447 HOMEWORK SET 2

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1

For any $a, b \in \mathbb{R}$, such that $a < b$, show that $a < \frac{a+b}{2} < b$.

Solution From $a < b$, add b to both sides of the inequality to get $a + b < 2b$, then multiply both sides by $\frac{1}{2}(> 0)$ to get $\frac{a+b}{2} < b$. Similarly, add a to both sides of $a < b$ to get $a + a = 2a < a + b$, then multiply both sides by $\frac{1}{2}$ to get $a < \frac{a+b}{2}$. These two statements imply $a < \frac{a+b}{2} < b$. $\Diamond$

2

Let $a, b \in \mathbb{R}$ and suppose that for every $\varepsilon > 0$, we have $a \leq b + \varepsilon$. Show that $a \leq b$.

Solution Assume that $a > b$, i.e. $a - b > 0$ and choose $\varepsilon = \frac{a-b}{2}$. Then, by the hypothesis, $a \leq b + \frac{a-b}{2} \Leftrightarrow \frac{a}{2} \leq \frac{b}{2} \Leftrightarrow a \leq b$, a contradiction. Hence, $a \leq b$. $\Diamond$

3

Prove that for any $a, b \in \mathbb{R}$:

$$(\frac{a+b}{2})^2 \leq \frac{a^2 + b^2}{2}$$

Show that equality holds if and only if a = b.

Solution Let $a, b \in \mathbb{R}$. Then, by expanding the square, multiplying both sides by $4(> 0)$, subtracting everything on the left side of the inequality from both sides, respectively, we get:

$$(\frac{a+b}{2})^2 \leq \frac{a^2 + b^2}{2}$$

$$\Leftrightarrow \frac{a^2 + 2ab + b^2}{4} \leq \frac{a^2 + b^2}{2}$$

$$\Leftrightarrow a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$$

$$\Leftrightarrow 0 \leq a^2 - 2ab + b^2 = (a - b)^2$$

Since any square of a real number is ≥ 0, and $a - b \in \mathbb{R}$, we are done. From the last line, we see that equality in the original statement holds if and only if $0 = (a-b)^2 \Leftrightarrow 0 = a - b \Leftrightarrow a = b$. $\Diamond$

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4

(a) Suppose that $0 < c < 1$. Show that $0 < c^2 < c < 1$. Show also that for all $n \in \mathbb{N}$, $c^n \leq c$.

Solution We have $c - 1 < 0$, so we may multiply both sides by $c > 0$ to get $c(c-1) = c^2 - c < 0 \Leftrightarrow c^2 < c$. Since $c < 1$, we may conclude $0 < c^2 < c < 1$.
Now, for the base case, we have $c^1 = c \leq c$. Now assume $c^n \leq c$ for some $n \in \mathbb{N}$. Then we have $c^n - c \leq 0$, and using the induction hypothesis and the first result of this problem, $c(c^n - c) = c^{n+1} - c^2 \leq 0 \Leftrightarrow c^{n+1} \leq c^2 \leq c$. Thus the statement holding for $n \Rightarrow$ it holds for $n+1$, so by induction, it holds for all $n \in \mathbb{N}$. $\Diamond$

(b) Suppose that $c > 1$. Show that $c^2 > c > 1$. Show also that for all $n \in \mathbb{N}$, $c^n \geq c$.

Solution $c > 1 \Leftrightarrow c - 1 > 0$. Multiplying both sides by c, which is positive, gives $c(c-1) = c^2 - c > 0 \Leftrightarrow c^2 > c > 1$.
Now, for the base case, we have $c^1 = c \geq c$. Now assume $c^n \geq c$ for some $n \in \mathbb{N}$. Then we have $c^n - c \geq 0 \Leftrightarrow c(c^n - c) \geq 0 \Leftrightarrow c^{n+1} - c^2 \geq 0 \Leftrightarrow c^{n+1} \geq c^2 \geq c$, using the induction hypothesis and the first result of this problem. Thus the statement holding for $n \Rightarrow$ it holds for $n+1$, so by induction, it holds for all $n \in \mathbb{N}$. $\Diamond$

5

(a) Show that if $a \in \mathbb{R}$, then $|a| = \sqrt{a^2}$.

Solution If $a = 0$, $|0| = 0 = \sqrt{0^2}$. If not, since we have $|a| \geq 0$, and $a^2 \geq 0$ (since we used it to prove the AM-GM Inequality), $|a| = \sqrt{a^2} \Leftrightarrow |a|^2 = \sqrt{a^2}^2 = a^2$, which is true since $|a|^2 = a^2, \forall a \in \mathbb{R}$. $\Diamond$

(b) If (i) $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$.

