The G_2–Hitchin component for hyperbolic triangle groups: dimension, integer points and thin subgroups of $G_2^{4,3}$

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Abstract

The image of $\mathrm{PSL}(2,\mathbb{R})$ under the irreducible representation into $\mathrm{PSL}(7,\mathbb{R})$ is contained in the split real form $G_2^{4,3}$ of the exceptional Lie group G_2 . This irreducible representation therefore gives a representation ρ of a hyperbolic triangle group $\Gamma(p,q,r)$ into $G_2^{4,3}$, and the *Hichin component* of the representation variety $\mathrm{Hom}(\Gamma(p,q,r)\,,\,G_2^{4,3})$ is the component of $\mathrm{Hom}(\Gamma(p,q,r)\,,\,G_2^{4,3})$ containing ρ .

In this article we give a simple, elementary proof of a formula for the dimension of this Hitchin component, this formula having been obtained earlier in [ALS22] as part of a wider investigation using Higgs bundle techniques. Then we specialize to the $(2,4,6)$–triangle group and give two infinite sequences of integer points on its G_2–Hitchin component, yielding infinitely many non-conjugate thin subgroups of $G_2^{4,3}$.

1 Introduction

The (p,q,r)*–triangle group* $\Gamma(p,q,r)$ is the orbifold fundamental group of a 2–sphere with three cone points, of respective orders p,q,r . Here we assume throughout that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, so that $\Gamma(p,q,r)$ may be regarded as the group of orientation preserving symmetries of a tiling of the hyperbolic plane $\mathbb{H}^2$ by triangles with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$: the quotient of $\mathbb{H}^2$ by the action of this group of isometries is the abovementioned orbifold. One of the reasons for the interest in triangle groups is that they contain surface groups as subgroups of finite index. From the Seifert – van Kampen theorem (extended to the context of orbifolds) a triangle group admits a presentation as follows:

$$\Gamma(p,q,r) \;=\; \langle a\,,\,b\,,\,c \mid a^p = b^q = c^r = abc = 1\rangle \;,$$

the generators being represented by rotations about the vertices of a constituent triangle of the tiling.

Recall that for $n \geq 3$ there is an irreducible representation, unique up to conjugacy, of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL(n,\mathbb{R})}$. Since we may identify the group of orientation preserving isometries of $\mathbb{H}^2$ with $\mathrm{PSL}(2,\mathbb{R})$, the restriction of this irreducible representation to a triangle group $\Gamma(p,q,r)$ is a representation ρ_n : $\Gamma(p,q,r) \to \mathrm{PSL(n,\mathbb{R})}$, also irreducible [ALS22]. The *Hitchin component* of the representation variety $\mathrm{Hom}(\Gamma(p,q,r)\,,\,\mathrm{PSL(n,\mathbb{R})})$ is defined to be the component of that variety that contains ρ_n . The term “Hitchin component” was originally coined by F. Labourie [Lab06] for representations of surface groups of negative Euler characteristic, but it is appropriate to use the term for representations of hyperbolic triangle groups or other orbifold fundamental groups.

There has been considerable study of Hitchin components, but often the case $n = 7$ has been regarded as a technical inconvenience to be avoided, on account of the existence of the 7–dimensional representation in $\mathrm{SL}(7, \mathbb{C})$ of the exceptional Lie group G_2, giving rise to representations of its real forms in $\mathrm{SL}(7, \mathbb{R})$. It so happens that the image of $\mathrm{PSL}(2, \mathbb{R})$ under the irreducible representation into $\mathrm{PSL}(7, \mathbb{R})$ is contained in the split real form $G_2^{4,3}$, this notation reflecting the fact that it is contained in $\mathrm{SO}(4, 3)$ (the other real form of G_2 is compact and is contained in $\mathrm{SO}(7)$) [Agr08, Draper17].

If $n = 2k + 1$ is odd, the image of $\mathrm{PSL}(2, \mathbb{R})$ under the irreducible representation into $\mathrm{PSL(n, \mathbb{R})} = \mathrm{SL(n, \mathbb{R})}$ is contained in the split orthogonal group $\mathrm{SO(k + 1, k)}$, whereas if $n = 2k$ is even, the image is contained in the (projectivized) symplectic group $\mathrm{PSp(k)}$. The determination of the dimension of the Hitchin component of $\Gamma(p, q, r)$ in $\mathrm{PSL(n, \mathbb{R})}$ was carried out in [LT18b], and for $\mathrm{SO(k + 1, k)}$ and $\mathrm{PSp(k)}$ independently by [Weir18] and [ALS22]. The methods of [LT18b] and [Weir18] are quite elementary in nature. On the other hand [ALS22] uses Higgs bundle techniques to obtain results of greater generality, including in particular the formula of 2.1 below. The independent proof of the formula given here is elementary, and is along the lines of [LT18b].

For the second part of this note we focus on the G_2–Hitchin component $\mathcal{H}_{2,4,6}$ of the $2, 4, 6$–triangle group, which we were able to calculate exactly. Within $\mathcal{H}_{2,4,6}$ we found a sequence (ρ_n) of representations, each of which can be conjugated to be over the integers. Interestingly, at the time that we proved the integrality of the representations ρ_n, we had been unable to find a formula generating conjugates of ρ_n in $\mathrm{SL}(7, \mathbb{Z})$. Eventually, after a lapse of more than a year, a formula was found for n a multiple of 4. We did however find an *ad hoc* process which yielded integer representations conjugate to the first twelve representations in the sequence, and which appears to be effective beyond that.

2 The dimension of $\mathcal{H}_{G_2}$ for $\Gamma(p, q, r)$

Theorem 2.1. *Let $S = \{n \in \mathbb{N} \mid n \geq 2\}$, and let $f : S \to \mathbb{N}$ be defined as follows:*

$$f(n) = \begin{cases} 8 & (n = 2) \\ 10 & (n = 3, 4, 5) \\ 12 & (n \geq 6) \end{cases}$$

The dimension of the Hitchin component $\mathcal{H}_{G_2}$ of $\Gamma(p, q, r)$ in $G_2^{4,3}$ is $f(p) + f(q) + f(r) - 14$, and that of the corresponding component of the character variety is $f(p) + f(q) + f(r) - 28$.

From the formula given in Theorem 1, we see that the two triangle groups $\Gamma(2, 4, 5)$, $\Gamma(2, 5, 5)$ are rigid in $G_2^{4,3}$, and that all other hyperbolic triangle groups are “flexible”, with even-dimensional Hitchin components. We were able to calculate the two-dimensional $G_2^{4,3}$–Hitchin components for $\Gamma(3, 3, 4)$, $\Gamma(3, 4, 4)$

and $\Gamma(2,4,6)$ exactly, using techniques based broadly on those of [LT18a]. The exact tautological representation for $\Gamma(2,4,6)$ is given in the Appendix.

