Bismash Product Commuting Squares

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Abstract

Let L be a finite group with subgroups F, G such that $L = FG$ is an exact factorization of L. This factorization gives maps $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$ such that $(F, G, \triangleleft, \triangleright)$ is a matched pair of groups. We introduce *bismash product commuting squares* having in one corner the bismash product Hopf algebra $\mathbb{C}^G \# \mathbb{C}[F]$ constructed from $(F, G, \triangleleft, \triangleright)$, and in the opposite corner its dual $\mathbb{C}^F \# \mathbb{C}[G]$.

The defect of a commuting square was introduced in [4] as an upper bound for the number of linearly independent sequential deformations of the commuting square, in the space of commuting squares. We compute defects of bismash product commuting squares in the cases of exact factorizations arising from direct products $L = F \times G$, semidirect products $L = F \rtimes G$ with G abelian, and a more exotic Zappa-Szép product which is not a semidirect product.

1 Introduction

Let L be a finite group with proper subgroups F, G such that $L = FG$ is an exact factorization of L. This factorization gives group actions $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$ making $(F, G, \triangleleft, \triangleright)$ a matched pair of groups. It follows that we can construct from $(F, G, \triangleleft, \triangleright)$ a bismash product Hopf algebra $\mathbb{C}^G \# \mathbb{C}[F]$. In Sections 2 and 3 we recall the definition of an exact factorization and of the bismash product, and we give several examples.

The algebra $\mathbb{C}^G \# \mathbb{C}[F]$ and its dual $\mathbb{C}^F \# \mathbb{C}[G]$ have canonical embeddings in $M_n(\mathbb{C})$, for $n = |L|$. We can thus consider the square of inclusions

$$\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]} = \begin{pmatrix} \mathbb{C}^F \# \mathbb{C}[G] & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} I_n & \subset & \mathbb{C}^G \# \mathbb{C}[F] \end{pmatrix}.$$

In Section 4 we prove that this is a commuting square. Note that this requires a proof, since the corners of $\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]}$ are embedded in a different way than the Hopf algebra commuting squares considered in [6]. We also use Section 4 to give some concrete examples of such commuting squares.

In Section 5 we present computations of defects for bismash product commuting squares $\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]}$. The (undephased) defect of a commuting square was introduced in [4] as an upper bound for the number of linearly independent directions in which one can sequentially deform the commuting square in the space of commuting squares.

We compute the defect of $\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]}$ when $L = F \times G$ is a direct product of groups. Next we compute the defect when $L = F \rtimes G$ is a semidirect product of groups with G abelian. We then consider a more exotic example of a Zappa-Szép product which is not a semidirect product. In all our computations the defect of the bismash product commuting square $\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]}$ is the same as the defect of the group L.

The framework of our computations hints to quasi-canonical bases for the tangent space at the corresponding commuting square, in the manifold of commuting squares. In the case of group commuting squares, such bases can be exponentiated to construct analytic deformations in the manifold of commuting squares ([5]). In particular, for group commuting squares the defect is not just an upper bound but an exact formula for the number of linearly independent directions of sequential (or analytic) deformations by commuting squares. Our computations for smash product commuting squares hint to the existence of similar deformations, which we plan to explore in a future paper. By iterating Jones' basic construction ([3]), such deformations would yield $d(L)$ families of (possibly non-isomorphic) subfactors.

2 Exact Factorizations of Finite Groups

Definition 1. *An exact factorization of a group L is a pair of proper subgroups F, G such that $L = FG$ and $F \cap G = \{e\}$. We say that F and G factor L exactly.*

Note that if F and G are proper subgroups of a group L, then F and G factor L exactly if and only if $|L| = |F| \cdot |G|$ and $F \cap G = \{e\}$. Furthermore, $L = FG$ is an exact factorization of L if and only if each $l \in L$ can be uniquely expressed as $l = fg$ for some unique $f \in F$ and $g \in G$. This is equivalent to saying that L is the internal Zappa-Szép product of F, G.

