Midterm I

Math 447: Honors Advanced Calculus I

October 03, 2014

Name: ______________________________

Student Number. :______________________

Signature:____________________________

Problem No	1	2	3	4	Total
Points					

INSTRUCTIONS:

1. The use of a calculator, cell phone, or any other electronic device is not permitted during this examination.
2. The use of notes of any kind is not permitted during this examination.
3. Each student should be prepared to produce his/her ID upon request.
4. Read and observe the following rules:
   (a) Students are not permitted to ask questions of the proctors, except in cases of supposed errors or ambiguities in examination questions.
   (b) CAUTION - Students guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
   - Making use of any books, papers or memoranda, other than those authorized by the examiners.
   - Speaking or communicating with other students.
   - Purposely exposing written papers to the view of other students. The plea of accident or forgetfulness shall not be received.

1. (a) (5 Pts.) State the Archimedean Property of $\mathbb{R}$. (Be precise!)

For every $x \in \mathbb{R}$, there is a natural number n_x such that $x < n_x$.

(b) (10 Pts.) Prove that if a and b are real numbers and $0 \leq a < b$, then there exist $m, n \in \mathbb{N}$ such that $a < \dfrac{m}{10^n} < b$.

(Hint: You may use proof by contradiction argument.)

Let $0 \leq a < b$ be given. Suppose the conclusion of the statement is false. That is, suppose that for all $n, m \in \mathbb{N}$ either $a \geq \dfrac{m}{10^n}$ or $b \leq \dfrac{m}{10^n}$. Now if $a \geq \dfrac{m}{10^n}$ for all $n, m \in \mathbb{N}$, then in particular, $a \geq \dfrac{m}{10^1}$ for all $m \in \mathbb{N}$. This implies that $10\,a$ is an upper bound for $\mathbb{N}$, which contradicts the Archimedean Property. If $b \leq \dfrac{m}{10^n}$ for all $n, m \in \mathbb{N}$, then in particular, $b \leq \dfrac{1}{10^n}$, for all $n \in \mathbb{N}$. That is, the positive number b is a lower bound for $\left\{\dfrac{1}{10^n} : n \in \mathbb{N}\right\}$. But again this contradicts the Archimedean Property since $\inf\left\{\dfrac{1}{10^n} : n \in \mathbb{N}\right\} = 0$. Therefore, the conclusion of the statement is true.

2. (a) (5 Pts.) State the Completeness Property of $\mathbb{R}$. (Be precise!)

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Or

Every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

(b) (10 Pts.) Suppose that A, B are nonempty subsets of $\mathbb{R}$ and $E = A \cup B$. Assume that E has a supremum. Show that $\sup A$ and $\sup B$ both exist. Moreover, $\sup E$ is one of the numbers $\sup A$ or $\sup B$.

To show that $\sup A$ and $\sup B$ both exist, we only need to check that the sets are bounded above by the Completeness Property, since A and B are nonempty. But since $\sup E$ exists, $\sup E \geq s$, for all $s \in E$. In particular $\sup E \geq a$ for all $a \in A$, since $a \in A \implies a \in E$. That is A is bounded above and $\sup E$ is a upper bound for A. Similarly, $\sup E$ is an upper bound for B.

To show that $\sup E$ is one of the numbers $\sup A$ or $\sup B$. it suffices to show that $\sup E = \max\{\sup A, \sup B\}$. To that end, let us assume without loss of generality that $\max\{\sup A, \sup B\} = \sup A$. Clearly $\sup A \leq \sup E$. If, on the other hand, $\sup A < \sup E$, there $\sup A$ is not an upper bound for E. Then there exists $s \in E$ such that $s > \sup A$. This implies that $s \notin A$. Also $s \notin B$, since $s > \sup A \geq \sup B$. That is $s \notin A \cup B = E$, a contradiction. Therefore $\sup A \geq \sup E$. and so $\sup E = \sup A$.

3. For given real numbers a and r, $r \neq 1$, consider the sequence (x_n), where

$$x_n = \sum_{k=1}^{n} a\, r^{k-1} = a(1 + r + r^2 + \cdots + r^{n-1}).$$

(a) (5 Pts.) Use induction to show that for all $n \in \mathbb{N}$, $x_n = \dfrac{a(1-r^n)}{1-r}$.

For $n = 1$, $x_1 = a = \dfrac{a(1-r^1)}{1-r}$, the statement is true. Assume now that the statement is true for n. Let us show it is also true for $n+1$.

$$x_{n+1} = \sum_{k=1}^{n+1} ar^{k-1} = \sum_{k=1}^{n} ar^{k-1} + ar^n = \frac{a(1-r^n)}{1-r} + a\, r^n$$

where we have used the induction assumption in the last equality. Simplifying the last expression we have

$$x_{n+1} = \frac{a(1-r^n)}{1-r} + a\, r^n = \frac{a(1-r^n) + a\, r^n(1-r)}{1-r} = \frac{a(1-r^{n+1})}{1-r}$$

which is exactly what we wanted to show. Then by the Principle of Mathematical Induction, the statement is true for all $n \in \mathbb{N}$.

(b) (5 Pts.) When $-1 < r < 1$, show that $\lim\limits_{n\to\infty} x_n = \dfrac{a}{1-r}$. From part a)

$$x_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} r^n.$$

Then for each $n \in \mathbb{N}$,

$$\left| x_n - \frac{a}{1-r} \right| \leq C|r|^n, \quad (C = \frac{a}{1-r})$$

Since $|r| < 1$, $\lim\limits_{n\to\infty} |r|^n = 0$. (We proved this in class using Bernoulli's inequality.) Therefore,

$$\lim_{n\to\infty} |x_n - \frac{a}{1-r}| = 0,$$

which is equivalent to $\lim\limits_{n\to\infty} x_n = \dfrac{a}{1-r}$.

