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Math 447: Homework 3

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Due date: Wednesday, September 17, 2014.

1. Find the infimum and supremum, whenever they exist, of the following sets. Justify (or explain) your answer.

   (a) $S_1 = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

   Answer. Note that $S_1 = \{2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5}, \frac{5}{6}, ...\}$. Elements have been written corresponding to $n \in \mathbb{N}$. It seems that 2 is the supremum and 1/2 is the infimum. Let this be demonstrated.

   - (2 is the supremum.) It must be shown that 2 is an upper bound for S_1 and that 2 is the least upper bound of S_1. The inequality $1 - \frac{(-1)^n}{n} \leq 2$ is true exactly when $-(-1)^n \leq n$ since $n > 0$ by $n \in \mathbb{N}$. Obviously, this implies the inequality is true for all $n \in \mathbb{N}$. Thus, 2 is an upper bound. It remains to be shown that 2 is the least upper bound. Consider u, any upper bound of S_1. Thus, for every element x in S_1, $u \geq x$, but $2 \in S_1$, so $2 \leq u$ for any upper bound u of S_1. Thus, 2 is the supremum of S_1.
   - ($\frac{1}{2}$ is the infimum.) It must be shown that 1/2 is an upper bound of S_1 and that 1/2 is the least upper bound of S_1. The inequality $1 - \frac{(-1)^n}{n} \geq \frac{1}{2}$ is true exactly when $n \geq 2(-1)^n$. Clearly, this inequality is true for all $n \in \mathbb{N}$, implying that 1/2 is a lower bound of S_1. It must be shown that 1/2 is the greatest lower bound. Suppose l is a lower bound of S_1. Thus, for any $x \in S_1$, $l \leq x$. Thus, since $1/2 \in S_1$, $l \leq 1/2$ for any lower bound of S_1 and 1/2 is the infimum.

   (b) $S_2 = \left\{x \in \mathbb{R} : x < \frac{1}{x}\right\}$

   Answer. Solving the inequality $x < \frac{1}{x}$ implies that $S_2 = \left\{x \in \mathbb{R} : -\infty < x < -1 \text{ or } 0 < x < 1\right\}$. Since the set has no lower bound, the infimum does not exist by the Completeness Property for Infima. Since an upper bound exists for the set, however, the supremum does exist. Consider 1. For every element $x \in S_2$, $x < 1$, so 1 is an upper bound. Is it the least upper bound? Thus, suppose $v < 1$ for some $v \in \mathbb{R}$. Thus, either $0 \leq v$ or $v > 0$. Should $0 \leq v$, choose any $s' \in \{x \in \mathbb{R} : 0 < x < 1\} \subseteq S_2$ and $v < s'$. In the other case, where $v > 0$, choose $s' = \frac{v+1}{2}$. Since $0 < v < 1$, $2v < v + 1$, implying that $v < \frac{v+1}{2}$. By Lemma 2.3.3, 1 is the supremum of S_2.

(c) $S_3 = \left\{x \in \mathbb{R} : x + 2 \geq x^2\right\}$

Answer. Solving the inequality $x+2 \geq x^2$ implies that $S_3 = \{x \in \mathbb{R} : -1 \leq x \leq 2\}$. Thus, the set is bounded and by the Completeness Property, the supremum and infimum exist. Consider 2. Since $x \leq 2$ for all $x \in S_3$, 2 is an upper bound. For any upper bound $b \in \mathbb{R}$, $x \leq b$ for all $x \in S_3$ so $2 \leq b$ for any upper bound b of S_3. Therefore, 2 is the supremum. Consider -1. $-1 \leq x$ for all $x \in S_3$, so it is a lower bound. For any lower bound b of S_3, b is less than every element of S_3, so $b \leq -1$. Thus, -1 is the infimum.

