Math 447. Homework 1 Pavel Grzegrzolka Due 09/03/2014

20/20

V. good!

① For each $n \in \mathbb{N}$, let $A_n = \{(n-1)k : k \in \mathbb{N}\}$

a) $\forall n \in \mathbb{N}, A_n \subseteq \mathbb{N}$

let $n=1$. Then, $A_1 = \{(1-1)k : k \in \mathbb{N}\} = \{0\}$

But $\{0\} \not\subseteq \mathbb{N}$. Therefore, $A \not\subseteq \mathbb{N}$. Thus, the truth value of the statement $\forall n \in \mathbb{N}, A_n \subseteq \mathbb{N}$ is $\boxed{\text{false}}$

b) $A_2 \cap A_3 = \{(2-1)k : k \in \mathbb{N}\} \cap \{(3-1)k : k \in \mathbb{N}\} =$

$= \{k : k \in \mathbb{N}\} \cap \{2k, k \in \mathbb{N}\} =$

$= \mathbb{N} \cap \{2k, k \in \mathbb{N}\} = \{2k, k \in \mathbb{N}\} =$

$=$ all even natural numbers

c) $\bigcup_{n \in \mathbb{N}} A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = \{0\} \cup \{k, k \in \mathbb{N}\} \cup \{2k, k \in \mathbb{N}\} \cup$

$\cup \{3k, k \in \mathbb{N}\} \cup \{4k, k \in \mathbb{N}\} \cup \ldots \cup \{nk : k \in \mathbb{N}\} \cup \ldots$

Since for any $n \geq 2$, $A_n \subseteq A_2$, $A_2 \cup A_3 \cup A_4 \ldots = A_2$

Therefore, $\bigcup_{n \in \mathbb{N}} A_n = A_1 \cup A_2 = \{0\} \cup \{k, k \in \mathbb{N}\} = \boxed{\mathbb{N} + \{0\}}$

All natural numbers plus 0

we use $\cup$ for union

$\bigcap_{n \in \mathbb{N}} A_n = A_1 \cap A_2 \cap \ldots = \{0\} \cap \{k, k \in \mathbb{N}\} \cap \ldots$

Since $\{0\}$ and $\{k, k \in \mathbb{N}\}$ have no elements in common, then $\{0\} \cap \{k, k \in \mathbb{N}\} = \emptyset$. Therefore, $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$

② $A = B := \{x \in \mathbb{R} : 0 \le x \le 1\}$. Consider the subset $C = \{(x,y) : x = y^2 + 1\}$ of $A \times B$. Is set C a function?

C is a function if for each $x \in A$, $\exists$ unique $y \in B$ such that $(x,y) \in C$.
However, if we pick $x = 0$, then there is no such y in B such that $x = y^2 + 1$, since $x = y^2 + 1 \Rightarrow y = \pm\sqrt{x-1}$ and if we substitute x with 0, then $y = \pm\sqrt{-1} \notin B$. Therefore, C is not a function from A to B

③ $f: A \to B$ is a function

a) $E \subseteq A$ and $F \subseteq A$

1) $f(E \cup F) = f(E) \cup f(F)$

⇒ direction let $y \in f(E \cup F)$. Then $\exists\, y^{-1} \in (E \cup F)$, where y^{-1} is the pre-image of y. Since $y^{-1} \in (E \cup F)$, $y^{-1} \in E$ or $y^{-1} \in F$. If $y^{-1} \in E$, then $f(y^{-1}) = y \in f(E)$. If $y^{-1} \notin E$, then $y^{-1} \in F$ and therefore $f(y^{-1}) = y \in f(F)$.
Thus, y belongs to either $f(E)$ or $f(F)$. Thus, $y \in f(E) \cup f(F)$

⇐ direction let $y \in f(E) \cup f(F)$. Then $y \in f(E)$ or $f(F)$. If $y \in f(E)$, then $\exists\, y^{-1} \in E$ such that $f(y^{-1}) = y$. Since $y^{-1} \in E$, $y^{-1} \in (E \cup F)$. Thus, $y \in f(E \cup F)$. If $y \notin f(E)$, then $y \in f(F)$. Consequently, $\exists\, y^{-1} \in F$ such that $f(y^{-1}) = y$. Since $y^{-1} \in F$, $y^{-1} \in (E \cup F)$. Thus, $y \in f(E \cup F)$.