The proof of Theorem 1 presented here is a close adaptation of the corresponding proof in [LT18b], to which the reader is referred. The idea is first to observe that under a continuous deformation the images α, β, γ in $G_2^{4,3}$ of the generators a, b, c of $\Gamma(p,q,r)$ can only move within their conjugacy classes in $G_2^{4,3}$, on account of their being of finite order. Then, writing $[\alpha]$, $[\beta]$, $[\gamma]$ for the respective conjugacy classes of α, β, γ, we have a smooth map

$$\Phi : [\alpha] \times [\beta] \times [\gamma] \ \to \ G_2^{4,3} \quad , \quad \Phi : (\alpha', \beta', \gamma') \mapsto \alpha'\beta'\gamma' \ ,$$

which the argument of [LT18b] shows is a submersion close to (α, β, γ), by invoking the irreducibility of the representation ρ_7. It follows from the defining relation $abc = 1$ that

$$\dim \mathcal{H}_{G_2} = \dim [\alpha] + \dim [\beta] + \dim [\gamma] - \dim \ G_2^{4,3} = \dim [\alpha] + \dim [\beta] + \dim [\gamma] - 14 \ .$$

It remains to determine the dimensions of the conjugacy classes in $G_2^{4,3}$ of the elliptic generators of $\Gamma(p,q,r)$, and this is the ingredient that inevitably has to use specific properties of that Lie group.

Proposition 2.2. *Let g be an elliptic generator of a triangle group, of order k say. The dimension of the conjugacy class in $G_2^{4,3}$ of $\rho_7(g)$ is $f(k)$, where f is the function defined in the statement of Theorem 1.*

Proof. Let γ, $[\gamma]$ denote $\rho_7(g)$ and its conjugacy class in $G_2^{4,3}$, respectively. Since the elements of $[\gamma]$ are in bijective correspondence with cosets of the centralizer $C_{G_2^{4,3}}(\gamma)$ of γ in $G_2^{4,3}$, we have

$$\dim [\gamma] \ = \ \dim \ G_2^{4,3} - \dim \ C_{G_2^{4,3}}(\gamma) \ = \ 14 - \dim \ C_{G_2^{4,3}}(\gamma) \ .$$

We would like to relegate the computation of $\dim \ C_{G_2^{4,3}}(\gamma)$ to linear algebraic considerations of the Lie algebra $\mathfrak{g}_2^{4,3}$, and to this end we make use of the following facts from basic Lie theory (these facts are also used in [LT18b]):

(i) There exists a neighbourhood of $0 \in \mathfrak{g}_2^{4,3}$ on which the exponential map is a diffeomorphism to a neighbourhood of the identity element of $G_2^{4,3}$;

(ii) $\exp(\mathrm{Ad}_\theta \xi) \ = \ \theta \exp(\xi) \theta^{-1} \quad (\theta \in G_2^{4,3} \ , \ \xi \in \mathfrak{g}_2^{4,3})$.

The identity (ii) may be checked merely by expanding the left hand side as a power series. It follows that the dimension of $C_{G_2^{4,3}}(\gamma)$ is equal to the dimension of the subspace

$$W \ = \ \{\xi \in \mathfrak{g}_2^{4,3} \mid \exp(\mathrm{Ad}_\gamma \xi) = \exp(\xi)\} \ = \ \{\xi \in \mathfrak{g}_2^{4,3} \mid \mathrm{Ad}_\gamma \xi = \xi\}$$

of the Lie algebra $\mathfrak{g}_2^{4,3}$.

The dimension of the real vector space W evidently depends only on of the conjugacy class in $\mathrm{PSL}(2,\mathbb{R})$ of our elliptic g, and indeed it will be most convenient to take a temporary excursion into $\mathrm{PSL}(2,\mathbb{C})$ by taking the conjugate represented by the diagonal matrix

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \quad (z = e^{\frac{i\pi}{k}})$$

whose image under ρ_7 is the following diagonal matrix:

$$\begin{pmatrix} z^{-6} &&&&&& \\ & z^{-4} &&&&& \\ && z^{-2} &&&& \\ &&& 1 &&& \\ &&&& z^{2} && \\ &&&&& z^{4} & \\ &&&&&& z^{6} \end{pmatrix} \tag{1}$$

Thus we shall in effect be computing the complex dimension of the complexification of W, but this is not an issue as this complex dimension is equal to the real dimension that we seek.

For the determination of $\dim W$, we are fortunate in having the very useful, explicit description of the Lie algebra of G_2 given in [Draper17]:

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & -2y^t & -2x^t \\ x & a & \ell_y \\ y & \ell_x & -a^t \end{pmatrix} \ (a \in \mathfrak{sl}_3(\mathbb{C}),\ x, y \in \mathbb{C}^3) \right\} \quad , \tag{2}$$

where for $v \in \mathbb{C}^3$ the 3×3 matrix ℓ_v represents the map $w \mapsto v \times w$ with respect to the canonical basis:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \implies \ell_v = \begin{pmatrix} 0 & -v_3 & -v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} .$$

A specification of the split real form $\mathfrak{g}_2^{4,3}$ is obtained by replacing $\mathbb{C}$ by $\mathbb{R}$ in the above description.

An important characterization of G_2 is that it is the isotropy group of a generic 3–form on the complex vector space $\mathbb{C}^7$, the term “generic” signifying that its orbit under the action of $\mathrm{GL}(7,\mathbb{C})$ is open in the space of 3–forms [Agr08, Draper17]. The representation of G_2 in $\mathrm{SL}(7,\mathbb{C})$ is of course only defined up to conjugacy, and different choices of conjugate representation will preserve different 3–forms in this orbit.

The 3–form preserved by the version of G_2 given above is (up to scalar multiple)

$$e_1 \wedge (e_2 \wedge e_5 \ + \ e_3 \wedge e_6 \ + \ e_4 \wedge e_7) \ + \ 2\,(e_2 \wedge e_3 \wedge e_4 \ - \ e_5 \wedge e_6 \wedge e_7) \ ,$$

whereas that preserved by the image of PSL$(2,\mathbb{C})$ under the irreducible representation given in the usual way by action on homogeneous polynomials is

$$\mathfrak{f} = e_1 \wedge e_4 \wedge e_7 - 3\,e_1 \wedge e_5 \wedge e_6 - 3\,e_2 \wedge e_3 \wedge e_7 + 6\,e_2 \wedge e_4 \wedge e_6 - 15\,e_3 \wedge e_4 \wedge e_5 \tag{3}$$

The action of the following matrix was found (by trial and error) to send the second of these 3–forms to the first:

$$\frac{1}{6\,(5^{1/3})}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -30 & 0 & 0 & 0 \end{pmatrix},$$

and conjugation of the diagonal matrix **(1)** by this matrix yields the diagonal matrix

$$\gamma = \begin{pmatrix} 1 &&&&&& \\ & z^2 &&&&& \\ && z^4 &&&& \\ &&& z^{-6} &&& \\ &&&& z^{-2} && \\ &&&&& z^{-4} & \\ &&&&&& z^6 \end{pmatrix}. \tag{4}$$

Using **(2)**, it is now an almost trivial matter to compute the dimension of the subspace

$$W = \{\xi \in \mathfrak{g}_2^{4,3} \mid \mathrm{Ad}_\gamma \xi = \xi\} \ .$$

For example, taking $k = 2$ and taking ξ to be the general element of $\mathfrak{g}_2$ given in **(2)**, we have

$$\gamma\xi - \xi\gamma = \begin{pmatrix} 0 & -4y_1 & 0 & -4y_3 & -4x_1 & 0 & -4x_3 \\ -2x_1 & 0 & -2a_{12} & 0 & 0 & 2y_3 & 0 \\ 0 & 2a_{21} & 0 & 2a_{23} & 2y_3 & 0 & -2y_1 \\ -2x_3 & 0 & -2a_{32} & 0 & 0 & -2y_1 & 0 \\ -2y_1 & 0 & 2x_3 & 0 & 0 & 2a_{21} & 0 \\ 0 & 2x_3 & 0 & -2x_1 & -2a_{12} & 0 & -2a_{32} \\ -2y_3 & 0 & -2x_1 & 0 & 0 & 2a_{23} & 0 \end{pmatrix}.$$

The condition $\mathrm{Ad}_\gamma \xi = \xi$ requires us to set the eight parameters

$$x_1\,,\ x_3\,,\ y_1\,,\ y_3\,,\ a_{12}\,,\ a_{21}\,,\ a_{23}\,,\ a_{32}$$

to zero, leaving $14-8=6$ degrees of freedom. Therefore the dimension of the centralizer of γ is 6, and the dimension of the conjugacy class of γ is $f(2)=14-6=8$.

