Lecture 7

Gram-Schmidt Orthogonalization

The "best" basis we can have for a vector space is an orthogonal basis. That is because we can most easily find the coefficients that are needed to express a vector as a linear combination of the basis vectors $v_1, ..., v_n$:

$$v = \frac{(v, v_1)}{\|v_1\|^2} v_1 + ... + \frac{(v, v_n)}{\|v_n\|^2} v_n.$$

But usually we are not given an orthogonal basis. In this section we will show how to find an orthogonal basis starting from an arbitrary basis.

7.1 Procedure

Let us start with two linear independent vectors v_1 and v_2 (i.e. not on the same line through zero). Let $u_1 = v_1$. How can we find a vector u_2 which is perpendicular to u_1 and that the span of u_1 and u_2 is the same as the span of v_1 and v_2? We try to find a number $a \in \mathbb{R}$ such that:

$$u_2 = a u_1 + v_2, \quad u_2 \perp u_1$$

Take the inner product with u_1 to get:

$$\begin{aligned} 0 = (u_2, u_1) &= a(u_1, u_1) + (v_2, u_1) \\ &= a \|u_1\|^2 + (v_2, u_1) \end{aligned}$$

or

$$a = -\frac{(v_2, u_1)}{\|u_1\|^2}$$

What if we have a third vector v_3? Then, after choosing u_1, u_2 as above, we would look for u_3 of the form:

$$u_3 = a_1 u_1 + a_2 u_2 + v_3$$

Take the inner product with u_1 to find a_1:

$$0 = (u_3, u_1) = a_1 \|u_1\|^2 + (v_3, u_1)$$

or

$$a_1 = -\frac{(v_3, u_1)}{\|u_1\|^2}$$

and the inner product with u_2 to find a_2:

$$0 = (u_3, u_2) = a_2 \|u_2\|^2 + (v_3, u_2)$$

or

$$a_2 = -\frac{(v_3, u_2)}{\|u_2\|^2}$$

Thus:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ u_3 &= v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} u_2 \end{aligned}$$

7.2 Examples

Example. Let $v_1 = (1, 1), v_2 = (2, -1)$. Then, we set $u_1 = (1, 1)$ and

$$\begin{aligned} u_2 &= (2, -1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (2, -1) - \frac{2 - 1}{2}(1, 1) \\ &= \frac{3}{2}(1, -1) \end{aligned}$$

Example. Let $v_1 = (2, -1), v_2 = (0, 1)$. Then, we set $u_1 = (2, -1)$ and

$$\begin{aligned} u_2 &= (0,1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (0,1) - \frac{-1}{5}(2,-1) \\ &= \frac{2}{5}(1,2) \end{aligned}$$

Note We could have also started with $v_2 = (0, 1)$, and get first basis vector to be $(0, 1)$ and second vector to be:

$$(2,-1) - \frac{(2,-1)\cdot(0,1)}{\|(0,1)\|^2}(0,1) = (2,0)$$

Example. Let $v_1 = (0, 1, 2), v_2 = (1, 1, 2), v_3 = (1, 0, 1)$. Then, we set $u_1 = (0, 1, 2)$ and

$$\begin{aligned} u_2 &= (1,1,2) - \frac{(0,1,2)\cdot(1,1,2)}{\|(0,1,2)\|^2}(0,1,2) \\ &= (1,1,2) - \frac{5}{2}(0,1,2) \\ &= (1,0,0) \end{aligned}$$

$$\begin{aligned} u_3 &= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0) \\ &= \frac{1}{5}(0,-2,1) \end{aligned}$$

Example. Let $v_0 = 1, v_1 = x, v_2 = x^2$. Then, v_0, v_1, v_2 is a basis for the space of polynomials of degree $\leq$. But they are not orthogonal, so we start with $u_0 = v_0$ and $u_1 = v_1 - \frac{(v_1,u_0)}{\|u_0\|^2} u_0$. So we need to find:

$$\begin{aligned} (v_1, u_0) &= \int_0^1 x\,dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2} \\ \|u_0\|^2 &= \int_0^1 1\;dx = [x]_0^1 = 1 \end{aligned}$$

Hence, $u_1 = x - \frac{1}{2}$. Then:

$$u_1 = v_2 - \frac{(v_2, u_0)}{\|u_0\|^2} u_0 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \,.$$

We also find that:

$$\begin{aligned} (v_2, u_0) &= \int_0^1 x^2 \, dx = \frac{1}{3} \\ (v_2, u_1) &= \int_0^1 x^2 (x - \frac{1}{2}) \, dx = \frac{1}{12} \\ \|u_1\|^2 &= \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{12}. \end{aligned}$$

Hence, $u_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$.

7.3 Theorem

Theorem. (Gram-Schmidt Orthogonalization)Let V be a vector space with inner product $(.,.)$. Let $v_1, ..., v_k$ be a linearly independent set in V. Then, there exists an orthogonal set $u_1, ..., u_k$ such that $(v_i, u_i) > 0$ and $\text{span}\{v_i, ..., v_i\} = \text{span}\{u_1, ..., u_i\}$ for all $i = 1, ..., k$.

Proof. See the book, p.129 − 131. □