(d) $S_4 = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$

Answer. Note that should $n = 1$ and m be large, $\frac{1}{n} - \frac{1}{m}$ will be very close to 1. Similarly, should $m = 1$ and n be large, this difference will be close to -1. As n and m vary in $\mathbb{N}$ otherwise, this difference lies between -1 and 1. Thus, it is conjectured that 1 is the supremum and -1 is the infimum. It has already been discussed that they are upper and lower bounds respectively. It remains to be determined whether they are the least upper bound and greatest lower bound, however. Consider 1. Let $\epsilon > 0$. Let $n = 1$ and choose $m \in \mathbb{N}$ by the Archimedean Principle such that $m > \frac{1}{\epsilon}$. Let $d = \frac{1}{n} - \frac{1}{m} \in S_4$. Thus, $1 - \epsilon = \frac{1}{n} - \epsilon < \frac{1}{n} - \frac{1}{m} = d$. Thus, by Lemma 2.3.4, 1 is the supremum of this set. Consider -1, and let $\epsilon > 0$. Let $m = 1$ and choose $n \in \mathbb{N}$ by the Archimedean Principle such that $n > \frac{1}{\epsilon}$. Let $d = \frac{1}{n} - \frac{1}{m} \in S_4$. Thus, $d = \frac{1}{n} - \frac{1}{m} < \epsilon - 1 = -1 + \epsilon$. Thus, by the corresponding lemma to Lemma 2.3.4 found in Exercise 14 on page 40, -1 is the infimum.

2. (Compatibility of sup/inf with algebraic operations)
Given nonempty subsets A and B of $\mathbb{R}$ and $k \in \mathbb{R}$, we define the following subsets of $\mathbb{R}$:

$$\begin{aligned} kA &:= \{k \cdot a : a \in A\} \\ k + A &:= \{k + a : a \in A\} \\ A + B &:= \{a + b : a \in A, b \in B\}. \end{aligned}$$

Assume that A and B are nonempty bounded subsets of $\mathbb{R}$. Prove (**any two of**) the following:

(a) If $k > 0$, then $\inf(kA) = k\inf(A)$, $\sup(kA) = k\sup(A)$.

Proof. Let $k > 0$. It is to be shown that that infimum of kA is $k \inf A$ and the supremum of kA is $k \sup A$.

- ($\inf kA = k \inf A$):
  - ($k \inf A$ is a lower bound): Note that for every $a \in A$, $\inf A \leq a$. Since $k > 0$, $k \inf A \leq ka$. Thus for any element ka of kA, $k \inf A$ is less than or equal to that element, i.e., $k \inf A$ is a lower bound of kA.
  - ($k \inf A$ is the greatest lower bound): Let l be a lower bound of kA, implying that $l \leq ka$ for any $a \in A$. Thus, since $k > 0$, $\frac{l}{k} \leq a$. This

implies that $\frac{l}{k}$ is a lower bound of A. Thus, $\frac{l}{k} \leq \inf A$. Again, since $k > 0$, $l \leq k \inf A$. Thus, $k \inf A$ is the greatest lower bound of kA.

Therefore, $\inf kA = k \inf A$.

- ($\sup kA = k \sup A$):
  - ($k \sup A$ is an upper bound): Note that for any $a \in A$, $\sup A \geq a$. Thus, since $k > 0$, $k \sup A \geq ka$. Thus, $k \sup A$ is an greater than or equal to any element ka in kA, i.e., $k \sup A$ is an upper bound of kA.
  - ($k \sup A$ is the least upper bound): Let u be an upper bound of kA, implying that $u \geq ka$ for all $a \in A$. Thus, since $k > 0$, $\frac{u}{k} \geq a$. This implies that $\frac{u}{k}$ is an upper bound for A. Thus, $\frac{u}{k} \geq \sup A$, implying that $u \geq k \sup A$ since $k > 0$. Thus, $k \sup A$ is the least upper bound of kA.

  Therefore, $\sup kA = k \sup A$.

□

(b) If $k < 0$, then $\inf kA = k \sup A$, $\sup kA = k \inf A$.

Proof. Let $k < 0$. It is to be shown that that infimum of kA is $k \sup A$ and the supremum of kA is $k \inf A$.

