Lecture 5

Inner Product

Let us start with the following problem. Given a point $P \in \mathbb{R}^2$ and a line $L \subsetneqq \mathbb{R}^2$, how can we find the point on the line closest to P?

Answer: Draw a line segment from P meeting the line in a right angle. Then, the point of intersection is the point on the line closest to P.

Let us now take a plane $L \subsetneqq \mathbb{R}^3$ and a point outside the plane. How can we find the point $u \in L$ closest to P?

The answer is the same as before, go from P so that you meet the plane in a right angle.

Observation

In each of the above examples we needed two things:

A1 We have to be able to say what the length of a vector is.

B1 Say what a right angle is.

Both of these things can be done by using the dot-product (or inner product) in $\mathbb{R}^n$.

Definition. Let $(x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n) \in \mathbb{R}^n$. Then, the **dot-product** of these vectors is given by the number:

$$((x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n)) = x_1 y_1 + x_2 y_2 + ... + x_n y_n.$$

The **norm** (or length) of the vector $\vec{u} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the non-negative number:

$$\|u\| = \sqrt{(u,u)} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

Examples

Example. (a) $((1,2,-3),(1,1,1)) = 1 + 2 - 3 = 0$

(b) $((1,-2,1),(2,-1,3)) = 2 + 2 + 3 = 7$

Perpendicular

Because,

$$|x_1y_1 + x_2y_2 + ... + x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + ... + x_n^2}\sqrt{y_1^2 + y_2^2 + ... + y_n^2}$$

or

$$|(u,v)| \leq \|u\| \cdot \|v\|$$

we have that (for $u, v \neq 0$)

$$-1 \leq \frac{(u,v)}{\|u\| \cdot \|v\|} \leq 1.$$

Hence we can define:

$$\cos(\angle(u,v)) = \frac{(u,v)}{\|u\| \cdot \|v\|}.$$

In particular, $u \perp v$ (u is **perpendicular** to v) if and only if $(u,v) = 0$.

Questions

Example. Let L be the line in $\mathbb{R}^2$ given by $y = 2x$. Thus,

$$L = \{r(1,2) : r \in \mathbb{R}\}.$$

Let $P = (2,1)$. Consider the following questions.

Question 1: What is the point on L closest to P?

Answer: Because $u \in L$, we can write $\vec{u} = (r, 2r)$. Furthermore, $v - u = (2 - r, 1 - 2r)$ is perpendicular to L. Hence,

$$0 = ((1,2), (2 - r, 1 - 2r)) = 2 - r + 2 - 4r = 4 - 5r.$$

Hence, $r = \frac{4}{5}$ and $\vec{v} = (\frac{4}{5}, \frac{8}{5})$. Question 2: What is the distance of P from the line?

Answer: The length of the vector $v - u$, i.e. $\|v - u\|$. First we have to find out what $v - u$ is. We have done almost all the work:

$$v - u = (2, 1) - (\frac{4}{5}, \frac{8}{5}) = (\frac{6}{5}, \frac{-3}{5}).$$

The distance therefore is:

$$\sqrt{\frac{36}{25} + \frac{9}{25}} = \frac{3\sqrt{5}}{5}.$$

Properties of the Inner Product

1. (**positivity**)To be able to define the norm, we used that $(u, u) \geq 0$.
2. (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u, u) = 0$ only if $u = 0$.
3. (**linearity**)If the vector $v \in \mathbb{R}^n$ is fixed, then a map $u \mapsto (u, v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is,
$$(ru + sw, v) = r(u, v) + s(w, v).$$
4. (**symmetry**) For all $u, v \in \mathbb{R}^n$ we have: $(u, v) = (v, u)$.

We will use the properties above to define an inner product on arbitrary vector spaces.

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

1. (**positivity**) $(u, u) \geq 0$, for all $v \in V$.
2. (**zero length**) $(u, u) = 0$ only if $u = 0$.

3. (**linearity**)If $v \in V$ is fixed, then a map $u \mapsto (u, v)$ from V to $\mathbb{R}$ is linear.
4. (**symmetry**) $(u, v) = (v, u)$, for all $u, v \in V$.