Example 1. *Let $L = A_5$ be the alternating group with 60 elements. Embed $F = A_4$ (the alternating group with 12 elements) in L, by defining F as the set of all permutations in L that leave 5 invariant. Consider also the subgroup $G = \mathbb{Z}_5$ generated by the 5-cycle $\langle(12345)\rangle$. We have $A_4 \cap \mathbb{Z}_5 = \{(1)\}$ and $|A_4| \cdot |\mathbb{Z}_5| = 12 \cdot 5 = 60 = |A_5|$. It follows that $A_4\mathbb{Z}_5$ is an exact factorization of A_5. Note that since A_n is a simple group for $n \geq 5$, neither A_4 nor $\mathbb{Z}_5$ are normal subgroups of A_5. In particular, this factorization is not a direct or semidirect product.*

Let $L = FG$ be an exact factorization. For every $g \in G$ and $f \in F$, define the maps $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$ by the following relation:

$$gf = (g \triangleright f)(g \triangleleft f).$$

We recall some basic facts about $\triangleright, \triangleleft$, which we will use in our computations.

Proposition 1. *For an exact factorization $L = FG$, the maps $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$ defined via $(g \triangleright f)(g \triangleleft f) = gf \in L$ for every $g \in G$ and $f \in F$ are group actions. Furthermore, we have that for every $a, b \in F$ and $x, y \in G$,*

1. $x \triangleright (ab) = (x \triangleright a)((x \triangleleft a) \triangleright b)$
2. $(xy) \triangleleft a = (x \triangleleft (y \triangleright a))(y \triangleleft a)$

Proof. Let $x \in G$ and $a, b \in F$. We have that

$$x(ab) = (x \triangleright ab)(x \triangleleft ab)$$

and

$$\begin{aligned}(xa)b &= ((x \triangleright a)(x \triangleleft a))b = (x \triangleright a)((x \triangleleft a)b) \\ &= (x \triangleright a)((x \triangleleft a) \triangleright b)(x \triangleleft a \triangleleft b) = (x \triangleright a)((x \triangleleft a) \triangleright b)(x \triangleleft ab)\end{aligned}$$

from which it follows that

$$(x \triangleright ab) = (x \triangleright a)((x \triangleleft a) \triangleright b).$$

A similar calculation proves the second claim. □

Definition 2. *Let F, G be two finite groups. Consider maps $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$ satisfying the conditions 1,2 from Proposition 1. We call $(F, G, \triangleright, \triangleleft)$ a matched pair of groups.*

Example 2. *For each odd $n \in \mathbb{N}$, we have that $A_n = A_{n-1}\mathbb{Z}_n$ is an exact factorization of the alternating group A_n, where $\mathbb{Z}_n = \langle(12\cdots n)\rangle$. It follows that $(A_{n-1}, \mathbb{Z}_n, \triangleright, \triangleleft)$ is a matched pair of groups.*

The following lemma will be used several times throughout the paper.

Lemma 1. *Let $(F, G, \triangleright, \triangleleft)$ be a matched pair of groups coming from the exact factorization $L = FG$. For $g \in G$ and $f \in F$, we have that $(g \triangleright f)^{-1} = (g \triangleleft f) \triangleright f^{-1}$ and $(g \triangleleft f)^{-1} = g^{-1} \triangleleft (g \triangleright f)$.*

Proof. Since for any $g \in G$ and $f \in F$ we have $gf = (g \triangleright f)(g \triangleleft f)$, it follows that

$$\begin{aligned}(g \triangleright f)^{-1} &= (g \triangleleft f)f^{-1}g^{-1}\\ &= ((g \triangleleft f) \triangleright f^{-1})((g \triangleleft f) \triangleleft f^{-1})g^{-1}\\ &= ((g \triangleleft f) \triangleright f^{-1})(g \triangleleft e)g^{-1}\\ &= (g \triangleleft f) \triangleright f^{-1}.\end{aligned}$$

A similar calculation shows $(g \triangleleft f)^{-1} = g^{-1} \triangleleft (g \triangleright f)$. □

Remark 1. *In the latter sections of this paper we will identify L with the set $F \times G$, and the subgroups F and G with $F \times \{1_L\}$ and $\{1_L\} \times G$ respectively. The group structure on $F \times G$ is given by:*

$$\begin{aligned}(f, g)(f', g') &= (f, 1_G)(1_F, g)(f', 1_G)(1_F, g')\\ &= (f, 1_G)(g \triangleright f', 1_G)(1_F, g \triangleleft f')(1_F, g')\\ &= (f(g \triangleright f'), (g \triangleleft f')g')\end{aligned}$$

for all $f, f' \in F$ and $g, g' \in G$.