The computation of $f(k)$ for values of k greater than 2 is similar, but it should be remarked that the fact that $f(3)$, $f(4)$, $f(5)$ are all equal should probably be regarded as a coincidence, as the computations for these three values are seemingly unrelated. □

3 Integer points in $\mathcal{H}_{G_2}(\Gamma(2,4,6))$

3.1 The character variety

From the formula of Theorem 1, $\mathcal{H}_{G_2}(\Gamma(2,4,6))$, the G_2–Hitchin component of the $(2,4,6)$–triangle group (in its character variety form) has dimension 2. Its exact formulation was obtained in a manner similar to that of [LT18a], the extra ingredient being a "filter" added with the purpose of steering the Newton process so that it converged (numerically) to a representation preserving a generic 3–form, specifically the form $\mathfrak{f}$ given in (3).

The exact version of $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ given in the appendix has generators a, b of orders 4, 6 respectively, and with ab of order 2. Each point of the character variety $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ is of course a conjugacy class of representations, but we shall identify this point with the specific representation determined by the given matrices a, b.

The entries of the matrices a, b are expressions in parameters $u=\mathrm{Tr}(aaab)+1$ and $v=\mathrm{Tr}(aabb)+1$. They lie in a quadratic extension $\mathbb{Q}(u,v)(\alpha)$ of $\mathbb{Q}(u,v)$, with

$$\begin{aligned}\alpha &= 4\,\mathrm{Tr}(abABaB)+4+2u^2-2u(2+v)\\ &= \pm\sqrt{2(-72u^2+72u^3-16u^4-u^5+v(-192u+64u^2+20u^3)+v^2(-192-32u+2u^2)-32v^3)}\ ,\end{aligned}$$

A, B denoting the respective inverses of a, b. Assigning appropriate values to u, v together with a sign for α determines a specific representation in the Hitchin component. The values chosen for u, v should be the coordinates of a point in the correct region of the (u,v)–plane as indicated in Figure 1. In that figure a component of the curve $\alpha=0$ is illustrated; the Hitchin component is homeomorphic to $\mathbb{R}^2$ [ALS22] and consists of two sheets glued along the illustrated curve. Thus a point of the Hitchin component is determined by a suitable pair of real numbers u,v together with a choice of sign for the corresponding value of α.

3.2 Finding integer points

For a representation in the Hitchin component to be written over the integers it is of course necessary for traces of all words in the generators a, b to be integers; thus we require u, $v\in\mathbb{Z}$, and moreover from $\alpha = 4\,\mathrm{Tr}(abABaB)+4+2u^2-2u(2+v)$ we require that α should be an even integer. A quick computer search in a reasonable range produces a scatterplot of points (u,v), $u,v\in\mathbb{Z}$ for which $\alpha(u,v)$

is also an integer, and then another search looks within these points for possible beginnings of sequences (u_n, v_n) such that $\alpha(u_n, v_n) \in \mathbb{Z}$ for all n. In this way, two such sequences were found, each lying on a curve in the Hitchin component:

$$\begin{array}{llll} \mathcal{S}_1 & (u_n\,,\, v_n) \;=\; (4(18+n^2)\,,\,(18+n^2)(21+n^2)) & \alpha \;=\; 8n(15+n^2)(18+n^2) & (5) \\ \mathcal{S}_2 & (u_n\,,\, v_n) \;=\; (4(22+3n^2)\,,\,(22+3n^2)(23+3n^2) & \alpha \;=\; 24n(21+3n^2)(22+3n^2) & \end{array}$$

The first few terms of these two sequences are illustrated in Figure 1, the points of $\mathcal{S}_1$ being marked as filled circles and those of $\mathcal{S}_2$ as stars.

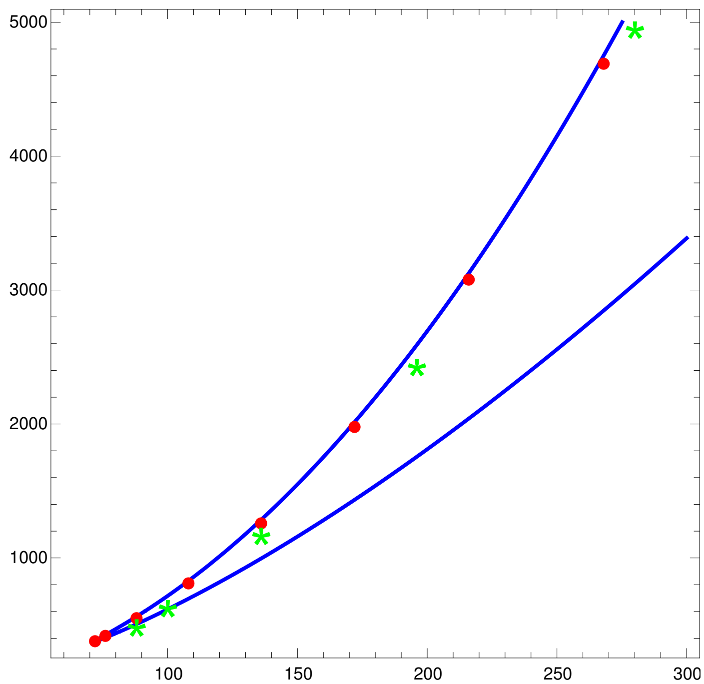

Figure 1:

Substitution of the values given in (5) produces generating matrices with entries in the rational numbers, and then it is incumbent upon the investigator to show somehow that the representations can be conjugated to be over the integers. Initially one applies *ad hoc* methods (involving solutions of diophantine equations) to individual representations, and although success was achieved for around twenty representations in each sequence, there was no discernable pattern in the results. Also the integer representations thus obtained were extremely unwieldy.

After approximately one year, finally, through persistent observation, solutions were found for the sequence $\mathcal{S}_1$, see Figure 2, where there are separate formulae for the generator b for the cases n even, n odd.