(c) (5 Pts.) When $r > 1$, show that (x_n) is an unbounded sequence.

From part a) $x_n = \dfrac{a}{1-r} - \dfrac{a}{1-r} r^n = \dfrac{a}{1-r} + \dfrac{a}{r-1} r^n$. To show that x_n is unbounded it suffices to show that r^n is unbounded. Now since $r > 1$, we can write $r = 1 + h$, for some $h > 0$. Then by Bernoulli's inequality, for all n

$$r^n = (1+h)^n \geq 1 + nh.$$

The right hand side is clearly unbounded, and therefore so is the left hand side.

4. (a) (5 Pts.) Prove that $\lim_{n\to\infty} y_n = y$ *if and only if* for all $\epsilon > 0$, $\lim_{n\to\infty} |y_n - y| \le \epsilon$.

($\Rightarrow$) If $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} |y_n - y| = 0$. It is then clear that for any $\epsilon > 0$, $\lim_{n\to\infty} |y_n - y| = 0 \le \epsilon$.
($\Leftarrow$) Assume that for all $\epsilon > 0$, $\lim_{n\to\infty} |y_n - y| < \epsilon$. Let $a = \lim_{n\to\infty} |y_n - y|$. We will show that $a = 0$. But this follows from the assumption and by what is proved in class that if $\forall \epsilon > 0$, $0 \le a \le \epsilon$, then $a = 0$. (Or just just say that $a \ge 0$ can not be positive for otherwise $0 < \frac{a}{2} < a$ violating the assumption.)

(b) (10 Pts.) Suppose that (x_n) is a sequence in $\mathbb{R}$ that converges to x_0. Let

$$a_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n).$$

Write the proof of the following steps to show that a_n converges to x_0.

- Show that for all $n \in \mathbb{N}$, $a_n - x_0 = \frac{1}{n}((x_1 - x_0) + (x_2 - x_0) + \cdots + (x_n - x_0))$.

$$\begin{aligned} a_n - x_0 = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) - x_0 &= \frac{1}{n}(x_1 + x_2 + \cdots + x_n - n\, x_0) \\ &= \frac{1}{n}((x_1 - x_0) + \cdots + (x_n - x_0)) \end{aligned}$$

where in the last equality the n-copies of x_0 are distributed.

- Prove that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$|a_n - x_0| \le \frac{1}{n}(|x_1 - x_0| + |x_2 - x_0| + \dots |x_N - x_0|) + \epsilon, \quad \text{for all } n \ge N_\epsilon,$$

Let $\epsilon > 0$. Since $x_n \to x_0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$|x_n - x_0| < \epsilon, \quad \forall n \ge N_\epsilon.$$

Using the first step and triangular inequality, for all $n \ge N_\epsilon$

$$\begin{aligned} |a_n - x_0| &= |\frac{1}{n}((x_1 - x_0) + \cdots + (x_{N_\epsilon - 1} - x_0) + (x_{N_\epsilon} - x_0) + \cdots + (x_n - x_0))| \\ &\le \frac{1}{n}(|x_1 - x_0| + \cdots + |x_{N_\epsilon - 1} - x_0| + |x_{N_\epsilon} - x_0| + \cdots + |x_n - x_0)| \\ &\le \frac{1}{n}(|x_1 - x_0| + \cdots + |x_{N_\epsilon - 1} - x_0| + \epsilon(n - N_\epsilon)) \\ &= \frac{1}{n}(|x_1 - x_0| + \cdots + |x_{N_\epsilon - 1} - x_0|) + \epsilon\frac{(n - N_\epsilon)}{)} \\ &\le \frac{1}{n}(|x_1 - x_0| + \cdots + |x_{N_\epsilon - 1} - x_0|) + \epsilon, \end{aligned}$$

as desired.

- Use part a) and the above steps to prove that $\lim_{n\to\infty} a_n = x_0$.

Using part a), it suffices to show that for all $\epsilon > 0$, $\lim_{n\to\infty} |a_n - x_0| \leq \epsilon$. Now let $\epsilon > 0$. By the above step, there exists a natural number $N_\epsilon \in \mathbb{N}$ such that

$$|a_n - x_0| \leq \frac{1}{n}(|x_1 - x_0| + |x_2 - x_0| + \ldots |x_N - x_0|) + \epsilon, \quad \text{for all } n \geq N_\epsilon,$$

Observe that for a fixed ϵ, $(|x_1 - x_0| + |x_2 - x_0| + \ldots |x_N - x_0|)$ is a constant, that we denote by C_ϵ. Thus

$$|a_n - x_0| \leq C_\epsilon \frac{1}{n} + \epsilon, \quad \text{for all } n \geq N_\epsilon,$$

This tell us that the tail of the sequence $|a_n - x_0|$ is not more than $C_\epsilon \frac{1}{n} + \epsilon$. Therefore,

$$\lim_{n\to\infty} |a_n - x_0| \leq \lim_{n\to\infty} C_\epsilon \frac{1}{n} + \epsilon = \epsilon,$$

and that is exactly what we wanted to show.

(c) (5 Pts.) Demonstrate that the converse of the statement in part b) is false.

Let us consider the sequence $(x_n) = ((-1)^n)$. Then $a_n = 0$ if n is even and $a_n = \frac{-1}{n}$ when n is odd. Clearly the sequence (a_n) converges to 0. But (x_n) is not convergent.