Note that if L is the direct product of F and G then the actions $\triangleleft, \triangleright$ are trivial and thus the group structure defined above on $F \times G$ is just the group structure of the direct product of groups $F \times G$. It is easy to check that the group structures also agree when L is a semidirect product ($L = F \rtimes G$ or $L = F \ltimes G$), and when L is a Zappa-Szép product $L = F \bowtie G$ (see [1]).

3 The Bismash Product Hopf Algebra

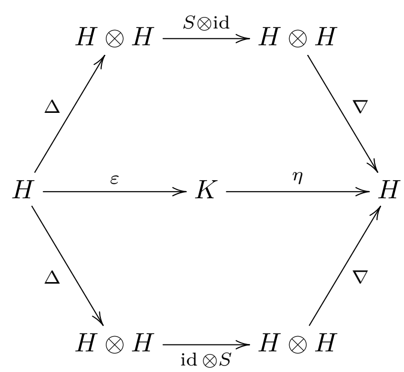
In this section we recall the definition of bismash product Hopf algebras, from which we will construct commuting squares. We begin by recalling the definition of a Hopf algebra.

Definition 3. *A Hopf Algebra is a K-vector space H equipped with five (bi)linear maps*

$$\nabla : H \otimes H \to H, \qquad \Delta : H \to H \otimes H$$

$$\eta : K \to H, \qquad \epsilon : H \to K, \qquad S : H \to H$$

such that $(H, \Delta, \nabla, \eta, \varepsilon)$ forms a bialgebra and such that the following diagram commutes.



The linear maps $\nabla, \eta, \Delta, \varepsilon, S$ are called the product, unit, co-product, co-unit, and antipode, respectively.

For $h \in H$, we write

$$\Delta(h) = \sum h_1 \otimes h_2$$

where by standard notation the index of the summation is suppressed.

3.1 Definition of the Bismash Product Hopf Algebra

Let L be a finite group and let $(F, G, \triangleright, \triangleleft)$ be a matched pair of groups arising from an exact factorization $L = FG$. Let $\{\rho_g | \ g \in G\}$ be the canonical basis for $\mathbb{C}^G$. Abusing notation for simplicity, let $\{\rho_f | \ f \in F\}$ be the canonical basis for $\mathbb{C}^F$. We have that $\mathbb{C}^F$ is a left $\mathbb{C}[G]$-module via the action defined by

$$g \cdot \rho_f := \rho_{g \triangleright f}$$

and $\mathbb{C}^G$ is a left $\mathbb{C}[F]$-module via the action defined by

$$f \cdot \rho_g := \rho_{g \triangleleft f^{-1}}$$

for all $g \in G, f \in F$

Definition 4. *Let $L = FG$ be an exact factorization and let $(F, G, \triangleright, \triangleleft)$ be its matched pair of groups. The bismash product Hopf algebra $\mathbb{C}^G \# \mathbb{C}[F]$ associated to (L, F, G) is the Hopf algebra having underlying vector space $\mathbb{C}^G \otimes \mathbb{C}[F]$, multiplication given by*

$$(\rho_x \# a)(\rho_y \# b) = \delta_x^{y \triangleleft a^{-1}} \rho_x \# ab$$

and unit given by

$$u(a) = \Sigma_{g \in G} \rho_g \# a,$$

for all $x, y \in G$ and $a, b \in F$, where δ is the Kronecker delta function.

The algebra structure of $(\mathbb{C}^G \# \mathbb{C}[F])^* = \mathbb{C}^F \# \mathbb{C}[G]$ then dualizes to give the comultiplication and counit of $\mathbb{C}^G \# \mathbb{C}[F]$.

