Homework 2. Advanced Calculus. Pawel Grzegrzolka 09/10/14

20/20

Problem 1

Let $a, b \in \mathbb{R}$ such that $a < b$.

Want to show: $a < \frac{a+b}{2} < b$

Let us first show that $a < \frac{a+b}{2}$.

We know that $a < b$. Thus:

$$a + a < b + a$$
$$2a < b + a$$
$$a < \frac{b+a}{2}$$

Then we want to show that $\frac{a+b}{2} < b$

We know that $a < b$. Thus;

$$a + b < b + b$$
$$a + b < 2b$$
$$\frac{a+b}{2} < b$$

Therefore, we can conclude that $a < \frac{a+b}{2} < b$

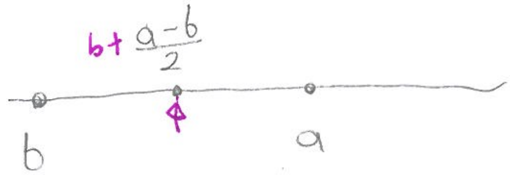
Problem 2

Let $a, b \in \mathbb{R}$. Suppose that $\forall \varepsilon > 0$, $a \leq b + \varepsilon$.

Want to show: $a \leq b$

Let us prove it by contradiction. Thus, assume that $\forall \varepsilon > 0$, $a \leq b + \varepsilon$

and $a > b$



Let $\varepsilon = \frac{a-b}{2}$. Since $a > b$, then $\varepsilon > 0$ and thus $a \leq b + \varepsilon$ should

be satisfied. But then

$$b + \varepsilon = b + \frac{a-b}{2} = \frac{2b + a - b}{2} = \frac{b+a}{2}$$

But from the previous problem we know that if $a > b$, then

$$b < \frac{a+b}{2} < a$$

Good! like that!

Thus, $b + \varepsilon = \frac{b+a}{2} < a$

But according to our assumption, $a \leq b + \varepsilon$. Thus, we have

a contradiction, and therefore we proved that

$$\forall \varepsilon > 0, \ a \leq b + \varepsilon \implies a \leq b$$

Problem 3

Let $a, b \in \mathbb{R}$

We want to show that $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$

We know that if $a, b \in \mathbb{R}$, then

$$(a-b)^2 \geq 0$$ (since any number squared is positive or 0)

$$a^2-2ab+b^2 \geq 0$$

$$-a^2+2ab-b^2 \leq 0$$

$$a^2+2ab+b^2 \leq 2a^2+2b^2$$

$$\frac{a^2+2ab+b^2}{4} \leq \frac{a^2+b^2}{2},$$

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$$, which finishes the proof.

if $a=b$, then

$$\left(\frac{a+b}{2}\right)^2 = \left(\frac{a+a}{2}\right)^2 = a^2 \quad , \quad \frac{a^2+b^2}{2} = \frac{a^2+a^2}{2} = \frac{2a^2}{2} = a^2$$

thus, equality holds.

Conversely, if equality holds, then

$$\left(\frac{a+b}{2}\right)^2 = \frac{a^2+b^2}{2}$$

$$\frac{a^2+2ab+b^2}{4} = \frac{a^2+b^2}{2}$$

$$a^2+2ab+b^2 = 2a^2+b^2$$

$$a^2-2ab+b^2 = 0$$

$\rightarrow (a-b)^2 = 0$, which implies that $a=b$.

Thus, we can claim that the equality holds if and only if $a=b$

Problem 4

a) Let $0<c<1$,

We want to show that $0<c^2<c<1$.

First, let us notice that $c^2 = c \cdot c$, and since c is positive, $c \cdot c$ is positive (by properties of positive real numbers). Thus, $c^2>0$

Also, we want to show that $c^2<c$. But this will happen if $c-c^2 \in \mathbb{P}$. So let us check. $c-c^2 = c(1-c)$. Since $c<1$, $1-c$ is positive. We know that c is positive.