Each of the two formulae in Figure 2 gives a parametrization of a curve $\mathcal{C}_1$ in the Hitchin component. A representative $\rho(s) (s \in \mathbb{R})$ of an arbitrary point on this curve can be defined unambiguously as follows:

$$a_{\text{even}} = a_{\text{odd}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$b_{\text{even}}(t) = \begin{pmatrix} 3 & 1 & -49+10t & -50+12t & -13+2t & 1+2t & 0 \\ -2(3+t) & 0 & 63-4t & 65-4t & 16 & -5-t & 4 \\ -2(4+t) & 0 & 98-22t+t^2 & 102-22t+t^2 & 25-4t & -6 & 6-t \\ 17+4t & 0 & -170+27t-t^2 & -174+27t-t^2 & -43+4t & 15+t & -11+t \\ -38-6t & 0 & 288-36t+t^2 & 287-36t+t^2 & 72-4t & -2(19+t) & 20-t \\ -2(5+2t) & 0 & 126-8t & 130-8t & 32 & -9-2t & 8 \\ -18-5t-2t^2 & 0 & 142-t-t^2 & 142-t-t^2 & 2(18+t) & -18-3t-t^2 & 9+t \end{pmatrix}$$

$$b_{\text{odd}}(t) = \begin{pmatrix} 3 & 1 & -62+12t & -62+12t & 2(-7+t) & -4+12t & 2(-1+t) \\ -5-2t & 0 & 72-4t & 76-4t & 16 & -6-4t & 2 \\ 2(8+t+t^2) & 0 & -125+(-1+t)^2 & -131-3t+t^2 & -2(14+t) & 58+4t+4t^2 & 2 \\ -2(4+t^2) & 0 & 43+7t-t^2 & 45+8t-t^2 & 2(5+t) & -34-4t^2 & -2 \\ -39-7t & 0 & 373-41t+t^2 & 396-42t+t^2 & 83-4t & -121-12t & -1 \\ -2-t & 0 & 36-2t & 38-2t & 8 & -2(1+t) & 1 \\ 5(3+t) & 0 & -234+31t-t^2 & -247+32t-t^2 & 4(-13+t) & 19+8t & -6 \end{pmatrix}$$

Figure 2: In b_{even} , $s = 2t$, and in b_{odd} , $s = 2t - 1$.

Definition. Let a , b be the elliptic generators of respective orders 2 , 6 of the triangle group $\Gamma(2,4,6)$. For $s \in \mathbb{R}$, $\rho_s : \Gamma(2,4,6) \to \mathrm{SL}(7,\mathbb{R})$ is the representation $\rho_s : a \longmapsto a_{\text{even}}(s)$, $b \longmapsto b_{\text{even}}(s)$ with $s = 2t$.

Recall that we have defined the Lie group G_2 to be the isotropy group in $\mathrm{SL}(7,\mathbb{R})$ of a generic 3–form on $\mathbb{R}^7$. Thus G_2 is actually only defined up to conjugacy, and the image of the representation ρ_s lies in a representative of this equivalence class, or informally a "copy" of G_2 . That this copy is unique for given s follows from direct calculation: using the formulae for a_{even} , b_{even} , one finds that the 3–form preserved by both of these generators is unique up to scalar multiple. A 1–parameter family of 3–forms preserved by the images of the representations ρ_s is given in Table 3 in the Appendix (for convenience a corresponding parametrized family is also given for the family $\langle a_{\text{odd}} , b_{\text{odd}} \rangle$). We should check that these forms are generic, although we are already reasonably confident that they are, as the Newton process used in the initial computation of the Hitchin component was designed to obtain numerical representations preserving the standard 3–form (3) whose isotropy group is the "standard" copy of $G_2^{4,3}$. The purpose of the next proposition is to give an independent proof of genericity not tainted by approximations inherent in numerical work. It involves a short computation that is easily carried out on a computer algebra system such as Mathematica.

Proposition 3.1. *The 3–forms listed in* (3) *for* $\langle a_{even}\,,\,b_{even}\rangle$ *are generic for all* $s \in \mathbb{R}$.

Proof. Let $\mathfrak{f}_{s_0}$ be any of the 3–forms listed in (3). The action of $\mathrm{GL}(7,\mathbb{R})$ on the space $\mathfrak{F}$ of 3–forms induces a smooth map

$$\phi_{s_0} : \mathrm{GL}(7,\mathbb{R}) \to \mathfrak{F} \quad , \quad g \longmapsto g.\mathfrak{f}_{s_0} \ .$$

For distinct i, j let E_{ij} be the 7×7 matrix whose (i, j) entry is 1 and all other entries 0 ; then let $H_{ij} \subset \mathrm{GL}(7,\mathbb{R})$ be the 1–parameter subgroup $I + kE_{ij}$ $(k \in \mathbb{R})$. There are 42 such subgroups, and a subset of size 35 (chosen randomly in fact), which we denote for convenience $\{H_1\,,\,\ldots\,,\,H_{35}\}$, was found to have the following property: the determinant of the matrix whose ith column is $H_i(k).\mathfrak{f}(s)$ is

$$-126100789566373888\, k^{34}\, (9+2x^2)^{26}\, (15+4x^2)^{26}\, (90-5x+14x^2)\, p(s)\, q(s) \ ,$$

where $p(s)\,,\,q(s)$ are irreducible polynomials in $\mathbb{Z}[s]$ of degrees $6\,,\,11$ respecitvely. It follows that if we exclude the real roots of the polynomials $p(s)\,,\,q(s)$ (there are three in number), the elements $\phi_s(H_i(k))\,,\,\ldots\,,\,\phi_s(H_{35}(k))$ span the space $\mathfrak{F}$ of 3–forms for all nonzero k . We deduce that the differential $d\phi_s$ maps the subspace of the Lie algebra of $\mathrm{GL}(7,\mathbb{R})$ spanned by the tangents to the H_i at the identity diffeomorphically to the tangent space of $\mathfrak{F}$ at $\mathfrak{f}(s)$, and the conclusion follows for all s except for the excluded three real roots of $p(s)\,,\,q(s)$. The same computation was carried out for a different subset of size 35 of the collection of 42 1–parameter subgroups, and this time the excluded values of s had no overlap with the previous ones. The conclusion of the proposition follows. ☐

It should be verified that we are in the correct orbit of generic 3–forms, not that corresponding to the compact real form G_2^c . One checks that the image of the representation ρ_s respects a bilinear form which is non-singular for all $s \in \mathbb{R}$, whence the signature is constant over all real s . Since at $s = 0$ the signature is $(4, 3)$ the verification is complete.

Definition. (i) $H(s)$ denotes the image of the representation ρ_s of the $2, 4, 6$–triangle group.

(ii) $G_2(s)$ denotes the isotropy group of the 3–form $\mathfrak{f}(s)$ given in Table 3. Thus $G_2(s)$ is the copy of $G_2^{4,3}$ containing the subgroup $H(s)$.

3.3 Zariski dense subgroups and thin subgroups arising from $\mathcal{S}_1$

In [LT23] a detailed investigation of the representations in the sequence $\mathcal{S}_1$ yielded the following result:

Theorem 3.2. *[LT23] The image groups of the representations* ρ_s *are Zariski dense in* G_2 *for all nonzero real* s .

Here we present an alternative method for determining Zariski denseness. It is applicable to representations over the integers, and is a straightforward application of Proposition 1 and Theorem 2 of [Lub99].

The following will be useful in our (brief) discussion of thin groups.

Proposition 3.3. *For each* $s \in \mathbb{Z}$ *the Lie group* $G_2(s)$ *is defined over* $\mathbb{Z}$ *.*

Proof. The coordinates of the 3–form $\mathfrak{f}(s)$ are all integers, and the condition $g.\mathfrak{f} = \mathfrak{f}$ can be set out as a system of polynomial equations over $\mathbb{Z}$ in the entries of the matrix g . The group $G_2(s)$ is the set of solutions over $\mathbb{R}$ of this system. □

We are now in a position to exploit the results of [Lub99].

Proposition 3.4. *Let* $G_2(s)$ *be as in 3.3; suppose that for some prime* p *the matrices obtained by reducing* $a_{even}(s)$, $b_{even}(s)$ *modulo* p *generate the Chevalley finite simple group* $G_2(p)$ *. Then the group generated by* $a_{even}(s)$, $b_{even}(s)$ *is Zariski dense in* $G_2(s)$ *.*

Proof. This follows at once from Proposition 1 and Theorem 2 of [Lub99]. □

The main result of [Lub99] is actually stronger, in that Strong Approximation is used to show that if the hypothesis of 3.4 is satisfied for some prime p , then it is satisfied for all but finitely many p .

