Math 447: Quiz 2

Name:SOLUTION

1. If $x, y, z \in \mathbb{R}$ and $x \leq z$, show that $x \leq y \leq z$ if and only if $|x-y|+|y-z|=|x-z|$.

   Proof. ($\Rightarrow$) If $x \leq y \leq z$, then $y-x \geq 0$, $z-y \geq 0$ and $z-x \geq 0$. Therefore

   $$|x-y|+|y-z|=y-x+z-x=z-x=|x-z|$$

   ($\Leftarrow$) Suppose now that $x \leq z$ and $|x-y|+|y-z|=|x-z|$. Let us show that $x \leq y \leq z$ by contradiction. If $y < x$, then $y < z$ (since $x \leq z$) and therefore, $|x-z|=z-x<z-y=|y-z|$. This is a contradiction as $|x-z| \geq |y-z|$.

   If $z < y$, then $x < y$ and therefore, $|x-z|=z-x<y-x=|y-x|$. But again that is impossible as $|x-z| \geq |y-x|$.

   Therefore, $x \leq y \leq z$. □

2. Find all values of x satisfying the inequality $4 < |x+2|+|x-1| < 5$.

   Solution. Let us considers cases that enable us to remove the absolute value. We will look for solutions in the intervals $(-\infty,-2]$, $(-2,1]$ and $[1,\infty)$ separately.

   On $(-\infty,-2]$, $|x+2|=-(x+2)$ and $|x-1|=-(x-1)$. Then the inequality simplifies to

   $$4 < -x-2-x+1 < 5 \iff 5 < -2x < 6 \iff -3 < x < -5/2$$

   That is all $x \in (-3,-5/2) \subset (-\infty,-2]$ satisfy the inequality.

   On $(-2,-1]$, the inequality is simplified to $4 < 3 < 5$, which is never true. That is no element in $(-2,-1]$ satisfies the inequality.

   On $(1,\infty)$, the inequality becomes

   $$4 < 2x+1 < 5 \iff 3/2 < x < 2.$$

   That is all $x \in (3/2,2) \subset (1,\infty)$ satisfy the inequality. Combining the above 3 cases we see that the inequality $4 < |x+2|+|x-1| < 5$ is satisfied if and only if $x \in (-3,-5/2) \cup (3/2,2)$. □