Since from ⇒ direction we know that $f(E \cup F) \subseteq f(E) \cup f(F)$ and from ⇐ direction we know that $f(E) \cup f(F) \subseteq f(E \cup F)$, we can conclude that $f(E \cup F) = f(E) \cup f(F)$

ii) $f(E \cap F) \subseteq f(E) \cap f(F)$

let $y \in f(E \cap F)$. Then, $\exists\, y^{-1} \in (E \cap F)$ such that $f(y^{-1}) = y$
Since $y^{-1} \in (E \cap F)$, $y^{-1} \in E$ and $y^{-1} \in F$. Therefore,
$f(y^{-1}) \in f(E)$ and $f(y^{-1}) \in f(F)$. So, in other words,
$y \in f(E)$ and $y \in f(F)$. Thus, $y \in f(E) \cap f(F)$ and
we can conclude that $f(E \cap F) \subseteq f(E) \cap f(F)$

b) $G \subseteq B$, $H \subseteq B$

i) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

$\Rightarrow$ direction

let $x \in f^{-1}(G \cup H)$. Then, $\exists\, y \in (G \cup H)$ such that $f(x) = y$
Since $y \in (G \cup H)$, $y \in G$ or $y \in H$. If $y \in G$, then
$x \in f^{-1}(y) \subseteq f^{-1}(G)$. If $y \notin G$, then $y \in H$ and consequently
$x \in f^{-1}(y) \subseteq f^{-1}(H)$. Therefore, since $x \in f^{-1}(G)$ or $x \in f^{-1}(H)$,
$x \in f^{-1}(G) \cup f^{-1}(H)$

$\Leftarrow$ direction

let $x \in f^{-1}(G) \cup f^{-1}(H)$. Then $x \in f^{-1}(G)$ or $x \in f^{-1}(H)$.
If $x \in f^{-1}(G)$, then $\exists\, y \in G$ such that $f(x) = y$. Since $y \in G$,
$y \in (G \cup H)$. Therefore, $x \in f^{-1}(y) \subseteq f^{-1}(G \cup H)$.
If $x \notin f^{-1}(G)$, then $x \in f^{-1}(H)$. Then, $\exists\, y \in H$ such that $f(x) = y$.
Since $y \in H$, $y \in (G \cup H)$. Therefore, $x \in f^{-1}(y) \subseteq f^{-1}(G \cup H)$

Since from $\Rightarrow$ direction we know that $f^{-1}(G \cup H) \subseteq f^{-1}(G) \cup f^{-1}(H)$
and from $\Leftarrow$ direction we know that $f^{-1}(G) \cup f^{-1}(H) \subseteq f^{-1}(G \cup H)$
we can conclude that $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

ii) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

$\Rightarrow$ direction

let $x \in f^{-1}(G \cap H)$. Then, $\exists\, y \in (G \cap H)$ such that $f(x) = y$. Since $y \in (G \cap H)$, $y \in G$ and $y \in H$. Since $y \in G$, $x \in f^{-1}(y) \subseteq f^{-1}(G)$. Since $y \in H$, $x \in f^{-1}(y) \subseteq f^{-1}(H)$. Thus, since $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, $x \in f^{-1}(G) \cap f^{-1}(H)$.

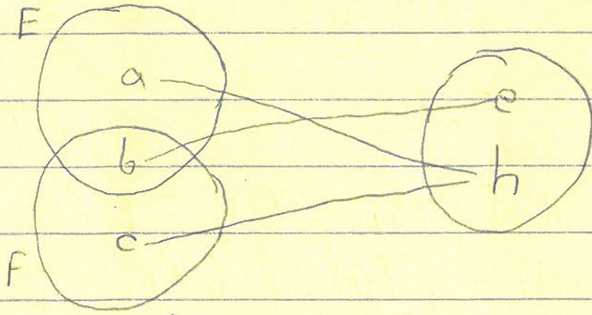
$\Leftarrow$ direction

let $x \in f^{-1}(G) \cap f^{-1}(H)$. Then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Since $x \in f^{-1}(G)$, $\exists\, y \in G$ such that $f(x) = y$. Since $x \in f^{-1}(H)$, $y \in H$ as well. Thus, $y \in (G \cap H)$ and concequently $x \in f^{-1}(y) \subseteq f^{-1}(G \cap H)$