Thus, a product of positive numbers is positive. Thus $c-c^2 \in \mathbb{P}$ and we can conclude that $c>c^2$. By transitivity we can sum up all the results and conclude that $0<c^2<c<1$

Also, we want to show that $\forall n \in \mathbb{N}$, $c^n \leq c$.

We use induction.

$\boxed{\text{base case}}$ $n=1$

$c^1 = c \leq c$

thus, base case is satisfied.

$\boxed{\text{inductive step}}$

assume it is true for some $k \in \mathbb{N}$. Thus $c^k \leq c$.

We want to show that $c^{k+1} \leq c$

We know that $c^k \leq c$. Then we can multiply both sides by c and the sign will not change, since $c>0$

$c^k \cdot c \leq c \cdot c \implies c^{k+1} \leq c \cdot c$.

But from the part a) of this problem we know that $c \cdot c < c$. Thus, we can conclude that $c^{k+1} \leq c \cdot c < c$. So, by PMI, $c^k \leq c$ $\forall n \in \mathbb{N}$
$c<1$

b) let $c > 1$

We want to show that $c^2 > c > 1$

$c^2 > c$ iff $c^2 - c$ is positive. But we know that $c^2 - c = c(c-1)$, and since c is positive, $c-1$ is positive (because $c > 1$), and the product of two positive numbers is positive, then $c^2 - c$ is positive and we can conclude that $c^2 > c$. Consequently, by transitivity, we can conclude that $c^2 > c > 1$

Also, we want to show that $\forall n \in \mathbb{N}$, $c^n \geq c$

We use induction.

$\boxed{\text{base case}}$ $n = 1$

$c^1 = c \geq c$

thus, base case is satisfied

$\boxed{\text{inductive step}}$

assume it is true for some $k \in \mathbb{N}$. Then, $c^k \geq c$

We want to show that $c^{k+1} \geq c$.

We know that $c^k \geq c$. Then we can multiply both sides by c and the sign will not change, since $c > 0$

$$c^k \cdot c \geq c \cdot c \implies c^{k+1} \geq c \cdot c$$

But from part b) of this problem we know that $c \cdot c > c$. Thus we can conclude that $c^{k+1} \geq c^2 > c$.

So, by PMI, $c^k \geq c$ $\forall n \in \mathbb{N}$, $c > 1$

Problem 5

a) let $a \in \mathbb{R}$

We want to show that $|a| = \sqrt{a^2}$

We have three cases

Case 1, $a = 0$

$$\left.\begin{array}{l} |0| = 0 \\ \sqrt{0^2} = \sqrt{0} = 0 \end{array}\right\} \text{ so } |0| = \sqrt{0^2}$$

Case 2 $a > 0$. Since $a > 0$, then $a^2 > 0$.

$|a| = a$

$\sqrt{a^2} = \sqrt{|a^2|} = \sqrt{|aa|} = \sqrt{|a||a|} = \sqrt{|a|^2} = |a| = a$.

Case 3, $a < 0$, let $b = -a$. Thus, $b > 0$

$|a| = -a = b$

$\sqrt{a^2} = \sqrt{((-1)b)^2} = \sqrt{(-1)^2 b^2} = \sqrt{(-1)^2}\sqrt{b^2}$

we know from b) of theorem 2.2.2 (p.32) that $|a|^2 = a^2 \; \forall a \in \mathbb{R}$

thus, $(-1)^2 = |-1|^2$ and therefore

$\sqrt{(-1)^2}\sqrt{b^2} = \sqrt{|-1|^2}\sqrt{b^2} = \sqrt{1}\sqrt{b^2} = \sqrt{b^2} = b$

from case 2

Thus, $|a| = \sqrt{a^2}$ for all $a \in \mathbb{R}$

b) let $a < x < b$
$a < y < b$

We want to show that $|x - y| < b - a$

(continued on the next page)

By theorem 2.1.7 c) (page 27) we can conclude that

$$a < y < b \implies a(-1) > y(-1) > b(-1)$$
$$-a > -y > -b$$

Now we add

$$\begin{array}{rl} & a < x < b \\ + & -b < -y < -a \\ \hline & -b+a < x-y < b-a \end{array}$$