Armed with 3.4 we can recover part of Theorem 3.2. More could be recovered by the use of additional primes larger than 5 , but there seems to be little point as it is unlikely that by this method one can prove the conclusion of Theorem 3.2 for all nonzero integers.

Corollary 3.4.1. *For* $s \neq 0 \pmod 5$ *the group* $\langle a_{even}(s), b_{even}(s) \rangle$ *is Zariski dense in* $G_2(s)$ *.*

Proof. This is a straightforward computation in GAP [Gap]. For $s = 2, 4, 6, 8$ it was found that the generators $a_{\text{even}}(s)$, $b_{\text{even}}(s)$ reduced modulo 5 generate a finite simple group of order 5859000000 , for which $G_2(5)$ is the only possibility. □

We end this subsection with a note regarding matrix groups that are *thin*, as defined in [Sarnak13], see also [KLLR19].

Recall from Proposition 3.3 that for $s \in \mathbb{Z}$ the copy $G_2(s)$ of $G_2^{4,3}$ containing the image $H(s)$ of $\rho(s)$ is defined over $\mathbb{Z}$. It follows from Major Theorem 5.1.11 of [Morris99] that $G_2(s) \cap \mathrm{SL}(7, \mathbb{Z})$ is a lattice in $G_2(s)$. We denote this lattice $G_2\mathbb{Z}(s)$.

The lattice $G_2\mathbb{Z}(s)$ is irreducible on account of $G_2(s)$ being simple, see [Morris99], §4.3. In light of Theorem 3.2 the conclusion of the next proposition holds for all nonzero integers s , but to keep the article self-contained we shall restrict here to integers that are not multiples of 5 .

Proposition 3.5. *Let* $s \in \mathbb{Z}$ *with* $s \neq 0 \pmod 5$ *. Then* $H(s)$ *is a thin matrix group.*

Proof. We have already shown that $H(s)$ is Zariski dense in $G_2(s)$, so in order to satisfy the thin group condition it remains to show that $H(s)$ has infinite index in $G_2\mathbb{Z}(s)$. Suppose to the contrary that $|G_2\mathbb{Z}(s) : H(s)|$ is finite. The triangle group $H(s)$ has a hyperbolic surface subgroup Σ of finite index, so

Σ is also of finite index in $G_2\mathbb{Z}(s)$ and is thus a lattice in $G_2(s)$. Let N be the commutator subgroup of Σ. Then N is an infinite normal subgroup of Σ with quotient Σ/N also infinite (it is the first homology group of the underlying surface). Since the rank of the Lie group $G_2(s)$ is 2 [Draper17], we have arrived at a contradiction to the Margulis Normal Subgroups Theorem, [Morris99] Theorem 17.1.1. □

Proposition 3.5 provides infinitely many pairwise non-conjugate thin groups in $\mathrm{SL}(7,\mathbb{Z})$, each with Zariski closure a copy of $G_2^{4,3}$. The argument of Theorem 1.5 of [LT18a] shows that amongst these groups there are infinitely many pairwise non-conjugate thin surface groups.

3.4 The representations in the sequence $\mathcal{S}_2$

As of this writing, generating matrices in $\mathrm{SL}(7,\mathbb{Z})$ for the representations of $\mathcal{S}_2$ have only been found for the first 20 representations in the sequence. This section will explain how to prove that the representations in $\mathcal{S}_2$ can all be written over the integers, without having explicit representations over $\mathbb{Z}$ to hand. It uses a technique from [BL15].

Looking at Table 4, which gives generators for the representations in $\mathcal{S}_2$ with rational entries, one sees that the only primes dividing the denominators are 2, $784 + 177t^2 + 9t^4$. Conjugating the representations so as to expunge the prime 2 from denominators is quite straightforward, as the conjugating matrix can be taken to be independent of the parameter t, see Table 5.

We use a different method for dealing with the remaining prime $784 + 177t^2 + 9t^4$. From the following extremely useful result of [Bass80], to prove that the representations in $\mathcal{S}_2$ can be written over $\mathbb{Z}$ it is sufficient to show that traces of all elements of $\langle a, b\rangle$ are integral. Currently we have established that $784 + 177t^2 + 9t^4$ is the only prime occurring in denominators of elements of $\langle a, b\rangle$, so exclusion of this prime from denominators of traces together with application of Bass's result will complete the proof of integrality of the representations.

Proposition 3.6. *([Bass80], Corollary 2.5.) Let Γ be a subgroup of* $\mathrm{SL}(\mathrm{n},\mathbb{Q})$ *satisfying:*

(i) *The action of Γ on $\mathbb{Q}^n$ is absolutely irreducible;*

(ii) $\chi(\Gamma) \subset \mathbb{Z}$*, i.e. each element of Γ has integer trace.*

Then there exists $t \in \mathrm{GL}(\mathrm{n},\mathbb{Q})$ with $t\Gamma t^{-1} \subset \mathrm{SL}(\mathrm{n},\mathbb{Z})$.

Proof. Let $\mathbb{Z}\Gamma$ denote the set of all finite $\mathbb{Z}$–linear combinations of elements of Γ. By Proposition 2.2(a) of [Bass80], $\mathbb{Z}\Gamma$ is a $\mathbb{Z}$–order of $M(n,\mathbb{Q})$.

The order $\mathbb{Z}\Gamma$ is contained in a maximal order $\mathcal{O}$ of $M(n,\mathbb{Q})$; from Theorem 21.6 of [Reiner75] such a maximal order is of the form $\{t \in M(n,\mathbb{Q}) \mid t(\Lambda) \subset \Lambda\}$ for some full lattice Λ in $\mathbb{Q}^n$. Since Λ is isomorphic (as a $\mathbb{Z}$–module) to the free module $\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n}$, we see that $\mathcal{O} \cong M(n,\mathbb{Z})$.

We may now apply the Skolem-Noether theorem (see Theorem 7.21 of [Reiner75]) to obtain $t \in \mathrm{GL(n, \mathbb{Q})}$ with $\phi_t(\mathcal{O}) = M(n, \mathbb{Z})$, ϕ_t being the inner automorphism $x \mapsto txt^{-1}$ of $M(n, \mathbb{Q})$.

$$\begin{array}{ccc} M(n,\mathbb{Q}) & \overset{\phi_t}{\dashrightarrow} & M(n,\mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathcal{O} & \xrightarrow{\cong} & M(n,\mathbb{Z}) \end{array}$$ □

In order that we may apply Proposition 3.6 we need to establish conditions (i), (ii) in the statement of that proposition, and for this we shall make use of the following construct.

Definition. Let k be a field, let $M(n, k)$ denote the algebra of $n \times n$ matrices over k, and let Γ be a multiplicative group of matrices in $M(n, k)$. A *Burnside Γ–basis for* $M(n, k)$ is a subset of Γ that is a basis for the n^2–dimensional k–vector space $M(n, k)$.

It is a classical result of W. Burnside that such a basis exists if Γ is *absolutely irreducible* in $M(n, k)$, *i.e.* Γ leaves no non-trivial, proper subspace of the $\overline{k}$–vector space $M(n, \overline{k})$ invariant, where $\overline{k}$ is an algebraic closure of k. The (much easier) converse is of immediate use to us.