Remark 2. *The formula for multiplication in $\mathbb{C}^G \# \mathbb{C}[F]$ follows from:*

$$(\rho_x \# a)(\rho_y \# b) = \rho_x (a \cdot \rho_y) \# ab = \rho_x \rho_{y \triangleleft a^{-1}} \# ab = \delta_x^{y \triangleleft a^{-1}} \rho_x \# ab.$$

The dual algebra $\mathbb{C}^F \# \mathbb{C}[G]$ has multiplication defined via

$$(\rho_a \# x)(\rho_b \# y) = \rho_a (x \cdot \rho_b) \# xy = \delta_a^{x \triangleright b} \rho_a \# xy$$

for all $x, y \in G$ and $a, b \in F$.

Remark 3. *The Hopf algebra structure of the bishmash product depends not only on the subgroups F, G, but also on their embeddings in $L = FG$. For example, consider $L = \mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$. In this case the matched pair of groups $(\mathbb{Z}_3, \mathbb{Z}_2)$ has trivial actions $\triangleright$ and $\triangleleft$. Consider also $L' = D_3$, the dihedral group with 6 elements, yielding the matched pair $(\mathbb{Z}_3, \mathbb{Z}_2)$ with actions $\triangleright'$ and $\triangleleft'$. Since $\mathbb{Z}_3$ is normal in L', the action $\triangleleft'$ is still trivial (but $\triangleright'$ is not). It follows that the bismash product $\mathbb{C}^{\mathbb{Z}_2} \# \mathbb{C}[\mathbb{Z}_3]$ has identical multiplications for both matched pairs of groups corresponding to L, L'. However, the dual algebras are not isomorphic and thus induce different comultiplications on $\mathbb{C}^{\mathbb{Z}_2} \# \mathbb{C}[\mathbb{Z}_3]$.*

4 Bismash product commuting squares

In this section we associate to any bismash product Hopf algebra a bismash product commuting square.

Definition 5. *Let $(F, G, \triangleleft, \triangleright)$ be a matched pair of groups arising from the exact factorization $L = FG$, where L is a group with n elements containing F, G as subgroups. Let*

$$\mathcal{C}^{L}_{\mathbb{C}^G\#\mathbb{C}[F]} = \begin{pmatrix} \mathbb{C}^F\#\mathbb{C}[G] & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}^G\#\mathbb{C}[F] \end{pmatrix}.$$

The embedding of $\mathbb{C}^G\#\mathbb{C}[F]$ in $M_n(\mathbb{C})$ is realized by using the left-regular representation. That is to say, we identify the basis element $\rho_g\#f \in \mathbb{C}^G\#\mathbb{C}[F]$ with the unitary $u_{(g,f)} = u_{\rho_g\#f} = \big(z_{(x,a),(y,b)}\big)_{\substack{x,y\in G\\ a,b\in F}} \in M_n(\mathbb{C})$, given by

$$z_{(x,a),(y,b)} = \begin{cases} 1, & \text{if } (\rho_g\#f)\cdot(\rho_y\#b) = (\rho_x\#a) \\ 0, & \text{otherwise.} \end{cases}$$

Let us identify $M_n(\mathbb{C}) = M_{|G|}(\mathbb{C}) \otimes M_{|F|}(\mathbb{C})$, by fixing some order on the elements of G, F respectively. Consider the matrix units $e_{(g,f),(g',f')} = e_{g,g'} \otimes e_{f,f'}$ of $M_n(\mathbb{C})$, with $f, f' \in F$ and $g, g' \in G$. Note that the pair (g, f) can be identified with the unique element $gf \in L$, since the factorization $L = GF$ is also exact. With these notations, we have:

$$u(g, f) = u_{\rho_g\#f} = \sum_{x\in F} e_{(g,fx),(g\triangleleft f,x)}.$$

The algebra $\mathbb{C}^F\#\mathbb{C}[G]$ is embedded in $M_n(\mathbb{C})$ by identifying each element $\rho_f\#g$ of its canonical basis with the unitary matrix $v(f, g)$ given by:

$$v(f, g) = v_{\rho_f\#g} = \sum_{y\in G} e_{(y,y^{-1}\triangleright f),(g^{-1}y,y^{-1}\triangleright f)}$$

We call $\mathcal{C}^{L}_{\mathbb{C}^G\#\mathbb{C}[F]}$ the bismash product commuting square associated to the exact factorization $L = FG$.