Since $b-a > 0$, from theorem 2.2.2 c) (page 32)
we conclude that $|x-y| < b-a$

Problem 6

$$|x| + |x+1| < 2$$

		-1		0	
$\vert x\vert$	$-$		$-$		$+$
$\vert x+1\vert$	$-$		$+$		$+$

for $x > 0$

$$|x| + |x+1| = x + x + 1 < 2$$
$$2x < 1$$
$$x < \frac{1}{2} \quad \text{, thus } x \in (0, \tfrac{1}{2})$$

for $x \in \langle -1, 0 \rangle$

$$|x| + |x+1| = -x + x + 1 < 2$$
$$1 < 2 \quad \text{, thus } x \in [-1, 0]$$

for $x < -1$

$$|x| + |x+1| = -x - x - 1 < 2$$
$$-2x < 3$$
$$x > -\frac{3}{2} \quad \text{thus } x \in (-\tfrac{3}{2}, -1)$$

thus, $x \in (-\frac{3}{2}, \frac{1}{2})$

Problem 7

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let $\varepsilon > 0$, $\delta > 0$, $a \in \mathbb{R}$

$V_\varepsilon(a) \cap V_\delta(a)$

$V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$

$V_\delta(a) = \{x \in \mathbb{R} : |x-a| < \delta\}$. Thus

$V_\varepsilon(a) \cap V_\delta(a) = \{x : \mathbb{R} : |x-a| < \varepsilon$ and $|x-a| < \delta\}$

a) $V_\gamma(a) = V_\varepsilon(a) \cap V_\delta(a)$ iff $\gamma = \min(\varepsilon, \delta)$

$V_\varepsilon(a) \cup V_\delta(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon$ or $|x-a| < \delta\}$. Thus

b) $V_\gamma(a) = V_\varepsilon(a) \cup V_\delta(a)$ iff $\gamma = \max(\varepsilon, \delta)$

a) explanation

$x \in V_\varepsilon(a) \cap V_\delta(a) \Rightarrow |x-a| < \varepsilon$ and $|x-a| < \delta$ but since $\gamma = \min(\varepsilon, \delta)$,

$|x-a| < \gamma$ and thus $x \in V_\gamma(a)$

$x \in V_\gamma(a) \Rightarrow |x-a| < \gamma \Rightarrow$ since $\gamma = \min(\varepsilon, \delta)$,

$|x-a| < \varepsilon$ and $|x-a| < \delta$ and thus $x \in V_\varepsilon(a) \cap V_\delta(a)$.

Therefore, $V_\gamma(a) = V_\varepsilon(a) \cap V_\delta(a)$

b) Explanation is exact same as in a), only "$\cap$" changes into "$\cup$", "min" changes into "max", and "and" changes into "or".

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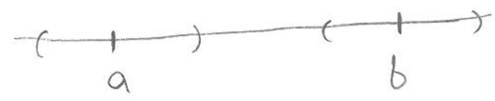


$a, b \in \mathbb{R} \quad a \neq b$

without loss of generality, assume $a < b$.

Let $\varepsilon = \frac{b-a}{2}$ $\quad$ ($\varepsilon > 0$, since $b > a$)

Then

$$U = V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x-a| < \varepsilon\}$$

$$\frac{a-b}{2} < x - a < \frac{b-a}{2}$$

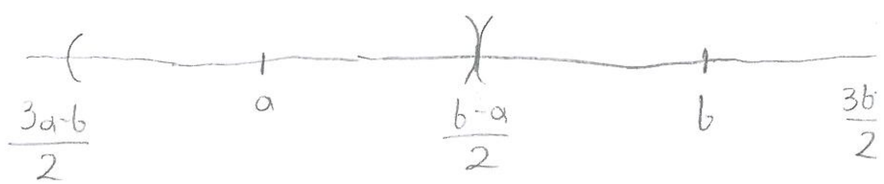
$$\frac{3a-b}{2} < x < \frac{b+a}{2} \qquad U = \{x \in (\tfrac{3a-b}{2}, \tfrac{b+a}{2})\}$$