Since from $\Rightarrow$ direction we know that $f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H)$ and from $\Leftarrow$ direction we know that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$, we can conclude that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

c) $f(E \cap F) \neq f(E) \cap f(F)$

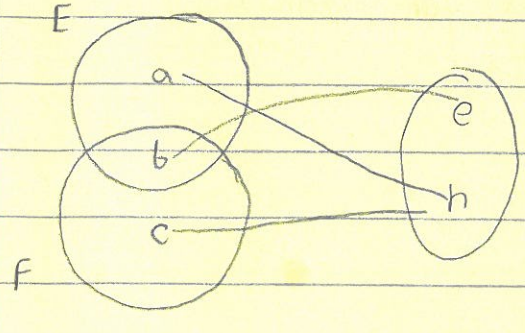
Example



Let $E = \{a, b\}$, $F = \{b, c\}$. Let $f(a) = h$, $f(b) = e$, $f(c) = h$. Then, $(E \cap F) = \{b\}$. So we have: $f(E) = \{e, h\}$, $f(F) = \{e, h\}$ which allows us to conclude that $f(E) \cap f(F) = \{e, h\}$. But $f(E \cap F) = \{e\}$ Thus, $f(E \cap F) \neq f(E) \cap f(F)$

d) $f(E\setminus F) \nsubseteq f(E)\setminus f(F)$

example :



Let $E=\{a,b\}$ and $F=\{b,c\}$. Let $f(a)=h$, $f(b)=e$, and $f(c)=h$

Then, $E\setminus F=\{a\}$. So we have $f(E\setminus F)=f(a)=h$.

On the other hand, $f(E)=\{e,h\}$ and $f(F)=\{e,h\}$.

So $f(E)\setminus f(F)=\emptyset$. And $h\notin\emptyset$. Therefore,

$f(E\setminus F) \nsubseteq f(E)\setminus f(F)$

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$\boxed{\text{16 on page 11}}$

$f(x) = x/\sqrt{x^2+1}$, $x\in\mathbb{R}$

Want to show: it is a bijection of $\mathbb{R}$ onto B, where $B=\{y: -1<y<1\}$

- injective

assume $f(x_1)=f(x_2)$. Then;

$$\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}}$$

$$x_1\left(\sqrt{x_2^2+1}\right) = x_2\left(\sqrt{x_1^2+1}\right)$$

since square root is always positive, we know that x_1 has the same sign as x_2. We square both sides:

$$x_1^2(x_2^2+1) = x_2^2(x_1^2+1)$$

$$(x_1x_2)^2 + x_1^2 = (x_1x_2)^2 + x_2^2 \qquad / -(x_1x_2)^2$$

$$x_1^2 = x_2^2$$

remembering that x_1 and x_2 have the same sign, we can conclude that $x_1 = x_2$, which completes the proof for injection

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

- surjective

$\forall y \in B, \exists x \in \mathbb{R}$ s.t $f(x) = y$

we know that

$$y = \frac{x}{\sqrt{x^2+1}}$$

we solve it for x:

$$y\sqrt{x^2+1} = x$$

square both sides $\quad y^2(x^2+1) = x^2$

$$y^2x^2 + y^2 - x^2 = 0$$

$$x^2(y^2-1) = -y^2 \qquad y \neq 1 \text{ and } y \neq -1 \text{, so we can divide}$$

$$x^2 = \frac{-y^2}{y^2-1}$$

$$x = \pm\sqrt{\frac{-y^2}{y^2-1}}$$

we want x to exist in $\mathbb{R}$, therefore,

$$\frac{-y^2}{y^2-1} \geq 0$$

since $y < 1$, $y^2 - 1 < 0$ for all y and thus multiplying both sides by y^2-1:

$$-y^2 \leq 0$$

$$y^2 \geq 0$$

which is satisfied for all $y \in B$. Therefore, $\forall y \in B, \exists x \in \mathbb{R}$, namely $x = \sqrt{\frac{-y^2}{y^2-1}}$, such that $f(x) = y$. Thus, it's surjective

Since f is injective and surjective, it is a bijection from $\mathbb{R}$ onto B

$\boxed{\text{19 on page 11}}$

a) $f: A \to B$ is injective. $E \subseteq A$.

Want to show: $f^{-1}(f(E)) = E$

Proof.