- ($\inf kA = k \sup A$):
  - ($k \sup A$ is a lower bound): Note that for every $a \in A$, $a \leq \sup A$. Since $k < 0$, $k \sup A \leq ka$. Thus for any element ka of kA, $k \sup A$ is less than or equal to that element, i.e., $k \sup A$ is a lower bound of kA.
  - ($k \sup A$ is the greatest lower bound): Let l be a lower bound of kA, implying that $l \leq ka$ for any $a \in A$. Thus, since $k < 0$, $\frac{l}{k} \geq a$. This implies that $\frac{l}{k}$ is an upper bound of A. Thus, $\frac{l}{k} \geq \sup A$. Again, since $k < 0$, $l \leq k \sup A$. Thus, $k \sup A$ is the greatest lower bound of kA.

  Therefore, $\inf kA = k \sup A$.
- ($\sup kA = k \inf A$):
  - ($k \inf A$ is an upper bound): Note that for any $a \in A$, $\inf A \leq a$. Thus, since $k < 0$, $k \inf A \geq ka$. Thus, $k \sup A$ is an greater than or equal to any element ka in kA, i.e., $k \sup A$ is an upper bound of kA.
  - ($k \inf A$ is the least upper bound): Let u be an upper bound of kA, implying that $u \geq ka$ for all $a \in A$. Thus, since $k < 0$, $\frac{u}{k} \leq a$. This implies that $\frac{u}{k}$ is an lower bound for A. Thus, $\frac{u}{k} \leq \inf A$, implying that $u \geq k \inf A$ since $k < 0$. Thus, $k \inf A$ is the least upper bound of kA.

  Therefore, $\sup kA = k \inf A$.

□

(c) $\sup (A + B) = \sup A + \sup B$, $\inf (A + B) = \inf A + \inf B$.

(d) $\sup (A \cup B) = \sup \{\sup (A), \sup (B)\}$, $\inf (A \cup B) = \inf\{\inf (A), \inf (B)\}$.

3. (The greatest integer function)

(a) Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $n \leq x < n+1$. *[n is the greatest integer less than or equal to x (sometimes called the floor of x) and is denoted by $\lfloor x \rfloor$. $n+1$ is the smallest integer greater than x (sometimes called the ceiling of x and is denoted by $\lceil x \rceil$)]*

Proof. Let $x \in \mathbb{R}$.

Case I: Suppose that $x \geq 0$. Thus, consider the subset $E_x := \{n \in \mathbb{N} : x < n\}$ of $\mathbb{N}$, which is nonempty by the Archimedean Property. By the Well-Ordering Principle, this E_x has a unique least element. Let this element be known as $n_x + 1$ for some $n_x \in \mathbb{Z}$. Thus, $n_x \notin E_x$, implying $n_x \leq x < n_x + 1$.