Definition. We say that u and v are perpendicular if $(u, v) = 0$.

Definition. If $(.,.)$ is an inner product on the vector space V, then the norm of a vector $v \in V$ is given by:

$$\|u\| = \sqrt{(u, u)}.$$

Properties of the Norm

Lemma. The norm satisfies the following properties:

1. $\|u\| \geq 0$, and $\|u\| = 0$ only if $u = 0$.
2. $\|ru\| = |r| \cdot \|u\|$.

Proof. We have that

$$\begin{aligned} \|ru\| &= \sqrt{(ru, rv)} \\ &= \sqrt{(r^2(u, v)} \\ &= |r|\sqrt{(u, v)} = |r| \cdot \|u\| . \end{aligned}$$

□

Examples

Example. Let $a < b$, $I = [a, b]$, and $V = PC([a, b])$. Define:

$$(f, g) = \int_a^b f(t)g(t)\, dt$$

Then, $(.,.)$ is an inner product on V.

Proof. Let $r, s \in \mathbb{R}$, $f, g, h \in V$. Then:

1. $(f, f) = \int_a^b f(t)^2\, dt$. As $f(t)^2 \geq 0$, it follows that $\int_a^b f(t)^2\, dt \geq 0$.
2. If $(f, f) = 0$, then $f(t)^2 = 0$ for all t, i.e $f = 0$.

3. $\int_a^b (rfsg)(t)h(t)\,dt = \int_a^b rf(t)h(t) + sg(t)h(t)\,dt$

$$\begin{aligned} &= r\int_a^b f(t)h(t)\,dt + s\int_a^b g(t)h(t)\,dt \\ &= r(f,h) + s(g,h). \end{aligned}$$

   Hence, linear in the first factor.

4. As $f(t)g(t) = g(t)f(t)$, it follows that $(f,g) = (g,f)$.

Notice that the norm is:

$$\|f\| = \sqrt{\int_a^b f(t)^2\,dt}.$$

□

Example. Let $a = 0$, $b = 1$ in the previous example. That is, $f(t) = t^2$ and $g(t) = t - 3t^2$. Then:

$$\begin{aligned} (f,g) &= \int_0^1 t^2(t-3t^2)\,dt \\ &= \int_0^1 t^3 - 3t^4)\,dt \\ &= \frac{1}{4} - \frac{3}{5} \\ &= -\frac{7}{20}. \end{aligned}$$

Also, the norms are:

$$\|f\| = \sqrt{\int_0^1 t^4\,dt} = \frac{1}{\sqrt{5}}.$$

$$\begin{aligned} \|g\| &= \sqrt{\int_0^1 (t-3t^2)^2\,dt} \\ &= \sqrt{\int_0^1 t^2 - 6t^3 + 9t^4\,dt} \\ &= \sqrt{\frac{1}{3} - \frac{3}{2} + \frac{9}{5}} \\ &= \sqrt{\frac{19}{30}}. \end{aligned}$$

Example. Let $f(t) = \cos 2\pi t$ and $g(t) = \sin 2\pi t$. Then:

$$\begin{aligned} (f,g) &= \int_0^1 \cos 2\pi t \sin 2\pi t \, dt \\ &= \frac{1}{4\pi} \left[(\sin 2\pi t)^2 \right]_0^1 = 0. \end{aligned}$$

So, $\cos 2\pi t$ is perpendicular to $\sin 2\pi t$ on the interval $[0,1]$.

Example. Let $f(t) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ and $g(t) = \chi_{[0,1)}$. Then:

$$\begin{aligned} (f,g) &= \int_0^1 (\chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t))(\chi_{[0,1)}) \, dt \\ &= \int_0^1 \chi_{[0,1/2)}(t) \, dt - \int_0^1 \chi_{[1/2,1)}(t) \, dt \\ &= \int_0^{1/2} dt - \int_{1/2}^1 dt = \frac{1}{2} = \frac{1}{2} = 0. \end{aligned}$$

One can also easily show that $\|f\| = \|g\| = 1$.