($\Rightarrow$ direction) let $x \in f^{-1}(f(E))$. Then $\exists\, f(x) \in B$ such that $f(x) \in f(E)$

Since f is a function, $\exists\, x_1 \in E$ such that $f(x_1) = f(x)$. But since f is injective, $f(x_1) = f(x) \Rightarrow x_1 = x$, so since $x_1 \in E$, $x \in E$

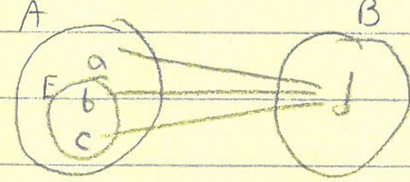
($\Leftarrow$ direction) let $x \in E$. Then $f(x) \in f(E)$. But since f is a function, $\exists\, x_1 \in A$ such that $x_1 \in f^{-1}(f(x)) \in f^{-1}(f(E))$, where $f(x_1) = f(x)$. But since f is injective, $f(x_1) = f(x) \Rightarrow x_1 = x$, so $x \in f^{-1}(f(E))$

From ($\Rightarrow$) direction we know that $f^{-1}(f(E)) \subseteq E$

From ($\Leftarrow$) direction we know that $E \subseteq f^{-1}(f(E))$

Thus, we conclude that $f^{-1}(f(E)) = E$ when f is injective

Example of a non-injective function for which $f^{-1}(f(E)) \neq E$



$A = \{a, b, c\}$

$B = \{d\}$

Let $E = \{b, c\}$ and let $f(a) = f(b) = f(c) = d$. Then $f(E) = \{d\}$ but $f^{-1}(f(E)) = \{a, b, c\}$. Thus, $E \neq f^{-1}(f(E))$

b) $f: A \to B$ is surjective , $H \subseteq B$

want to show : $f(f^{-1}(H)) = H$.

$\Rightarrow$ direction

Let $y \in f(f^{-1}(H))$. Then, by def of a function, $y = f(x)$ for some $x \in f^{-1}(H)$. By def of an inverse , $f(x) \in H$, but since $y = f(x)$, $y \in H$

$\Leftarrow$ direction

Let $y \in H$. Then; since f is surjective, $y = f(x)$ for some $x \in A$

By def. of an inverse , $x = f^{-1}(y) \in f^{-1}(H)$

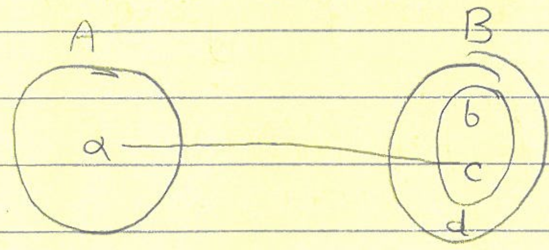
so, $f(x) = f(f^{-1}(y)) \in f(f^{-1}(H))$

Since $\Rightarrow$ direction tells us that $f(f^{-1}(H)) \subseteq H$

and $\Leftarrow$ direction tells us that $H \subseteq f(f^{-1}(H))$

we can conclude that $f(f^{-1}(H)) = H$ when f is surjective

Example of a non-surjective function for which $f(f^{-1}(H)) \neq H$



Let $A = \{a\}$, $B = \{b, c, d\}$, $H = \{b, c\}$, $f(a) = c$.

Then $f^{-1}(H) = \{a\}$ and $f(f^{-1}(H)) = \{c\}$.

So, $f(f^{-1}(H)) \neq H$

$\boxed{\text{20 on page 11}}$

a) $f \to$ injection

Want to show: $f^{-1} \circ f(x) = x \ \forall x \in D(f)$

Proof: let $x \in D(f)$. Then, by def of f^{-1}, $f^{-1}(f(x)) = x_1$ such that $f(x_1) = f(x)$. But since f is injective, then $x_1 = x$ and thus

$f^{-1} \circ f(x) = x$

Want to show: $f \circ f^{-1}(y) = y \ \forall y \in R(f)$

Proof: let $y \in R(f)$. Then, by def of f^{-1}, $f(f^{-1}(y)) = f(x)$ for some x satisfying $f(x) = y$. But then we know that

$f(f^{-1}(y)) = f(x) = y$ and thus $f \circ f^{-1}(y) = y$

b) $f \to$ bijection $A \to B$

Want to show: f^{-1} is a bijection

- first prove it is an injection

Let $f^{-1}(y_1) = f^{-1}(y_2)$ for some y_1, y_2 in B

Since f is a surjection, $\exists\ x_1, x_2 \in A$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Thus, $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$.

By the def of inverse function, $f^{-1}(f(x_1)) = x_3$ and $f^{-1}(f(x_2)) = x_4$ where $f(x_1) = f(x_3)$ and $f(x_2) = f(x_4)$.