$$V = V_\varepsilon(b) = \{x \in \mathbb{R} \mid |x-b| < \varepsilon\}$$

$$\frac{a-b}{2} < x - b < \frac{b-a}{2} \qquad V = \{x \in (\tfrac{b+a}{2}, \tfrac{3b-a}{2})\}$$

$$\frac{b+a}{2} < x < \frac{3b-a}{2}$$

Thus $U \cap V = \emptyset$



18. p 36

$a, b \in \mathbb{R}$

a) $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

f $\max(a,b) = a \Rightarrow a > b$, then $\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = a$ $\quad$ (since $a-b > 0$

f $\max(a,b) = b \Rightarrow b > a$, then $\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b-a+b) = b$ $\quad$ (since $a-b < 0$

thus, $\frac{1}{2}(a+b+|a-b|) = \max\{a,b\}$

$\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$

if $\min\{a,b\} = a$, then $a<b$ and thus $a-b<0$. Thus

$$\tfrac{1}{2}(a+b-|a-b|) = \tfrac{1}{2}(a+b-(b-a)) = a$$

if $\min\{a,b\} = b$, then $a>b$ and thus $a-b>0$. Thus

$$\tfrac{1}{2}(a+b-|a-b|) = \tfrac{1}{2}(a+b-(a-b)) = b$$

Thus, $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$

b) $\min\{a,b,c\} = \min\{\min\{a,b\}, c\}$

*i) if $\min\{a,b,c\} = a$, then $\min\{a,b\} = a$ and $\min\{a,c\} = a$

*ii) if $\min\{a,b,c\} = b$, then, $\min\{a,b\} = b$ and $\min\{b,c\} = b$

*iii) if $\min\{a,b,c\} = c$, then $\min\{a,b\} = a$ a $\min\{a,b\} = b$

if $\min\{a,b\} = a$, then $\min\{a,c\} = c$

if $\min\{a,b\} = b$, then $\min\{b,c\} = c$

*i) $\min\{a,b,c\} = a$ means $a>b$ & $a>c$

*ii) $\min\{a,b,c\} = b$ means $b>a$ & $b>c$

*iii) $\min\{a,b,c\} = c$ means $c>a$ & $c>b$

(19. p 36)

$a, b, c \in \mathbb{R}$

You could have assumed just $a \le b \le c$.
The other cases is just the renaming of this case.

want to show :

$\text{mid}\{a,b,c\} = \min\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\}$

(I) if $\text{mid}\{a,b,c\} = a$, then $\underset{①}{b<a<c}$ or $\underset{②}{c<a<b}$

① if $b<a<c$, then $\max\{a,b\}=a$, $\max\{b,c\}=c$, $\max\{c,a\}=c$,

and thus $\min\{a,c,c\}=a$

② if $c<a<b$, then $\max\{a,b\}=b$, $\max\{b,c\}=b$, $\max\{c,a\}=a$,

and thus $\min\{b,b,a\}=a$

(II) if $\text{mid}(a,b,c\} = b$, then $\underset{③}{a<b<c}$ or $\underset{④}{c<b<a}$

③ if $a<b<c$, then $\max\{a,b\}=b$, $\max\{b,c\}=c$, $\max\{c,a\}=c$,

and thus $\min\{b,c,c\}=b$

④ if $c<b<a$, then $\max\{a,b\}=a$, $\max\{b,c\}=b$, $\max\{c,a\}=a$,

and thus $\min\{a,b,a\}=b$

(III) if $\text{mid}(a,b,c\} = c$, then $\underset{⑤}{a<c<b}$ or $\underset{⑥}{b<c<a}$

⑤ if $a<c<b$, then $\max\{a,b\}=b$, $\max\{b,c\}=b$, $\max\{c,a\}=c$,

and thus $\min\{b,b,c\}=c$

⑥ if $b<c<a$, then $\max\{a,b\}=a$, $\max\{b,c\}=c$, $\max\{c,a\}=a$,

and thus $\min\{a,c,a\}=c$

Thus, we can conclude that

$\text{mid}\{a,b,c\} = \min\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\}$