin(-1).
\end{aligned}
$$

What we have used is the fact that the integral is a linear map $\mathcal{C}([a, b]) \longrightarrow \mathbb{R}$ and that

$$\int_a^b rf(x) + sg(x)\,dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx.$$

### 4.1.2 The Derivative

Another example is differentiation $Df = f'$. To differentiate the function $f(x) = x^4 - 3x + e^x - \cos x$, we first differentiate each term of the function and then add:

$$\begin{aligned} D(x^4 - 3x + e^x - \cos x) &= Dx^4 - 3Dx + De^x \\ &\quad -D\cos x \\ &= 4x^3 - 3 + e^x + \sin x. \end{aligned}$$

**Definition.** Let $V$ and $W$ be two vector spaces. A map $T : V \longrightarrow W$ is said to be linear if for all $v, u \in V$ and all $r, s \in \mathbb{R}$ we have:

$$T(rv + su) = rT(v) + sT(u).$$

Remark: This can also be written by using two equations:

$$T(v + u) = T(v) + T(u)$$

$$T(rv) = rT(v).$$

*Lemma.* Suppose that $T : V \longrightarrow W$ is linear. Then $T(\vec{0}) = \vec{0}$.

*Proof.* We can write $\vec{0} = 0v$, where $v$ is any vector in $V$. But then $T(\vec{0}) = T(0v) = 0T(v) = 0$ □

## 4.2 Linear Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

*Example.* Let us find all the linear maps from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Any arbitrary vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as:

$$(x_1, x_2) = x_1(1, 0) + x_2(0, 1).$$

Hence,

$$T(x_1, x_2) = x_1T(1, 0) + x_2T(0, 1).$$

Write $T(1, 0)$ and $T(0, 1)$ as:

$$T(1, 0) = (a_{11}, a_{12}), \quad T(0, 1) = (a_{21}, a_{22}), \quad \text{where} \quad a_{ij} \in \mathbb{R}.$$

$$\begin{aligned} T(x_1, x_2) &= x_1(a_{11}, a_{12}) + x_2(a_{21}, a_{22}) \\ &= (x_1a_{11} + x_2a_{21}, x_1a_{12} + x_2a_{22}) \\ &= (x_1, x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \end{aligned}$$

Thus, all the information about T is given by the matrix:
$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

*Example.* Next, let us find all the linear maps $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. As before we write $(x_1, x_2, x_3) \in \mathbb{R}^3$ as:

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$

where,

$$\begin{aligned} T(1, 0, 0) &= (a_{11}, a_{12}, a_{13}) \\ T(0, 1, 0) &= (a_{21}, a_{22}, a_{23}) \\ T(0, 0, 1) &= (a_{31}, a_{32}, a_{33}). \end{aligned}$$

Then,

$$\begin{aligned} T(x_1, x_2, x_3) &= x_1(a_{11}, a_{12}, a_{13}) + x_2(a_{21}, a_{22}, a_{23}) \\ &\quad + x_3(a_{31}, a_{32}, a_{33}) \\ &= (x_1a_{11} + x_2a_{21} + x_3a_{31}, x_1a_{12} + x_2a_{22} \\ &\quad + x_3a_{32}, x_1a_{13} + x_2a_{23} + x_3a_{33}) \\ &= (x_1, x_2, x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \end{aligned}$$

*Example.* All the linear maps from $\mathbb{R}^3 \longrightarrow \mathbb{R}$. Notice that $\mathbb{R}$ is also a vector space, so we can consider all the linear maps $\mathbb{R}^n$ to $\mathbb{R}$. We have :

$$\begin{aligned} T(x_1, x_2, ..., x_n) &= x_1T(1, 0, ..., 0) + x_2T(0, 1, ..., 0) \\ &\quad + ... + x_nT(0, 0, ..., 1) \\ &= x_1a_1 + x_2a_2 + ... + x_na_n \end{aligned}$$

where,

$$T(1, 0, ..., 0) = a_1,\ T(0, 1, ..., 0) = a_2, \ldots\ T(0, 0, ..., 1) = a_n.$$

### 4.2.1 Lemma

*Lemma.* A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear if and only if there exists numbers $a_{ij}, i = 1, ..., n,\ j = 1, ..., m$, such that:

$$\begin{aligned} T(x_1, x_2, ..., x_n) &= (x_1 a_{11} + x_2 a_{21} + ... + x_n a_{n1}, ..., \\ &\quad x_1 a_{1m} + x_2 a_{2m} + ... + x_n a_{nm}) \end{aligned}$$

This can also be written as:

$$T(x_1, x_2, ..., x_n) = (\sum_{i=1}^{n} x_i a_{i1}, \sum_{i=1}^{n} x_i a_{i2}, \sum_{i=1}^{n} x_i a_{im})$$

or by using matrix multiplication:

$$T(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n) \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{pmatrix}.$$