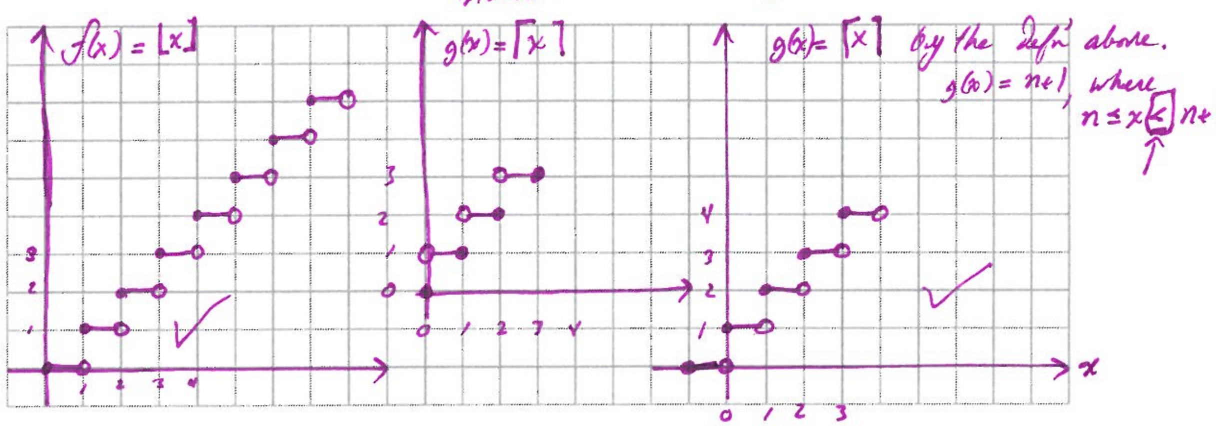
Case II: Suppose that $x < 0$. Consider the subset $-E_x := \{-n \in \mathbb{N} : -x \leq -n\}$ of $\mathbb{N}$ (note that $-n > 0$, since $n \in \mathbb{N}$), which is nonempty by the Archimedean Property. Thus, by the Well-Ordering Principle, this $-E_x$ has a unique least element $-n_x \in \mathbb{N}$ for $n_x \in \mathbb{Z}$. Thus, $-n_x - 1 \in \mathbb{Z}$ is not an element of $-E_x$, so $-n_x - 1 < -x \leq -n_x$, implying that $n_x \leq x < n_x + 1$.

Note that, in either case, the element n_x is unique by the Well-Ordering Principle. Thus, for any $x \in \mathbb{R}$, there exists a unique $n_x \in \mathbb{Z}$ such that $n_x \leq x < n_x + 1$. □

(b) Sketch the graphs of the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

Answer.

standard ceil where $g(x) = n+1$, where $n < x \leq n+1$



4. (Supremum/infimum of a function)

Definition 1. *Suppose D is a nonempty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ is a function.*

- *f is bounded above **(below)** if the range of f, $f(D)$, is bounded above **(below)**. f is a bounded function if $f(D)$ is a bounded set.*
- *Whenever they exist,* $\sup\limits_{x \in D} f(x) = \sup\{f(x) : x \in D\} := \sup(f(D))$; $\inf\limits_{x \in D} f(x) = \inf\{f(x) : x \in D\} := \inf(f(D))$.

Prove the following:

(a) f is bounded if and only if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$, for all $x \in D$.

Proof. Suppose that D is a nonempty subset of $\mathbb{R}$ and that $f : D \to \mathbb{R}$ is a function.

($\Longrightarrow$): Suppose that f is bounded. Thus, $f(D)$ is a bounded set, that is, $f(D)$ is bounded above and below. Let $x \in D$ Since $f : D \to \mathbb{R}$ is a function, $f(x) \in f(D)$. Since $f(D)$ is bounded, there exists $u, l \in \mathbb{R}$ such that $l \leq f(x) \leq u$. Choose $M := \max\{|l|, |u|\}$. By Theorem 2.2.2(d) in the text, $-M \leq l \leq f(x) \leq u \leq M$, implying that $-M \leq f(x) \leq M$. By the Fundamental Theorem of Absolute Value (Theorem 2.2.2(c)), $|f(x)| \leq M$. Therefore, if f is bounded, there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$, for all $x \in D$.

($\Longleftarrow$): Suppose that there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$. Thus, by Theorem 2.2.2(c), this implies that for all $x \in D$, $-M \leq f(x) \leq M$. Thus, M is an upper bound and $-M$ is a lower bound for the range of f, $f(D)$. Since the set is bounded above and below, i.e., $f(D)$ is bounded, f is bounded.