Proposition 3.7. *Let $k, \Gamma, M(n, k)$ be as above. If $M(n, k)$ admits a Burnside Γ–basis, then Γ is absolutely irreducible in $M(n, k)$.*

Proof. Suppose that Γ leaves invariant a non-trivial, proper subspace W of $M(n, \overline{k})$. We choose a basis for the subspace W and extend to a basis of $M(n, \overline{k})$. With respect to this basis the matrix for every element of Γ (regarded as a group of linear transformations of $M(n, \overline{k})$) has a block of zeros in the bottom left-hand corner, and so Γ is conjugate in $\mathrm{GL(n, \overline{k})}$ to a group Γ' of matrices of this form. Clearly the matrix whose $(n, 1)$–entry is 1 and all other entries 0 is not in the $\overline{k}$–span of Γ', so the subspace of $M(n, \overline{k})$ consisting of all $\overline{k}$–linear combinations of elements of Γ' has dimension strictly less than n^2. This dimension is equal to that of the subspace of $M(n, k)$ consisting of all k–linear combinations of elements of Γ, and the conclusion follows. □

Finding a Burnside basis, like many tasks undertaken in the context of integer representations, is not algorithmic; however in the current situation it was relatively easy to find one, valid for all integers t, for the images of the representations in $\mathcal{S}_2$. One just keeps adding words in the generators, checking linear independence at each stage, until one arrives at the requisite number 49. A Burnside basis that was effective in eliminating $784 + 177t^2 + 9t^4$ from denominators is given in Table 6 in the Appendix. In that table, A, B denote the inverses of the generators a, b, respectively.

One now proceeds as in [BL15]. Let $\Gamma = \langle a(t), b(t) \rangle$, where the matrix generators $a(t)$, $b(t)$ for the representations in $\mathcal{S}_2$ are those given in Table 4, and let $\mathcal{B} \subset M(7, \mathbb{Q}(t))$ be a Burnside basis for Γ. The group Γ acts on the algebra $M(7, \mathbb{Q}(t))$ by left multiplication, this action giving rise to a representation

Φ (with respect to $\mathcal{B}$) of Γ into $\mathrm{SL}(49, \mathbb{Q}(t))$. We now take the basis $\mathcal{B}^*$ that is dual to $\mathcal{B}$ with respect to the *trace form* $\langle X , Y \rangle = \mathrm{tr}(XY)$ on $M(7, \mathbb{Q}(t))$ and calculate generators for the representation $\Phi^* : \Gamma \to \mathrm{SL}(49, \mathbb{Q}(t))$ with respect to this dual basis $\mathcal{B}^*$. It is found that $784 + 177t^2 + 9t^4$ does not occur as a factor of the denominator of any matrix entry of these generators, whereas it is shown in [BL15] that any prime dividing the denominator of the trace of an element of Γ must so occur.

In this way the prime $784 + 177t^2 + 9t^4$ has been eliminated from denominators of traces of elements of Γ , and since we have already eliminated the prime 2 , this concludes the proof that the representations in $\mathcal{S}_2$ can be written over the integers.

Acknowledgment

We take pleasure in acknowledging several valuable conversations with Darren Long during the preparation of this article.

4 Appendix

4.1 The tautological representation for $\mathcal{H}_{G_2}(\Gamma(2,4,6))$

By means of a *tautological representation*, we can express the entire Hitchin component by means of a single matrix for each generator. The matrix entries are algebraic expressions in the parameters u , v of the variety, and assigning values to u , v together with a sign for α determines a specific representation in the variety. The choice of values for u , v should give a point in the correct region of the (u, v)–plane as indicated in Figure 1. Recall that in Figure 1 the Hitchin component consists of two sheets glued along the curve $\alpha = 0$, and the sign of α determines the particular sheet where the representation lies.

Here matrices with entries in the field $\mathbb{Q}(u, v)(\alpha)$,

$$\alpha \;=\; \pm\sqrt{2(-72u^2 + 72u^3 - 16u^4 - u^5 + v(-192u + 64u^2 + 20u^3) + v^2(-192 - 32u + 2u^2) - 32v^3)} \;,$$

are given for the triangle group $\Gamma(2, 4, 6)$. a , b are generators of orders 4 , 6 respectively, and their product ab has order 2 . The *base representation* ρ_0 is the composition of the holonomy representation in $\mathrm{PSL}(2, \mathbb{R})$ (given by the hyperbolic structure) with the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{PSL}(7, \mathbb{R}) = \mathrm{SL}(7, \mathbb{R})$, and has coordinates $(u, v) = (72, 378)$, $\alpha = 0$.

$$a = \begin{pmatrix} 1 & 0 & -4 & \frac{u}{2} & 0 & 0 & 1 \\ 0 & 0 & 2u & -u & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & u & -u & 0 & 0 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & u & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 & 0 & -4 & 1 & \frac{p_{15}}{2u(u^2-8v)d} & \frac{p_{16}}{2(u^2-8v)d} & 0 \\ 0 & 1 & 2u & 0 & \frac{p_{25}}{2(u^2-8v)d} & \frac{p_{26}}{2(u^2-8v)d} & 0 \\ 0 & 0 & -1 & 0 & \frac{p_{35}}{(u^2-8v)d} & \frac{p_{36}}{(u^2-8v)d} & 0 \\ 0 & 0 & -2 & 0 & \frac{p_{45}}{d} & \frac{p_{46}}{d} & 0 \\ 0 & 0 & u & 0 & \frac{p_{55}}{2(u^2-8v)d} & \frac{p_{56}}{(u^2-8v)d} & 1 \\ 1 & 0 & -2 & 0 & \frac{p_{65}}{2u(u^2-8v)d} & \frac{p_{66}}{2(u^2-8v)d} & 0 \\ 0 & 0 & u & 0 & \frac{p_{75}}{(u^2-8v)d} & \frac{p_{76}}{(u^2-8v)d} & 0 \end{pmatrix}$$

$$p_{ij} = p_{ij0} + (p_{ij1})\alpha \quad (1 \leq i \leq 7 \,,\; 5 \leq j \leq 6)$$

$$d = 192u - 112u^2 + 8u^3 + u^4 + 192v + 32uv - 16u^2v + 32v^2$$

Table 1: Tautological representation for the Hitchin component of the 2, 4, 6–triangle group

In the above table a , b are the images of the elliptic generators of orders 4 , 6 respectively. The coefficients p_{ij0} , p_{ij1} are given in the next table.