Remark 4. *It is easy to check that the formula from Definition 5 gives an embedding of $\mathbb{C}^F\#\mathbb{C}[G]$ in $M_n(\mathbb{C})$. Indeed, let $g, g' \in G$ and $f, f' \in F$ and consider the matrices*

$$v(f, g) = \sum_{y\in G} e_{(y,y^{-1}\triangleright f),(g^{-1}y,y^{-1}\triangleright f)}$$

and

$$v(f', g') = \sum_{x\in G} e_{(x,x^{-1}\triangleright f'),(g'^{-1}x,x^{-1}\triangleright f')}$$

as in Definition 5. We show that

$$v(f,g)v(f'g') = v(f,gg')$$

if $f = g \triangleright f'$, *and* $v(f,g)v(f'g') = 0$ *otherwise.*

First note that $v(f,gg') = \sum_{y \in G} e_{(y,y^{-1} \triangleright f),(g'^{-1}g^{-1}y,y^{-1} \triangleright f)}$.

It follows that

$$v(f,g)v(f',g') = \sum_{x,y \in G} e_{(y,y^{-1} \triangleright f),(g^{-1}y,y^{-1} \triangleright f)} e_{(x,x^{-1} \triangleright f'),(g'^{-1}x,x^{-1} \triangleright f')}$$

This product is 0 except when

$$g^{-1}y = x \text{ and } y^{-1} \triangleright f = x^{-1} \triangleright f'.$$

Note that in this case, we have that

$$y^{-1} \triangleright f = x^{-1} \triangleright f' = (y^{-1}g) \triangleright f' = y^{-1} \triangleright (g \triangleright f'),$$

which implies that $f = g \triangleright f'$. *Therefore, we have that*

$$v(f,g)v(f',g') = \sum_{y \in G} e_{(y,y^{-1} \triangleright f),(g'^{-1}g^{-1}y,y^{-1} \triangleright f)} = v(f,gg').$$

Example 3. *Consider again the exact factorization* $L = A_5 = FG$, *where* $F = A_4$ *and* $G = \mathbb{Z}_5$.

Figure 1 shows the form of a typical element in the algebra $\mathbb{C}^{\mathbb{Z}_5} \# \mathbb{C}[A_4]$.

Figure 2 shows the form of a typical element in the dual algebra $\mathbb{C}^{A_4} \# \mathbb{C}[\mathbb{Z}_5]$.

Similar shadings represent identical matrix entries. The white shading represents the zero entries.

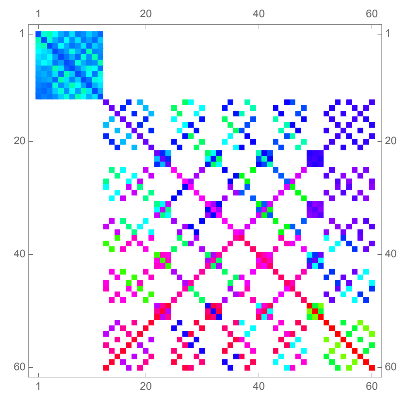


Figure 1: A typical element in the bismash product $\mathbb{C}^{\mathbb{Z}_5}\#\mathbb{C}[A_4]$ associated to the group A_5. Identical entries are represented by the same color.

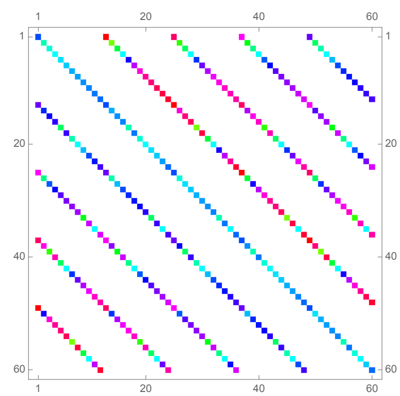


Figure 2: A typical element in the bismash product $\mathbb{C}^{A_4}\#\mathbb{C}[\mathbb{Z}_5]$, as the dual of $\mathbb{C}^{\mathbb{Z}_5}\#\mathbb{C}[A_4]$. Identical entries are represented by the same color.