Therefore, f is bounded if and only if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$, for all $x \in D$. □

(b) Suppose that f and g are bounded functions with common domain D. Assume that $f(x) \leq g(x)$, for all $x \in D$. Then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x).$$

Proof. Suppose that $f(x) \leq g(x)$, for all $x \in D$. Since f and g are bounded, $f(D)$ and $g(D)$ are bounded sets, implying that $\sup f(x) = \sup f(D)$ and $\inf f(x) = \inf f(D)$ exist by the Completeness Property. Recalling that a supremum is an upper bound, note that $f(x) \leq \sup f(x)$ and $g(x) \leq \sup g(x)$ for all $x \in D$. Thus, since $f(x) \leq g(x)$ for all $x \in D$, $f(x) \leq \sup g(x)$. Thus, $\sup g(x)$ is an upper bound for $f(D)$. However, since $\sup f(x)$ is a supremum, it is the least upper bound, implying that $\sup f(x) \leq \sup g(x)$. □

5. Do Exercise #8 in Section 2.4.

Problem 8, Section 2.4, page 45. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}$$

Give examples to show that each of these inequalities can be either equalities or strict inequalities.

Answer.

Proof. Suppose that X is a nonempty set, and f and g are defined on X, each having bounded ranges in $\mathbb{R}$.

- ($\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$):
Since f and g each have bounded ranges in $\mathbb{R}$, the Completeness Property guarantees the existence of $s_{f(x)+g(x)} := \sup\{f(x)+g(x) : x \in X\}$, $s_{f(x)} := \sup\{f(x) : x \in X\}$, and $s_{g(x)} := \sup\{g(x) : x \in X\}$. For all $x \in \mathbb{R}$, it is known by definition that $f(x)+g(x) \leq s_{f(x)+g(x)}$, $f(x) \leq s_{f(x)}$, and $g(x) \leq s_{g(x)}$. From the latter two of these inequalities, one may deduce that $f(x)+g(x) \leq s_{f(x)} + s_{g(x)}$, that is, that $s_{f(x)} + s_{g(x)}$ is an upper bound for $\{f(x)+g(x) : x \in X\}$. However, since $s_{f(x)+g(x)}$ is the supremum of $\{f(x)+g(x) : x \in X\}$, $s_{f(x)+g(x)} \leq s_{f(x)} + s_{g(x)}$. Therefore, $\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$.
- ($\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x)+g(x) : x \in X\}$):
Since f and g each have bounded ranges in $\mathbb{R}$, the Completeness Property for Infima (discussed in the text on page 39) guarantees the existence of $i_{f(x)+g(x)} := \inf\{f(x)+g(x) : x \in X\}$, $i_{f(x)} := \inf\{f(x) : x \in X\}$, and $i_{g(x)} := \inf\{g(x) : x \in X\}$. For all $x \in \mathbb{R}$, it is known by definition that $i_{f(x)+g(x)} \leq f(x)+g(x)$, $i_{f(x)} \leq f(x)$, and $i_{g(x)} \leq g(x)$. From the latter two of these inequalities, one may deduce that $i_{f(x)} + i_{g(x)} \leq f(x)+g(x)$, that is, that $i_{f(x)} + i_{g(x)}$ is a lower bound for $\{f(x)+g(x) : x \in X\}$. However, since $i_{f(x)+g(x)}$ is the infimum of $\{f(x)+g(x) : x \in X\}$, $i_{f(x)} + i_{g(x)} \leq i_{f(x)+g(x)}$. Therefore, $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x)+g(x) : x \in X\}$.

□

Examples. For sake of simplicity, a simple lemma will be proven.

Lemma. *Let $a, b \in \mathbb{R}$ such that $a \leq b$. For $E := \{x \in \mathbb{R} : a \leq x \leq b\}$, $\inf E = a$ and $\sup E = b$.*

Proof. Let $a, b \in \mathbb{R}$ such that $a \leq b$. Consider $E := \{x \in \mathbb{R} : a \leq x \leq b\}$. Note that b is an upper bound of E since for every $x \in E$, $x \leq b$. For any upper bound u of E, $x \leq u$ for all $x \in u$, so $b \leq u$ for all upper bounds u of E. Thus, $\sup E = b$. Similarly, a is a lower bound of E since for every $x \in E$, $a \leq x$. Furthermore, for any lower bound l of E, $l \leq x$ for all $x \in E$. Thus, $l \leq a$ and $\inf E = a$. □

In the following examples, the above lemma will be used. Since the ranges of the functions below are sets as described in the lemma (closed intervals), the maximums and minimums of these functions on the intervals (found by elementary calculus) are taken to be the suprema and infima.