Solution $a < y < b \Leftrightarrow -b < -y < -a$ (we multiplied by -1, which is negative). Adding this inequality together with (i) gives $a - b = -(b-a) < x - y < b - a$, and by property of absolute value gives $|x - y| < b - a$, since $b - a > 0$ by hypothesis. $\Diamond$

6

Find all $x \in \mathbb{R}$ such that $|x| + |x+1| < 2$.

Solution By cases:
If $x < -1$, then $|x| + |x+1| = -x - (x+1) = -2x - 1 < 2 \Leftrightarrow -2x < 3 \Leftrightarrow x > -\frac{3}{2}$, (We "flipped" the inequality since we multiplied both sides by $-\frac{1}{2} < 0$).
If $-1 \leq x < 0$, then $|x| + |x+1| = -x + x + 1 = 1 < 2 \Leftrightarrow 0 < 1$, a tautology.
If $x \geq 0$, then $|x| + |x+1| = x + x + 1 = 2x + 1 < 2 \Leftrightarrow 2x < 1 \Leftrightarrow x < \frac{1}{2}$.
Therefore our solution set is $\{x \in \mathbb{R} : -\frac{3}{2} < x < \frac{1}{2}\}$ $\Diamond$

7

16) Let $\varepsilon > 0, \delta > 0$, and $a \in \mathbb{R}$. Show that $I := V_\varepsilon(a) \cap V_\delta(a)$ and $U := V_\varepsilon(a) \cup V_\delta(a)$ are $\gamma-$neighborhoods of a for appropriate values of γ.

Proof If $\varepsilon = \delta$, in both I, U simply take $\gamma = \delta$, then the result is trivial.
WLOG, assume $\varepsilon > \delta$. Then, $x \in I \Leftrightarrow |x-a| < \delta < \varepsilon$, so let $\gamma = \delta$. Also, $x \in U \Leftrightarrow |x-a| < \delta$ or $|x-a| < \varepsilon \Leftrightarrow |a-x| < \varepsilon$, since $\varepsilon > \delta$, so take $\gamma = \varepsilon$. Thus, we have appropriate values for $\gamma-$nbhd's. $\Diamond$

17) Show that if $a, b \in \mathbb{R}$ and $a \neq b$, then there exists $\varepsilon-$nbhd's $U_\varepsilon(a), V_\varepsilon(b)$ such that $U \cap V = \phi$.

Proof WLOG, assume that $a > b$, set $\varepsilon = \frac{a-b}{2}$, and assume $\exists x \in U \cap V$. Then

$$|x-a| < \frac{a-b}{2} \quad ; \quad |x-b| = |b-x| < \frac{a-b}{2}$$
$$\Leftrightarrow \frac{b-a}{2} < x-a < \frac{a-b}{2}\, ; \, \frac{b-a}{2} < b-x < \frac{a-b}{2}$$
$$\Leftrightarrow b-a < b-a < a-b,$$

which is clearly a contradiction. (We deduced the third line in calculation by adding the two inequalities from the second line.)Therefore, The intersection of these two $\varepsilon-$nbhd's is ϕ. $\Diamond$

18a) Show that if $a, b \in \mathbb{R}$, then $max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ and $min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$.

Proof WLOG, assume $a \geq b$, then $\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = \frac{1}{2}(2a) = a$. Likewise, $\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(a+b-a+b) = \frac{1}{2}(2b) = b$.

18b) $min\{a,b,c\} = min\{min\{a,b\},c\}$

Proof Assume $a \geq b \geq c$. Then $min\{min\{a,b\},c\} = min\{\frac{1}{2}(a+b-|a-b|),c\} = min\{b,c\} = \frac{1}{2}(b+c-|b-c|) = \frac{1}{2}(b+c-b+c) = c = min\{a,b,c\}$ $\Diamond$

19) Show that if $a, b, c \in \mathbb{R}$, then the "middle number" is $mid\{a,b,c\} = min\{max\{a,b\}, max\{b,c\}, max\{a,c\}\}$

Proof Once again, assume $a \geq b \geq c$. Then $min\{max\{a,b\}, max\{b,c\}, max\{a,c\}\} = min\{a,b,a\} = min\{a,b\} = b = mid\{a,b,c\}$, by applying our previous results. $\Diamond$