KEY

MATH 2025
Exam 1

Name ____________________

October 7, 2010 Score: ________

Instructions: This is a closed book, closed notes exam. Please read all instructions carefully. Be sure to show your work in order to receive full credit. An answer with no supporting work will receive no credit.

1. Consider the signal $s = (3, 1, 0, 4, 8, 6, 9, 9)$.

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   (a) Find the In-place fast Haar Wavelet transform of the signal.

$$\vec{s}^{(3)} = (3, 1, 0, 4, 8, 6, 9, 9)$$

$$\vec{s}^{(3-1)} = (2, 1, 2, -2, 7, 1, 9, 0)$$

$$\vec{s}^{(3-2)} = (2, 1, 0, -2, 8, 1, -1, 0)$$

$$\vec{s}^{(3-3)} = (5, 1, 0, -2, -3, 1, -1, 0)$$

   (b) Modify the wavelet coefficients you obtained in part a) by setting the entries that are $-1, 0, 1$ to be zero and apply the In-Place wavelet inverse transform to obtain a corrected data.

$$\vec{s}^{(0)\prime} = (5, 0, 0, -2, -3, 0, 0, 0)$$

$$\vec{s}^{(1)\prime} = (2, 0, 0, -2, 8, 0, 0, 0)$$

$$\vec{s}^{(2)\prime} = (2, 0, 2, -2, 8, 0, 8, 0)$$

$$\vec{s}^{(3)\prime} = (2, 2, 0, 4, 8, 8, 8, 8)$$

2. Assume that for some sample with eight entries $s = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$, the In-place fast Haar Wavelet transform produces $(4, -1, -1, 2, 0, 1, -2, -2)$.

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(a) What is the average of the sample.

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(b) What is the change from the average over the first half to the average over the second half of the sample.

0

(c) Apply the In-Place inverse wavelet Haar transform to obtain the original signal.

$\vec{s}^{(0)} = (4, -1, -1, 2, 0, 1, -2, -2)$

$\vec{s}^{(1)} = (4, -1, -1, 2, 4, 1, -2, -2)$

$\vec{s}^{(2)} = (3, -1, 5, 2, 2, 1, 6, -2)$

$\boxed{\vec{s}^{(3)} = (2, 4, 7, 3, 3, 1, 4, 8)}$

(d) Write an approximating simple function for the signal.

$\hat{f} = 2\varphi_{[0,\frac{1}{8})} + 4\varphi_{[\frac{1}{8},\frac{1}{4})} + 7\varphi_{[\frac{1}{4},3/8)}$

$+ 3\varphi_{[3/8,\frac{1}{2})} + 3\varphi_{[\frac{1}{2},5/8)} + \varphi_{[5/8,3/4)}$

$+ 4\varphi_{[3/4,7/8)} + 8\varphi_{[7/8,1)}$

3. Let $V = \mathbb{R}^3$ and $W = \{c(1,1,-2) : c \in \mathbb{R}\}$. Let $P : \mathbb{R}^3 \to W$ be given by

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$$P(x,y,z) = (x+y-2z)(1,1,-2)$$

(a) Evaluate $P(1,0,2)$.

$$P(1,0,2) = (1+0-2\cdot 2)\ (1,1,-2)$$
$$= (-3)(1,1,-2) = (-3,-3,6)$$

(b) Show that P is a linear map.

(i) $P((x,y,z)+(u,v,w))$
$= P((x+u, y+v, z+w))$
$= [x+u+y+v-2(z+w)]\ (1,1,-2)$
$= (x+y-2z)(1,1,-2) + (u,v,-2w)(1,1,-2)$
$= P(x,y,z) + P(u,v,w)$

(ii) $P(k(x,y,z)) = P((kx,ky,kz)) = (kx+ky-2kz)(1,1,-2)$
$= k(x+y-2z)(1,1,-2)$
$= kP(x,y,z)$.

(c) Show that for any $\mathbf{w} \in W$, $P(\mathbf{w}) = 6\mathbf{w}$.

Let $w \in W$. Then $w = c(1,1,-2)$ for some c.

$\Rightarrow P(w) = P(c(1,1,-2))$
$= cP(1,1,-2) = c\ (1+1-2(-2))\ (1,1,-2)$
$= c\ (6)(1,1,-2) = 6\ c(1,1,-2)$
$= 6w$.

(d) Find the kernel of P.

$P(x,y,z) = 0 \iff x+y-2z = 0.$

$\ker P = \{(x,y,z) : x+y-2z = 0\}.$

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4. Consider the vector space $V = C^0[0,1]$, the set of continuous functions defined over $[0,1]$, together with the inner product

$$< f, g >= \int_0^1 f(x)g(x)dx.$$

(a) Show that $\mathcal{P}^2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, the set of polynomials of degree at most 2, is a subspace of V.

$$i) \quad (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)$$
$$= (a_0 + b_0) + a_1 x + b_1 x + a_2 x^2 + b_2 x^2$$
$$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 \in \mathcal{P}^2$$

$$(ii) \quad k(a_0 + a_1 x + a_2 x^2) = k a_0 + k a_1 x + k a_2 x^2 \in \mathcal{P}^2$$

(b) Let $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$. Explain why $\{p_1,\ p_2,\ p_3\}$ is a generating set of $\mathcal{P}^2$.

Because any element of $\mathcal{P}^2$ is a linear combination of $P_1, P_2, \&\ P_3$.

(c) Compute $< p_1, p_2 >$, $< p_1, p_3 >$, and $< p_2, p_3 >$. Are any of the pairs orthogonal?

$$\langle P_1, P_2 \rangle = \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\langle P_1, P_3 \rangle = \int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

$$\langle P_2, P_3 \rangle = \int_0^1 x \cdot x^2 dx = \int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}$$

None of the pairs are orthogonal!