(a) Suppose that $X = \{x \in \mathbb{R} : 0 \leq x \leq 5\}$ and suppose that $f(x) = x$ and $g(x) = 2x + 3$. Thus, $\sup\{f(x)+g(x) : x \in X\} = 18$, $\inf\{f(x)+g(x) : x \in X\} = 3$, $\sup\{f(x) : x \in X\} = 5$, $\sup\{g(x) : x \in X\} = 13$, $\inf\{f(x) : x \in X\} = 0$, and $\inf\{g(x) : x \in X\} = 3$. Thus, for this example,

$$\sup\{f(x)+g(x) : x \in X\} = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} = \inf\{f(x) + g(x) : x \in X\}.$$

(b) Suppose that $X = \{x \in \mathbb{R} : 0 \leq x \leq 3\}$ and suppose that $f(x) = x$ and $g(x) = -x^2$. Thus, $\sup\{f(x) + g(x) : x \in X\} = \frac{1}{4}$, $\inf\{f(x) + g(x) : x \in X\} = -6$, $\sup\{f(x) : x \in X\} = 3$, $\sup\{g(x) : x \in X\} = 0$, $\inf\{f(x) : x \in X\} = 0$, and $\inf\{g(x) : x \in X\} = -9$. Thus, for this example,

$$\sup\{f(x) + g(x) : x \in X\} = \frac{1}{4} < 3 = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} = -9 < -6 = \inf\{f(x) + g(x) : x \in X\}.$$

Thus, in the first example, strict equality holds, but in the second example, strict inequality holds.

6. Do **(any two of)** Exercise #9, 10, 11, 12 in Section 2.4.

9. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.

(a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; find $\inf\{f(x) : x \in X\}$.

Proof. Let $x \in X$. It is conjectured that $s := 2x + 1$ is the supremum of $\{h(x, y) : y \in Y\}$. Let this be demonstrated.

(s is an upper bound): Let $h \in \{h(x, y) : y \in Y\}$. Thus, for some $y \in Y$, $h = 2x + y$, but since $y \in Y$, $0 < y < 1$, implying that $2x + y < 2x + 1$. Thus, for any $h \in h \in \{h(x, y) : y \in Y\}$, $h \leq s$, and s is an upper bound.

(The upper bound s is the supremum (see Lemma 2.3.4, page 38)): Let $\epsilon > 0$. Choose $h := 2x + y \in \{h(x, y) : y \in Y\}$ for $y \in Y$ such that $y = \max\{1 - \frac{\epsilon}{2}, \frac{1}{2}\}$. Thus, for any ϵ, y is guaranteed to satisfy $0 < y < 1$. From this, it is known that $y - (1 - \epsilon) \geq 1 - \frac{\epsilon}{2} - 1 + \epsilon = \epsilon - \frac{\epsilon}{2} > 0$. Thus,

$$\begin{aligned} y - (1 - \epsilon) &> 0 \\ \implies 1 - \epsilon &< y \\ \implies 2x + 1 - \epsilon &< 2x + y \\ \implies s - \epsilon &< h. \end{aligned}$$

Thus, for any $\epsilon > 0$ there exists $h \in \{h(x, y) : y \in Y\}$ such that $s - \epsilon < h$. Therefore, by Lemma 2.3.4, s is the supremum and $f(x) = 2x + 1$.

Thus, since $0 < x < 1$, it is conjectured that $\inf\{f(x) : x \in X\} = 1$. This shall be demonstrated.

(1 is an lower bound): Since for all $x \in X$, $0 < x < 1$, $2x > 0$. This implies that $2x + 1 > 1$ (loosely, $2x + 1 \geq 1$) for all $x \in X$. Thus, 1 is a lower bound.