Problem

Problem: Find a polynomial $f(t) = a + bt$ that is perpendicular to the polynomial $g(t) = 1 - t$.

<u>Answer</u>: We are looking for numbers a and b such that:

$$\begin{aligned} 0 = (f,g) &= \int_0^1 (a+bt)(1-t) \, dt \\ &= \int_0^1 a + bt - at - bt^2 \, dt \\ &= a + \frac{b}{2} - \frac{a}{2} - \frac{b}{3} \\ &= \frac{a}{2} + \frac{b}{6}. \end{aligned}$$

Thus, $3a + b = 0$. So, we can take $f(t) = 1 - 3t$.

Important Facts

We state now two important facts about the inner product on a vector space V. Recall that in $\mathbb{R}^2$ we have:

$$\cos(\theta) = \frac{(u,v)}{\|u\| \cdot \|v\|}.$$

where u, v are two non-zero vectors in $\mathbb{R}^2$ and θ is the angle between u and v. In particular, because $-1 \leq \cos\theta \leq 1$, we must have:

$$\|(u,v)\| \leq \|u\| \cdot \|v\|.$$

We will show now that this comes from the positivity and linearity of the inner product.

Theorem. Let V be a vector space with inner product $(.,.)$. Then:

$$|(u,v)| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$.

Proof. We can assume that $u, v \neq 0$ because otherwise both the LHS and the RHS will be zero. By the positivity of the inner product we get:

$$\begin{aligned}
0 &\leq (v - \frac{(v,u)}{\|u\|^2}u, v - \frac{(v,u)}{\|u\|^2}u) \quad (\textit{positivity}) \\
&= (v,v) - \frac{(v,u)}{\|u\|^2}(u,v) - \frac{(v,u)}{\|u\|^2}(v,u) + \frac{(v,u)^2}{\|u\|^4}(u,u) \quad (\textit{linearity}) \\
&= \|v\|^2 - 2\frac{(u,v)^2}{\|u\|^2} + \frac{(v,u)^2}{\|u\|^2} \quad (\textit{symmetry}) \\
&= \|v\|^2 - \frac{(u,v)^2}{\|u\|^2}.
\end{aligned}$$

Thus,

$$\frac{(u,v)^2}{\|u\|^2} \leq \|v\|^2 \quad \text{or} \quad \|(u,v)\| \leq \|u\| \cdot \|v\|.$$

□

Note

Notice that:

$$0 = (v - \frac{(u,v)}{\|u\|^2}u, v - \frac{(u,v)}{\|u\|^2}u)$$

only if

$$v - \frac{(u,v)}{\|u\|^2}u = 0$$

i.e.

$$v = \frac{(u,v)}{\|u\|^2}u.$$

Thus, v and u have to be on the same line through 0.

A Lemma

We can therefore conclude:

Lemma. $\|(u,v)\| = \|u\| \cdot \|v\|$ if and only if u and v are on the same line through 0.

Theorem

The following statement is generalization of Pythagoras Theorem.

Theorem. Let V be a vector space with inner product $(.,.)$. Then:

$$\|u+v\| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$. Furthermore, $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $(u,v) = 0$.

Proof.

$$\begin{aligned} \|u+v\|^2 &= (u+v, u+v) \\ &= (u,u) + 2(u,v) + (v,v) \quad (*) \\ &\leq \|u\|^2 + 2\,\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$

If $(u,v) = 0$, then (*) reads:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

On the other hand, if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, we see from (*) that $(u,v) = 0$. □

Example. Let $u = (1,2,-1), v = (0,2,4)$. Then:

$$(u,v) = 4 - 4 = 0$$

and

$$\|u\|^2 = 1 + 4 + 1 = 6, \|v\|^2 = 4 + 16 = 20.$$

Also, $u + v = (1,4,3)$ and finally:

$$\|u+v\|^2 = 1 + 16 + 9 = 26 = 6 + 20 = \|u\|^2 + \|v\|^2.$$