We now check that the square of inclusions from Definition 5 is indeed a commuting square. Note that this requires a proof, since the corners of $\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]}$ are embedded in a different way than the traditional Hopf algebra commuting squares considered in [6].

Theorem 1. *Let $(F, G, \triangleleft, \triangleright)$ be a matched pair of groups arising from the exact factorization $L = FG$, where L is a group with n elements containing F, G as subgroups. Then the square of inclusions*

$$\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]} = \begin{pmatrix} \mathbb{C}^F\#\mathbb{C}[G] & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}^G\#\mathbb{C}[F] \end{pmatrix}$$

is a commuting square.

Proof. We recall the commuting square condition:

$$(\mathbb{C}^G\#\mathbb{C}[F])\ominus\mathbb{C}I_n \perp (\mathbb{C}^F\#\mathbb{C}[G])\ominus\mathbb{C}I_n$$

where the orthogonality and orthogonal complements are considered with respect to the inner product induced by the normalized trace of $M_n(\mathbb{C})$. Equivalently, we must show that $\tau(uv)=\tau(u)\tau(v)$ for all $u\in\mathbb{C}^G\#\mathbb{C}[F]$ and $v\in\mathbb{C}^F\#\mathbb{C}[G]$.

Consider general basis elements $u(g,f)=u_{\rho_g\#f}$, $v(f',g')=v_{\rho_{f'}\#g'}$ of $\mathbb{C}^G\#\mathbb{C}[F]$, $\mathbb{C}^F\#\mathbb{C}[G]$ respectively. We show $\tau(u_{\rho_g\#f}v^*_{\rho_{f'}\#g'})=\tau(u_{\rho_g\#f})\tau(v^*_{\rho_{f'}\#g'})$. Indeed, we have:

$$\begin{aligned}\tau(u(g,f)v(f'g')^*)&=\tau\left(\sum_{x\in F}\sum_{y\in G}e_{(g,fx),(g\triangleleft f,x)}e_{(g'^{-1}y,y^{-1}\triangleright f'),(y,y^{-1}\triangleright f')}\right)\\&=\tau\left(e_{(g,fx),(y,y^{-1}\triangleright f')}\right)\ \text{ with } y=g'(g\triangleleft f)\text{ and } x=y^{-1}\triangleright f'.\end{aligned}$$

It follows that

$$\begin{aligned}\tau(u(g,f)v(f'g')^*)&=\tau\left(e_{(g,fx),(y,y^{-1}\triangleright f')}\right)\\&=\frac{1}{|F||G|}\delta^g_{g'(g\triangleleft f)}\delta^{fx}_x\\&=\frac{1}{|F||G|}\delta^g_{g'(g\triangleleft f)}\delta^f_{1_F}\end{aligned}$$

Note that

$$\tau(u(g,f))=\tau\left(\sum_{x\in F}e_{(g,fx),(g\triangleleft f,x)}\right)=\frac{1}{|F|}\delta^g_{g\triangleleft f}\delta^f_{1_F}$$

and

$$\tau(v(f',g')^*)=\tau\left(\sum_{y\in G}e_{(g'^{-1}y,y^{-1}\triangleright f'),(y,y^{-1}\triangleright f')}\right)=\frac{1}{|G|}\delta^{g'}_{1_G}.$$

If $f\neq 1_F$ then we have

$$\tau(u(g,f)v(f'g')^*)=0=\tau(u(g,f))\tau(v(f',g')^*).$$

If $f=1_F$, then we have $g\triangleleft f=g$ and so $\delta^g_{g'(g\triangleleft f)}=\delta^{1_G}_{g'}$ and $\delta^g_{g\triangleleft f}=1$. Therefore, we obtain

$$\tau(u(g,f)v(f'g')^*)=\tau(u(g,f))\tau(v(f',g')^*)=\frac{1}{|F||G|}\delta^g_{g'(g\triangleleft f)}\delta^f_{1_F}=\frac{1}{|F||G|}\delta^{1_G}_{g'}\delta^f_{1_F}.$$

This shows that the commuting square condition holds.

□

5 The Undephased Defect of a Bismash Commuting Square

In this section we give computations of the undephased defect (in the sense of [4]) of some classes of bismash product commuting squares.