(1 is the greatest lower bound (see Theorem: Exercise 14, page 40)): Let $\epsilon > 0$. Let $y = f(x)$ be chosen in $\{f(x) : x \in X\}$ for $x := \min\{\frac{\epsilon}{4}, \frac{1}{2}\}$. Thus, since $x < \frac{\epsilon}{2}$, $y = 2x + 1 < 2(\frac{\epsilon}{2}) + 1 = 1 + \epsilon$. Thus, by the theorem referenced above, for every $\epsilon > 0$, there exists $y \in \{f(x) : x \in X\}$ such that $y < 1 + \epsilon$. Therefore, 1 is the infimum of $\{f(x) : x \in X\}$.

□

(b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

Proof. Let $y \in Y$. It is conjectured that $b := y$ is the infimum of $H := \{h(x, y) : x \in X\}$. Let this be demonstrated.

(b is an lower bound): Let $h \in H$. Thus, for some $x \in X$, $h = 2x + y$. Since $0 < x < 1$, $0 < 2x$, implying that $y < 2x + y$ for all $x \in X$. Thus, for all $h \in H$, $b \leq h$ and b is a lower bound.

(The lower bound b is the infimum): Let $\epsilon > 0$. Let $h = 2x + y \in H$ be chosen such that $x = \min\{\frac{\epsilon}{4}, \frac{1}{2}\}$. Thus, $x < \frac{\epsilon}{2}$. That is, $h = 2x + y < 2(\frac{\epsilon}{2}) + y = b + \epsilon$. Therefore, since $h < b + \epsilon$, b the infimum of H and $g(y) := y$.

Thus, since $0 < y < 1$, it is conjectured that 1 is the supremum of $G := \{g(y) : y \in Y\}$. Let this be shown.

(1 is an upper bound): Let $y \in Y$. By definition of Y, $y < 1$ and 1 is an upper bound.

(The upper bound 1 is the supremum): Let $\epsilon > 0$. Let $g = y \in G$ chosen such that $y = \max\{1 - \frac{\epsilon}{2}, \frac{1}{2}\}$. Thus, $1 - \epsilon = 1 - 2(\frac{\epsilon}{2}) < 1 - \frac{\epsilon}{2} \leq y = g$, implying $1 - \epsilon < g$. Therefore, since for any $\epsilon > 0$ there exists $g \in G$ such that $1 - \epsilon < g$, 1 is the supremum of G.

As a final note, it is interesting that $\inf\{f(x) : x \in X\} = \sup\{g(y) : y \in Y\}$.

□

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

Proof. It is conjectured that $f(x) := \sup\{h(x, y) : y \in Y\}$ is 1. By the definition of $h(x, y)$, it is known that $H := \{h(x, y) : y \in Y\} = \{0, 1\}$. Thus, $1 \geq 0$ and $1 \geq 1$, so 1 is an upper bound of H. Consider any other upper bound $b \in \mathbb{R}$. By definition, b is greater than or equal to all elements of H, so $b \geq 1$. Thus, $f(x) = 1$. Since $f(x)$ is constant, the set $G := \{f(x) : x \in X\} = \{1\}$. Thus, 1 is a lower bound since $1 \leq 1$ and is the infimum since it is a lower bound and for any $b \in \mathbb{R}$ such that $b \leq 1$, $b \leq 1$. It is further conjectured that $g(y) := \inf\{h(x, y) : x \in X\} = 0$. As above, $H := \{h(x, y) : y \in Y\} = \{0, 1\}$.

As to not tire the reader, recall the statement at the top of page 39 in the text that if a nonempty set has a finite number of elements, then it has a largest and smallest element. Furthermore, it states that the largest element is the supremum and the smallest element is the infimum. Thus, since 0 is the smallest element of H, $g(y) = 0$. Since $g(y)$ is constant, $\{g(y) : y \in Y\} = \{0\}$, implying by the statement from the text that $\sup\{g(y) : y \in Y\} = 0$. $\square$