But since f is injective, $x_1 = x_3$ and $x_2 = x_4$. Therefore,

$f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$. Thus, since

$f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$,

So, $f(x_1) = f(x_2)$. So, $y_1 = y_2$.

Thus, we have shown that $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow y_1 = y_2$

Therefore, $f^{-1}(y)$ is an injection

- now let us prove it is a surjection

since f is a function, $\forall x \in A$, $\exists y \in B$ such that $y = f(x)$

since f is surjective, $\forall y \in B$, $\exists x \in A$ such that $y = f(x)$

thus, for any $x \in A$, $\exists y \in B$ such that $f^{-1}(y) = f^{-1}(f(x)) = x$

"Therefore, it is a surjection.

(In other words, one can say that since from the def of a function all $x \in A$ are "used" by a function, the inverse of that function will have as a codomain all x. But since all x are already paired up with some ys, all the elements of a codomain will be used by f^{-1}. Or, in other words, codomain of f^{-1} = range f^{-1}. Thus, it is a surjection)

Since f^{-1} is injective and surjective, it is bijective.

$\boxed{\text{21 on page 11}}$

$f: A \to B$ is bijective

$g: B \to C$ is bijective

Want to show: $g \circ f$ is bijective

- proof for injection.

let $g \circ f(x_1) = g \circ f(x_2)$ for some $x_1, x_2 \in A$

Then, $g \circ f(x_1) = g \circ f(x_2)$

$g(f(x_1)) = g(f(x_2))$

since g is injective, then $f(x_1) = f(x_2)$

but since f is injective, then $x_1 = x_2$

Thus, $g \circ f(x_1) = g \circ f(x_2) \Rightarrow x_1 = x_2$, and therefore $g \circ f$ is injective

- proof for surjection

let $c \in C$. Since g is surjective, $g(b) = c$ for some $b \in B$

but since f is surjective, $f(a) = b$ for some $a \in A$

Thus, for any $c \in C$, $(g \circ f)(a) = g(b) = c$

Thus, $g \circ f$ is surjective

Therefore, since $g \circ f$ is injective and surjective, $g \circ f$ is bijective

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$f: A \to B \quad g: B \to C$

a) $g \circ f$ is injective. Let $x_1, x_2 \in A$

let $f(x_1) = f(x_2)$. But we also know that

$g(f(x_1)) = g(f(x_2))$. But since $g \circ f$ is injective:

$g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, so f in injective

b) $g \circ f$ is surjective. Let $c \in C$.

Since $g \circ f$ is surjective, $\exists\, a \in A$ such that

$(g \circ f)(a) = c$

By the def. of function composition, we know that

$(g \circ f)(a) = g(f(a))$

So, $g(f(a)) = c$.

Thus, g is surjective, since $\forall\, c \in C$ $\exists$ an element in B

(namely, $f(a)$) such that $g(f(a)) = c$

Therefore, g is surjective

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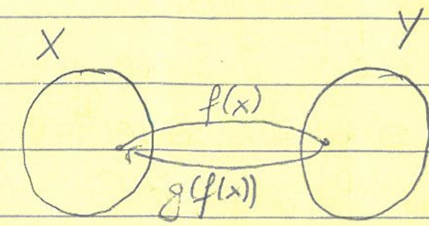

$(g \circ f)(x) = x \quad \forall\, x \in D(f)$
$(f \circ g)(y) = y \quad \forall\, y \in D(g)$
Want to show: $g = f^{-1}$!

$(g \circ f)(x) = g(f(x)) = x$ for all $x \in D(f)$
The function g takes all the outputs of f and assigns them the corresponding input. Therefore, domain$(g) \supseteq$ codomain(f).
Also, domain $(f) \subseteq$ codomain (g), since $g(f(x)) = x$ for all $x \in D(f)$

$(f \circ g)(y) = f(g(x)) = y$ for all $y \in D(g)$
The function f takes all the outputs of g and assigns them the corresponding input. Therefore, domain $(f) \supseteq$ codomain (g)
Also, domain $(g) \subseteq$ codomain (f), since $f(g(y)) = y$ for all $y \in D(g)$

Thus, $\left.\begin{array}{l} \text{domain}(g) \supseteq \text{codomain}(f) \\ \text{domain}(g) \subseteq \text{codomain}(f) \end{array}\right\} \Rightarrow \text{domain}(g) = \text{codomain}(f)$

