Math 447: Homework 3*

Due date: Wednesday September 17, 2014.

1. Find the infimum and supremum, whenever exist, of the following sets. Justify (or explain) your answer.

   (a) $S_1 = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

   (b) $S_2 = \left\{ x \in \mathbb{R} : x < \frac{1}{x} \right\}$

   (c) $S_3 = \{ x \in \mathbb{R} : x + 2 \geq x^2 \}$

   (d) $S_4 = \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}$

2. (Compatibility of sup/inf with algebraic operations)

   Given nonempty subsets A and B of $\mathbb{R}$ and $k \in \mathbb{R}$, we define the following subsets of $\mathbb{R}$:

$$kA := \{k \cdot a : a \in A\}$$
$$k + A := \{k + a : a \in A\}$$
$$A + B := \{a + b : a \in A, b \in B\}.$$

   Assume that A and B are nonempty bounded subsets of $\mathbb{R}$. Prove (**any two of**) the following.

   (a) If $k > 0$, then $\inf(kA) = k \inf A$, $\quad \sup kA = k \sup A$.

   (b) If $k < 0$, then $\inf(kA) = k \sup A$, $\quad \sup kA = k \inf A$.

   (c) $\sup(A + B) = \sup A + \sup B$, $\quad \inf(A + B) = \inf A + \inf B$.

   (d) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$, $\quad \inf(A \cup B) = \inf\{\inf(A), \inf(B)\}$.

3. (The greatest integer function)

   (a) Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.
   [n is the greatest integer less than or equal to x (sometimes called the floor of x) and is denoted by $\lfloor x \rfloor$. n+1 is the smallest integer greater than x (sometimes called the ceiling of x) and is denoted by $\lceil x \rceil$]

   (b) Sketch the graphs of the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

*This homework covers Section 2.3 and 2.4. Please do as many exercise as you can in the textbook on these section. And let me know if you have any question.

4. (Supremum/infimum of a function)

Definition 0.1. *Suppose D is a nonempty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ is a function.*

- *f is bounded above(**below**) if the range of f, $f(D)$, is bounded above (**below**). f is a bounded function if $f(D)$ is a bounded set.*
- *Whenever they exist, $\sup\limits_{x\in D} f(x) = \sup\{f(x) : x \in D\} := \sup f(D)$; $\inf\limits_{x\in D} f(x) = \inf\{f(x) : x \in D\} := \inf f(D)$.*

Prove the following.

(a) f is bounded if and only if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$, for all $x \in D$.

(b) Suppose that f and g are bounded functions with common domain D. Assume that $f(x) \leq g(x)$, for all $x \in D$. Then

$$\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x).$$

5. Do Exercise # 8 in Section 2.4.

6. Do (**any two of**) Exercise # 9, 10, 11, 12 in section 2.4.