(d) Compute $\|p_1\|$, $\|p_2\|$ and $\|p_3\|$.

$$\| P_1 \| = \sqrt{\int_0^1 P_1(x)^2 dx} = \sqrt{\int_0^1 dx} = 1$$

$$\| P_2 \| = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\| P_3 \| = \sqrt{\int_0^1 x^4 dx} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

(e) Let $w_2 = p_2 - \frac{<p_1,p_2>}{\|p_1\|^2} p_1$. As a function, what is w_2? Is $w_2 \in \mathcal{P}^2$? Explain.

$$W_2(x) = x - \tfrac{1}{2}\left(\tfrac{1}{1}\right)\cdot 1 = x - \tfrac{1}{2}.$$

Yes, $w_2 \in P^2$, as it is a poly of degree at most 2.

Let us rename p_1 to be w_1.

(f) Compute $\|w_2\|$, $< w_2, w_1 >$, and $< w_2, p_3 >$. Are w_1 and w_2 orthogonal? What about w_2 and p_3?

$$\|w_2\| = \sqrt{\int_0^1 (x-\tfrac{1}{2})^2 dx} = \sqrt{\int_0^1 x^2 - x + \tfrac{1}{4}\, dx}$$
$$= \sqrt{\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4}} = \sqrt{\tfrac{-1}{6} + \tfrac{1}{4}} = \sqrt{\tfrac{1}{12}} = \tfrac{1}{2\sqrt{3}}.$$

$$<w_2, w_1> = \int_0^1 (x-\tfrac{1}{2})\, dx = \tfrac{1}{2}x^2 - \tfrac{1}{2}x \Big|_0^1 = 0$$

$$< w_2, p_3 > = \int_0^1 (x-\tfrac{1}{2})x^2 dx = \int_0^1 x^3 - \tfrac{1}{2}x^2 dx = \tfrac{1}{4}x^4 - \tfrac{1}{6}x^3\Big|_0^1$$
$$= \tfrac{1}{4} - \tfrac{1}{6} = \tfrac{2}{24} = \tfrac{1}{12}$$

w_1 & w_2 are orthogonal.

w_2 & p_3 are not orthogonal.

(g) Let $w_3 = p_3 - \frac{<w_1,p_3>}{\|w_1\|^2} w_1 - \frac{<w_2,p_3>}{\|w_2\|^2} w_2$. As a function, what is w_3? Is $w_3 \in \mathcal{P}^2$? Explain.

$$w_3 = x^2 - \tfrac{1}{3}\cdot(1)\,1 - \tfrac{1}{12}\left(\tfrac{1}{\cancel{12}}\right)(x - \tfrac{1}{2})$$
$$= x^2 - \tfrac{1}{3} - (x - \tfrac{1}{2}) = x^2 - \tfrac{1}{3} - x + \tfrac{1}{2} = x^2 - x + \tfrac{1}{6}.$$

$\in P^2$.

(h) Show that w_3 is orthogonal to both w_1, and w_2.

$$\langle w_3, w_1 \rangle = \int_0^1 \left(x^2 - x + \tfrac{1}{6}\right) dx = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{-1}{6} + \tfrac{1}{6} = 0$$

$$\langle w_3, w_2 \rangle = \int_0^1 \left(x^2 - x + \tfrac{1}{6}\right)\left(x - \tfrac{1}{2}\right) dx$$

$$= \int_0^1 x^3 - \tfrac{1}{2}x - x^2 + \tfrac{1}{2}x + \tfrac{1}{6}x - \tfrac{1}{12}\, dx$$

$$= \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{12} - \tfrac{1}{2} = 0.$$

(i) Write $1 + x + x^2$ as a linear combination of w_1, w_2 and w_3. [*We will show in class that the set* $\{w_1, w_2, w_3\}$ *also generate* $\mathcal{P}^2$.]

$$1 + x + x^2 = c_0 + c_1\left(x - \tfrac{1}{2}\right) + c_2\left(x^2 - x + \tfrac{1}{6}\right)$$

$$= c_0 + c_1 x - \tfrac{1}{2}c_1 + c_2 x^2 - c_2 x + \tfrac{1}{6}c_2$$

$$= \left(c_0 - \tfrac{1}{2}c_1 + \tfrac{1}{6}c_2\right) + (c_1 - c_2)x + c_2 x^2.$$

$$\Rightarrow \quad c_2 = 1, \quad c_1 - c_2 = 1 \Rightarrow c_1 = 2.$$

and $c_0 - \tfrac{1}{2}c_1 + \tfrac{1}{6}c_2 = 1$

$$c_0 = 1 + \tfrac{1}{2}c_1 - \tfrac{1}{6}c_2 = 1 + \tfrac{1}{2}\cdot(2) - \tfrac{1}{6}\cdot 1 = 2 - \tfrac{1}{6} = \tfrac{11}{6}.$$

$$1 + x + x^2 = \tfrac{11}{6} + 2\left(x - \tfrac{1}{2}\right) + 1\left(x^2 - x + \tfrac{1}{6}\right).$$

(j) Show that w_1, w_2 and w_3 are linearly independent.

$$0 = c_0 + c_1\left(x - \tfrac{1}{2}\right) + c_2\left(x^2 - x + \tfrac{1}{6}\right)$$

$$= \left[c_0 - \tfrac{1}{2}c_1 + \tfrac{1}{6}c_2\right] + (c_1 - c_2)x + c_2 x^2$$

$$\Rightarrow \quad c_2 = 0, \quad c_1 = c_2 = 0, \quad c_0 = 0.$$