$\left.\begin{array}{l} \text{domain} f \subseteq \text{codomain}(g) \\ \text{domain} f \supseteq \text{codomain}(g) \end{array}\right\} \Rightarrow \text{domain}(f) = \text{codomain}(g)$

Thus, $f(x) = y$, $g(y) = x$, for all x, y
In other words, $y = f(x)$ iff $x = g(y)$

Therefore, g satisfies the conditions for an inverse of f, and we can conclude that $g = f^{-1}$

(5)

a) $\forall n \in \mathbb{N},\ 1^3+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Base case:

$n=1$

$1^3 = 1$

$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$

so $1^3 = \left(\frac{1(1+1)}{2}\right)^2$. Thus, it is true for $n=1$

Inductive step.

Assume $1^3+2^3+\ldots+m^3 = \left(\frac{m(m+1)}{2}\right)^2$

Want to show $1^3+2^3+\ldots+n^3+(m+1)^3 = \left(\frac{(m+1)(m+2)}{2}\right)^2$

$\underbrace{1^2+2^2+\ldots+m^3}_{\text{we know what this is by assumption}}+(m+1)^3 = \left(\frac{m(m+1)}{2}\right)^2+(m+1)^3 =$

$= \left(\frac{m^2+m}{2}\right)^2 + (m^2+2m+1)(m+1) =$

$= \frac{m^4+2m^3+m^2}{4} + \frac{4(m^3+2m^2+m+m^2+2m+1)}{4} =$

$= \frac{m^4+6m^3+13m^2+12m+4}{4} = \frac{(m^2+3m+2)(m^2+3m+2)}{4} =$

$= \left(\frac{(m+1)(m+2)}{2}\right)^2$

Thus, by the principle of mathematical induction,

$1^2+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \forall n \in \mathbb{N}$

b) $\forall n \in \mathbb{N},\ \sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} \frac{n(n+1)}{2}$

Base case $(n=1)$

$$\sum_{k=1}^{1} (-1)^{1+1} 1^2 = 1$$

$$(-1)^{1+1} \frac{1(1+1)}{2} = 1 \cdot \frac{2}{2} = 1$$

So, it is true that $\sum_{k=1}^{1} (-1)^{1+1} 1^2 = (-1)^{1+1} \frac{1(1+1)}{2}$. So it is true for $n=1$

Inductive step

Assume $\sum_{k=1}^{m} (-1)^{k+1} k^2 = (-1)^{m+1} \frac{m(m+1)}{2}$

Want to show: $\sum_{k=1}^{m+1} (-1)^{k+1} k^2 = (-1)^{m+2} \frac{(m+1)(m+2)}{2}$

$$\sum_{k=1}^{m+1} (-1)^{k+1} k^2 = \underbrace{\left(\sum_{k=1}^{m} (-1)^{k+1} k^2\right)}_{\text{we know what this is by assumption}} + (-1)^{m+1+1} (m+1)^2 =$$

$$(-1)^{m+1} \frac{m(m+1)}{2} + (-1)^{m+2} (m+1)^2 = (-1)^{m+1} \left(\frac{m^2+m}{2} + (-1)(m^2+2m+1) \right) =$$

$$= (-1)^{m+1} \left(\frac{m^2+m}{2} - \frac{2m^2+4m+2}{2} \right) = (-1)^{m+1} \left(\frac{-m^2-3m-2}{2} \right) =$$

$$= (-1)^{m+2} \left(\frac{m^2+3m+2}{2} \right) = (-1)^{m+2} \frac{(m+1)(m+2)}{2}$$

Thus, by the principle of mathematical induction,

$$\forall n \in \mathbb{N},\ \sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} \frac{n(n+1)}{2}$$

c) $\forall n \in \mathbb{N}, n \geq 4, 2^n < n!$

Base case:

$n = 4$

$2^4 = 16$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

Thus, $2^4 < 4!$ and the base case is true

Inductive step.

Assume $k \geq 4$, $2^k < k!$

Want to show $2^{k+1} < (k+1)!$

We know that $2^k < k!$

$$2^k \cdot 2 < k! \cdot 2$$

$$2^{k+1} < k! \cdot 2$$

but since $2 < (k+1)$, for any $k \geq 4$, then

$$2^{k+1} < k! \cdot 2 < k!(k+1)$$

$$2^{k+1} < (k+1)!$$

Thus, by the principle of mathematical induction,

$$\forall n \in \mathbb{N}, n \geq 4, 2^n < n!$$