$$
\begin{aligned}
p_{150} &= -8u^5 - u^6 + 16u^4(10+v) + 32u^2v(10+v) + 4u^3(-84-24v+v^2) + 512v(18+9v+v^2) \\
&\quad - 64uv(72+18v+v^2) \\
p_{151} &= 112u^2 + u^4 - 32v(6+v) - u^3(14+v) + 8u(-24+2v+v^2) \\
p_{160} &= 3u^6 + u^4(64-96v) - 3u^5(-10+v) - 128v^2(12+8v+v^2) + 16u^3(-54-7v+3v^2) \\
&\quad + 32uv(60+28v+3v^2) + 8u^2(144+4v+32v^2+v^3) \\
p_{161} &= (-4+u)(-16u^2+u^3+16u(3+v)-8v(6+v)) \\
p_{250} &= u(-4u^5 + 32u^3(-6+v) + 1536uv + 2u^4(22+v) + 64v(-36+v^2) - 8u^2(-36+44v+3v^2)) \\
p_{251} &= (-8+u)u(8u+u^2-12(2+v)) \\
p_{260} &= -(u^2(u^5 - u^4v - 8u^3(-14+3v) + 4u^2(-132-32v+5v^2) - 16v(-36+12v+7v^2) \\
&\quad + 16u(36+12v+13v^2))) \\
p_{261} &= u(-4u^3 + u^4 + 16(-6+v)v - 4u^2(8+3v) + 16u(6+5v)) \\
p_{350} &= -2(-6+2u-v)(4u^3 - 32uv - 2u^2(6+v) + 16v(6+v)) \\
p_{351} &= -2(-8+u)(-6+2u-v) \\
p_{360} &= -4u(-6+2u-v)(-14u^2+u^3-4(-6+v)v+4u(6+v)) \\
p_{361} &= 8(u-v)(-6+2u-v) \\
p_{450} &= -8(6-2u+v)^2 \\
p_{451} &= 0 \\
p_{460} &= -16u^2 - 8u^3 + u^4 + 32v(6+v) - 4u(-12+4v+v^2) \\
p_{461} &= 2(-6+2u-v) \\
p_{550} &= u(2u^5 - u^4(-26+v) - 64v(6+v)^2 - 8u^3(46+7v) + 32u(-36+24v+5v^2) \\
&\quad + 4u^2(300+48v+5v^2)) \\
p_{551} &= (-8+u)u(8u+u^2-12(2+v)) \\
p_{560} &= u(-1152u^2 + 1056u^3 - 248u^4 + 2u^5 + u^6 - 1152uv - 192u^2v + 312u^3v - 12u^4v \\
&\quad - 192uv^2 - 128u^2v^2 - 4u^3v^2 + 64uv^3) \\
p_{561} &= u(12u^2 - 4u^3 - 96v + 32uv + 2u^2v - 16v^2) \\
p_{650} &= 8u^5 + u^6 - 768uv(6+v) + 32u^3(18+v) - 8u^4(22+3v) - 128v(-36+v^2) \\
&\quad + 16u^2(-36+88v+9v^2) \\
p_{651} &= -8u^3 - u^4 - 32u(6+v) - 32v(6+v) + 16u^2(7+v) \\
p_{660} &= -3u^6 + u^5(-34+v) + 80u^4(6+v) + 256v^2(6+v) + 64uv(60+16v+v^2) \\
&\quad - 64u^2(-18+29v+5v^2) - 4u^3(348+40v+5v^2) \\
p_{661} &= -((-8+u)u(8u+u^2-12(2+v))) \\
p_{750} &= 24u^5 + u^6 - 256v^2(6+v) - 32u^4(12+v) + 48u^3(26+3v) \\
&\quad + 32uv(-84-8v+v^2) + 64u^2(-18+17v+v^2) \\
p_{751} &= 0 \\
p_{760} &= -4u^2(6-2u+v)(-14u^2+u^3-4(-6+v)v+4u(6+v)) \\
p_{761} &= -2u(-14u^2+u^3-4(-6+v)v+4u(6+v))
\end{aligned}
$$

Table 2: Numerators of entries for columns 5, 6 of the matrix b

4.2 Tables relating to $\mathcal{S}_1$, $\mathcal{S}_2$

$\mathfrak{f}_1$	$=$	$-8(9t+2t^3)$	$\mathfrak{g}_1$	$=$	$19-4t+4t^2$
$\mathfrak{f}_2$	$=$	$8(9t+2t^3)$	$\mathfrak{g}_2$	$=$	$-19+4t-4t^2$
$\mathfrak{f}_3$	$=$	0	$\mathfrak{g}_3$	$=$	0
$\mathfrak{f}_4$	$=$	$-2(45-72t+10t^2-16t^3)$	$\mathfrak{g}_4$	$=$	$133-256t+76t^2-48t^3$
$\mathfrak{f}_5$	$=$	$16(9+2t^2)$	$\mathfrak{g}_5$	$=$	$19-42t+12t^2-8t^3$
$\mathfrak{f}_6$	$=$	$4(-540-249t-126t^2-54t^3+4t^4)$	$\mathfrak{g}_6$	$=$	$362-168t+99t^2-22t^3$
$\mathfrak{f}_7$	$=$	$-4(135+36t+51t^2+8t^3+4t^4)$	$\mathfrak{g}_7$	$=$	$2(59-26t+25t^2-5t^3+2t^4)$
$\mathfrak{f}_8$	$=$	$-1620-1113t-402t^2-242t^3+12t^4$	$\mathfrak{g}_8$	$=$	$3074+1005t+343t^2+388t^3-20t^4$
$\mathfrak{f}_9$	$=$	$2(-483+174t-106t^2+44t^3)$	$\mathfrak{g}_9$	$=$	$512+195t+47t^2+72t^3-4t^4$
$\mathfrak{f}_{10}$	$=$	$-4(-36t+15t^2-8t^3+4t^4)$	$\mathfrak{g}_{10}$	$=$	$-2(-17+9t-14t^2+3t^3-2t^4)$
$\mathfrak{f}_{11}$	$=$	$-1440-969t-362t^2-210t^3+12t^4$	$\mathfrak{g}_{11}$	$=$	$2732+982t+291t^2+368t^3-20t^4$
$\mathfrak{f}_{12}$	$=$	$6(-191+46t-42t^2+12t^3)$	$\mathfrak{g}_{12}$	$=$	$474+184t+43t^2+68t^3-4t^4$
$\mathfrak{f}_{13}$	$=$	$-8(45+36t+13t^2+8t^3)$	$\mathfrak{g}_{13}$	$=$	$-2(-307-130t-38t^2-44t^3)$
$\mathfrak{f}_{14}$	$=$	$4(-72+15t-16t^2+4t^3)$	$\mathfrak{g}_{14}$	$=$	$-2(-53-24t-6t^2-8t^3)$
$\mathfrak{f}_{15}$	$=$	$-279-54t-62t^2-12t^3$	$\mathfrak{g}_{15}$	$=$	$-2(-19-53t+8t^2-12t^3)$
$\mathfrak{f}_{16}$	$=$	$-4(-540-249t-126t^2-54t^3+4t^4)$	$\mathfrak{g}_{16}$	$=$	$-362+168t-99t^2+22t^3$
$\mathfrak{f}_{17}$	$=$	$-4(-36t+15t^2-8t^3+4t^4)$	$\mathfrak{g}_{17}$	$=$	$-2(-17+9t-14t^2+3t^3-2t^4)$
$\mathfrak{f}_{18}$	$=$	$1440+969t+362t^2+210t^3-12t^4$	$\mathfrak{g}_{18}$	$=$	$-2732-982t-291t^2-368t^3+20t^4$
$\mathfrak{f}_{19}$	$=$	$-6(-191+46t-42t^2+12t^3)$	$\mathfrak{g}_{19}$	$=$	$-474-184t-43t^2-68t^3+4t^4$
$\mathfrak{f}_{20}$	$=$	$-4(135+36t+51t^2+8t^3+4t^4)$	$\mathfrak{g}_{20}$	$=$	$2(59-26t+25t^2-5t^3+2t^4)$
$\mathfrak{f}_{21}$	$=$	$1620+1113t+402t^2+242t^3-12t^4$	$\mathfrak{g}_{21}$	$=$	$-3074-1005t-343t^2-388t^3+20t^4$
$\mathfrak{f}_{22}$	$=$	$-2(-483+174t-106t^2+44t^3)$	$\mathfrak{g}_{22}$	$=$	$-512-195t-47t^2-72t^3+4t^4$
$\mathfrak{f}_{23}$	$=$	$8(45+36t+13t^2+8t^3)$	$\mathfrak{g}_{23}$	$=$	$-2(307+130t+38t^2+44t^3)$
$\mathfrak{f}_{24}$	$=$	$-4(-72+15t-16t^2+4t^3)$	$\mathfrak{g}_{24}$	$=$	$-2(53+24t+6t^2+8t^3)$
$\mathfrak{f}_{25}$	$=$	$-279-54t-62t^2-12t^3$	$\mathfrak{g}_{25}$	$=$	$-2(-19-53t+8t^2-12t^3)$
$\mathfrak{f}_{26}$	$=$	0	$\mathfrak{g}_{26}$	$=$	0
$\mathfrak{f}_{27}$	$=$	$-2(3555+366t+844t^2+52t^3-8t^4)$	$\mathfrak{g}_{27}$	$=$	$10648-3627t+2792t^2-364t^3+8t^4$
$\mathfrak{f}_{28}$	$=$	$-4(-1245+36t-298t^2+20t^3)$	$\mathfrak{g}_{28}$	$=$	$1504-381t+366t^2-20t^3$
$\mathfrak{f}_{29}$	$=$	$-900-153t-224t^2-28t^3$	$\mathfrak{g}_{29}$	$=$	$-2(-586+166t-145t^2+12t^3)$
$\mathfrak{f}_{30}$	$=$	$-2(-315-12t-76t^2)$	$\mathfrak{g}_{30}$	$=$	$-2(-82+16t-19t^2)$
$\mathfrak{f}_{31}$	$=$	$-2(-2961+486t-704t^2+142t^3-4t^4)$	$\mathfrak{g}_{31}$	$=$	$-2(2510-79t+479t^2+102t^3-8t^4)$
$\mathfrak{f}_{32}$	$=$	$900+153t+224t^2+28t^3$	$\mathfrak{g}_{32}$	$=$	$-2(586-166t+145t^2-12t^3)$
$\mathfrak{f}_{33}$	$=$	$-2(315+12t+76t^2)$	$\mathfrak{g}_{33}$	$=$	$-2(82-16t+19t^2)$
$\mathfrak{f}_{34}$	$=$	$-2(-2961+486t-704t^2+142t^3-4t^4)$	$\mathfrak{g}_{34}$	$=$	$-2(2510-79t+479t^2+102t^3-8t^4)$
$\mathfrak{f}_{35}$	$=$	$1521-144t+356t^2-48t^3$	$\mathfrak{g}_{35}$	$=$	$-4(274+7t+53t^2+14t^3)$
		$\langle a_{\text{even}}\,,\, b_{\text{even}}\rangle$			$\langle a_{\text{odd}}\,,\, b_{\text{odd}}\rangle$