Let

$$A = \operatorname{span}_{\mathbb{C}} \left\{ u_{(g,f)} = \sum_{a \in F} e_{(g,fa),(g \triangleleft f, a)} : f \in F, g \in G \right\},$$

$$A^* = \operatorname{span}_{\mathbb{C}} \left\{ v_{(f,g)} = \sum_{x \in G} e_{(x, x^{-1} \triangleright f),(g^{-1}x, x^{-1} \triangleright f)} : f \in F, g \in G \right\}$$

be the representations of $\mathbb{C}^G \# \mathbb{C}[F]$, $\mathbb{C}^F \# \mathbb{C}[G]$ respectively in $M_n(\mathbb{C})$.

Let

$$[A, A^*] = \operatorname{span}_{\mathbb{C}} \{ [u, v] = uv - vu : u = u(g,f), v = v(f', g'), g, g' \in G, f, f' \in F \}.$$

It follows that

$$\begin{aligned} \{[A, A^*] = &\operatorname{span}_{\mathbb{C}} \{ e_{(g, f((g \triangleleft f)^{-1} \triangleright f')),(g'^{-1}(g \triangleleft f),(g \triangleleft f)^{-1} \triangleright f')} \\ &- e_{(g'g, g^{-1}g'^{-1} \triangleright f'),(g \triangleleft f, f^{-1}(g^{-1}g'^{-1} \triangleright f'))} : f, f' \in F, g, g' \in G \} \end{aligned} \tag{1}$$

The defect of the associated bismash product commuting square is given by

$$d(A) := \dim_{\mathbb{C}} [A, A^*]^{\perp}.$$

This is the same as the dimension over $\mathbb{C}$ of the vector space

$$V := \left\{ \left(z_{(g,f),(g',f')} \right)_{\substack{g,g' \in G \\ f,f' \in F}} \in M_n(\mathbb{C}) : \sum_{\substack{g,g' \in G \\ f,f' \in F}} z_{(g,f),(g',f')} \left[u_{(g,f)}, v_{(f',g')} \right] = 0 \right\}.$$

Note that the condition

$$\sum_{\substack{g,g' \in G \\ f,f' \in F}} z_{(g,f),(g',f')} [u_{(g,f)}, v_{(f',g')}] = 0$$

is equivalent to

$$\sum_{\substack{g,g' \in G \\ f,f' \in F}} z_{(g,f),(g',f')} e_{(g, f((g \triangleleft f)^{-1} \triangleright f')),(g'^{-1}(g \triangleleft f),(g \triangleleft f)^{-1} \triangleright f')}$$

$$= \sum_{\substack{g,g' \in G \\ f,f' \in F}} z_{(g,f),(g',f')} e_{(g'g, g^{-1}g'^{-1} \triangleright f'),(g \triangleleft f, f^{-1}(g^{-1}g'^{-1} \triangleright f'))}$$

5.1 Computing $d(\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]})$ when $L = F \times G$

We show that when the exact factorization is a direct product, the defect of the commuting square $d(\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]})$ is just the defect of the group L (in the sense of [4]). Note that an exact factorization $L = FG$ yields $L \simeq F \times G$ if and only if both F, G are normal in G. This is because the elements of two normal subgroups with trivial intersection must commute with each other.

Theorem 2. *Let $L = FG$ be an exact factorization of a finite group L, where F and G are normal subgroups of L. We have*

$$d(\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]}) = d(L).$$

Proof. Since F, G are normal subgroups, we have that both $\triangleleft$ and $\triangleright$ are trivial, i.e. $g \triangleright f = f$ and $g \triangleleft f = g$ for all $f \in F$ and $g \in G$. Using these specific actions, the defect is given by the dimension of the vector space of matrices $z_{(g,f),(g',f')} \in M_{|G||F|}(\mathbb{C})$ satisfying:

$$\sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')} e_{(g,ff'),(g'^{-1}g,f')} - \sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')} e_{(g'g,f'),(g,f^{-1}f')} = 0$$