### 4.2.2 A Counterexample

*Example.* The map $T(x, y, z) = (2x + 3xy, z + y)$ is <u>not</u> linear because of the factor $xy$. Notice that:

$$T(1, 1, 0) = (5, 0)$$

but

$$T(2(1, 1, 0)) = T(2, 2, 0) = (16, 0)$$

and

$$2T(1, 1, 0) = (10, 0) \neq (16, 0)$$

### 4.2.3 Examples

*Example.* Evaluate the given following maps at a given point:

$$\begin{aligned} T(x, y) &= (3x + y, 3y), \quad (x, y) = (1, -1) \\ T(1, -1) &= (3 \cdot 1 - 1, 3(-1)) = (2, -3) \end{aligned}$$

$$\begin{aligned} T(x, y, z) &= (2x - y + 3z, 2x + z), \quad (x, z, y) = (2, -1, 1) \\ T(2, -1, 1) &= (4 + 1 + 3, 4 + 1) = (8, 5) \end{aligned}$$

*Example.* Some examples involving differentiation and integration:

$$D(3x^2 + 4x - 1) = 6x + 4$$

$$\begin{aligned}\int_1^2 x^2 - e^x \, dx &= \left[\frac{1}{3}x^3 - e^x\right]_1^2 \\ &= \frac{8}{3} - e^2 - \frac{1}{3} + e \\ &= \frac{7}{3} - e^2 + e\end{aligned}$$

## 4.3 Kernel

**Definition.** Let $V$ and $W$ be two vector spaces and $T : V \longrightarrow W$ a linear map.

- A1 The set $Ker(T) = \{v \in V : T(v) = 0\}$ is called the **kernel** of $T$.
- A2 The set $Im(T) = \{w \in W : \text{there exists a} v \in V : T(v) = w\}$ is called the **image** of $T$.

Remark: Notice that $Ker(T) \subseteq V$ and $Im(T) \subseteq W$.

*Theorem.* The kernel of T is a vector space.

*Proof.* Let $u, v \in Ker(T)$ and $r, s \in \mathbb{R}$ We have to show that $ru + sv \in Ker(T)$. Now,$u, v \in Ker(T)$ if and only if $T(u) = T(v) = 0$. Hence,

$$\begin{aligned}T(ru + sv) &= rT(u) + sT(v) \quad (T\text{is linear}) \\ &= r \cdot 0 + s \cdot 0 \quad (u, v \in Ker(T)) \\ &= 0\end{aligned}$$

This shows that $ru + sv \in Ker(T)$. □

Remark: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map:

$$T(x, y) = (x^2 + y, x + y).$$

Then,

$$T(1, -1) = (1 - 1, 1 - 1) = (0, 0).$$

But $T(2(1, -1)) = T(2, -2) = (4 - 2, 2 - 2) = (2, 0) \neq (0, 0)$.

So if $T$ is <u>not</u> linear, then the set $v \in V : T(v) = 0$ is in general <u>not</u> a vector space.

*Example.* Let $\mathbb{R}^2 \longrightarrow \mathbb{R}$ be the map: $T(x, y) = 2x - y$. Describe the kernel of $T$. We know that $(x, y)$ is in the kernel of $T$ if and only if $T(x, y) = 2x - y = 0$. Hence, $y = 2x$. Thus, the kernel of $T$ is a line through $(0, 0)$ with slope 2.

*Example.* Let $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the map: $T(x, y, z) = (2x-3y+z, x+2y-z)$. Describe the kernel of $T$.

We have that $(x, y, z) \in Ker(T)$ if and only if

$$2x - 3y + z = 0 \quad \text{and} \quad x + 2y - z = 0.$$

The equations describe planes through $(0, 0, 0)$ with normal vectors $(2, -3, 1)$ and $(1, 2, -1)$ respectively. The normal vectors are not parallel and therefore the planes are different. It follows that the intersection is a line.

Let us describe this line. Adding the equations we get:

$$3x - y = 0 \quad \text{or} \quad y = 3x.$$

Plugging this into the second equation we get:

$$0 = x + 2(3x) - z = 7x - z \quad \text{or} \quad z = 7x.$$

Hence, the line is given by: $x \cdot (1, 3, 7)$.

*Theorem.* Let $V$ and $W$ be vector spaces, and $T : V \longrightarrow W$ linear. Then, $Im(T) \subseteqq W$ is a vector space.

*Proof.* Let $w_1, w_2 \in Im(T)$. Then we can find $u_1, u_2 \in V$ such that $T(u_1) = w_1, T(u_2) = w_2$. Let $r, s \in \mathbb{R}$. Then,

$$\begin{aligned} rw_1 + sw_2 &= rT(u_1) + sT(u_2) \\ &= T(ru_1 + su_2) \in Im(T)). \end{aligned}$$

$\square$