Table 3: Coordinates of 3–form preserved by $\mathcal{S}_1$

$$a = \begin{pmatrix} 1 & 0 & -2 & 2r & 0 & 0 & r \\ 0 & 0 & 4 & -4 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{q_{15}}{4d} & \frac{q_{16}}{2d} & 0 \\ 0 & 0 & 0 & 0 & \frac{q_{25}}{2d} & \frac{q_{26}}{d} & 1 \\ 0 & 0 & -1 & 0 & \frac{q_{35}}{4d} & \frac{q_{36}}{2d} & 0 \\ 1 & 0 & 0 & 0 & \frac{q_{45}}{4d} & \frac{q_{46}}{2d} & 0 \\ 0 & 0 & 0 & 0 & \frac{q_{55}}{d} & \frac{q_{56}}{d} & 0 \\ 0 & 0 & 0 & 0 & \frac{q_{65}}{2d} & \frac{q_{66}}{d} & 0 \\ 0 & 1 & 0 & 0 & \frac{q_{75}}{2d} & \frac{q_{76}}{d} & 0 \end{pmatrix}$$

$$\begin{aligned}
r &= 22 + 3t^2 \\
d &= 784 + 177t^2 + 9t^4 \\
q_{15} &= -(34496 + 28368t + 20412t^2 + 12177t^3 + 3726t^4 + 1728t^5 + 216t^6 + 81t^7) \\
q_{16} &= -3(20384 + 13768t + 11220t^2 + 5307t^3 + 1944t^4 + 666t^5 + 108t^6 + 27t^7) \\
q_{25} &= -(10976 - 7200t + 4470t^2 - 2160t^3 + 603t^4 - 162t^5 + 27t^6) \\
q_{26} &= -784 + 5436t - 1260t^2 + 1719t^3 - 351t^4 + 135t^5 - 27t^6 \\
q_{35} &= -3(16 + 3t^2)(112 + 40t + 37t^2 + 6t^3 + 3t^4) \\
q_{36} &= -3(1792 + 976t + 1048t^2 + 327t^3 + 177t^4 + 27t^5 + 9t^6) \\
q_{45} &= 34496 - 4272t + 12492t^2 - 1179t^3 + 1458t^4 - 81t^5 + 54t^6 \\
q_{46} &= 7840 - 4272t + 3762t^2 - 1179t^3 + 567t^4 - 81t^5 + 27t^6 \\
q_{55} &= 784 + 960t + 357t^2 + 324t^3 + 36t^4 + 27t^5 \\
q_{56} &= 2(1568 + 1254t + 714t^2 + 369t^3 + 72t^4 + 27t^5) \\
q_{65} &= -3t(320 + 108t^2 + 9t^4) \\
q_{66} &= -(784 + 960t + 357t^2 + 324t^3 + 36t^4 + 27t^5) \\
q_{75} &= 10976 + 5280t + 4830t^2 + 1512t^3 + 657t^4 + 108t^5 + 27t^6 \\
q_{76} &= 3(7056 + 2544t + 2560t^2 + 681t^3 + 285t^4 + 45t^5 + 9t^6)
\end{aligned}$$

Table 4: Generators of orders $4, 2$ for the curve $\mathcal{C}_2$

This table gives matrices σ with the property that for the generators a, b of Table 4:

(i) $\sigma\, a\, \sigma^{-1} \in \mathrm{SL}(7, \mathbb{Z})$;
(ii) all denominators of entries of $\sigma\, b\, \sigma^{-1}$ are odd.

No conjugation is required for $t \equiv 0 \pmod 8$, as denominators are already odd.

$$t \equiv 1 \pmod 4 : \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

$$t \equiv 2 \pmod 4 : \quad \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$t \equiv 3 \pmod 4 : \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$t \equiv 4 \pmod 8 : \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Table 5: Conjugating matrices for expunging the prime 2 from the representations in $\mathcal{S}_2$

I	a	b	A	B	aa	ab
aB	ba	bA	bb	aab	aaB	abA
abb	aBa	aBB	baa	baB	bAb	bAB
Abb	AbA	$aabb$	$aabA$	$aaBa$	$aaBA$	$aaBB$
$abaB$	$abba$	$abbb$	$aBab$	$aBBa$	$baab$	$baaB$
$baBa$	$baBB$	$bbaa$	$bbaB$	$bbba$	$bbbA$	$bbAb$
$bAbb$	$bAbA$	$Abba$	$AbbA$	$AbAb$	$Baab$	$BaBab$

Table 6: A Burnside basis for the group of Table 4 $(A = a^{-1}\,,\ B = b^{-1})$

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