Using the change of variables $x = g, a = ff', y = g'^{-1}g,$ and $b = f'$ for the first sum, and $x = g'g, a = f', y = g,$ and $b = f^{-1}f'$ for the second sum, we have:

$$z_{(x,ab^{-1}),(xy^{-1},b)} = z_{(y,ab^{-1}),(xy^{-1},a)}.$$

Using another change of variables $g = x, f = ab^{-1}, g' = xy^{-1}$ and $f' = b$ it follows that

$$z_{(g,f),(g',f')} = z_{(g'^{-1}g,f),(g',ff')}.$$

Iterating this relationship, we see that, for all $n \in \mathbb{Z}_{>0}$

$$z_{(g,f),(g',f')} = z_{(g'^{-n}g,f),(g',f^nf')}.$$

The smallest n that gives $g = g'^{-n}g$ and $f' = f^nf'$ is $n = lcm(|g'|, |f|)$ for each $g \in G$ and $f' \in F$. Therefore, the dimension of the space of solutions $z_{(g,f),(g',f')}$ is equal to

$$\sum_{\substack{g\in G\\ f'\in F}} \frac{|G||F|}{lcm(|g'|, |f|)}.$$

In the above we denoted by $lcm(a, b)$ the least common multiple of a, b.

For each $l \in L$, $l = fg' = g'f$, the order of l is $|l| = lcm(|g'|, |f|)$. Using the formula for the defect of a group L, we obtain

$$d(\mathbb{C}^G\#\mathbb{C}[F]) = \sum_{\substack{g\in G\\ f'\in F}} \frac{|G||F|}{lcm(|g'|, |f|)} = \sum_{l\in L} \frac{|L|}{|l|} = d(L).$$

□

5.2 Computing $d(\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]})$ when $L = F \rtimes G$

We now look at semidirect products $L = F \rtimes G$, with G abelian. We show that it is again true that the defect of the bismash product commuting square is equal to the defect of the group L.

Theorem 3. *Let $L = FG$ be an exact factorization of a finite group L, where F is a normal subgroup of L and G is an abelian subgroup of L. We have*

$$d(\mathcal{C}^L_{\mathbb{C}^G\#\mathbb{C}[F]}) = d(L).$$

Proof. Since $L = F \rtimes G$, we have $g \triangleright f = gfg^{-1}$ and $g \triangleleft f = g$ for all $f \in F$ and $g \in G$. The defect of the bismash product commuting square is given by the dimension of the matrix vectors space V of solutions z to the equation

$$\sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')}[u_{(g,f)}, v_{(f',g')}] = 0.$$

Equivalently,

$$\sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')} e_{(g,f(g^{-1}f'g)),(g'^{-1}g,g^{-1}f'g)} = \sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')} e_{(g'g,g^{-1}g'^{-1}f'g'g),(g,f^{-1}(g^{-1}g'^{-1}f'g'g))}.$$

By changing the variable g to $g'^{-1}g$ in the second sum, we obtain the equivalent equation

$$\sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',f')} e_{(g,f(g^{-1}f'g)),(g'^{-1}g,g^{-1}f'g)} = \sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g'^{-1}g,f),(g',f')} e_{(g,g^{-1}f'g),(g'^{-1}g,f^{-1}(g^{-1}f'g))}.$$

After replacing f' with $gf'g^{-1}$ in both sums, we have

$$\sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g,f),(g',gf'g^{-1})} e_{(g,ff'),(g'^{-1}g,f')} = \sum_{\substack{g,g'\in G\\ f,f'\in F}} z_{(g'^{-1}g,f),(g',gf'g^{-1})} e_{(g,f'),(g'^{-1}g,f^{-1}f')}.$$

We will use the change of variables $x = g$, $a = ff'$, $y = g'^{-1}g$, and $b = f'$ in the first sum, and $x = g$, $a = f'$, $y = g'^{-1}g$, and $b = f^{-1}f'$ in the second sum. This gives

$$\sum_{\substack{x,y\in G\\ a,b\in F}} z_{(x,ab^{-1}),(xy^{-1},xbx^{-1})} e_{(x,a),(y,b)} = \sum_{\substack{x,y\in G\\ a,b\in F}} z_{(y,ab^{-1}),(xy^{-1},xax^{-1})} e_{(x,a),(y,b)}$$

The above is equivalent to

$$z_{(x,ab^{-1}),(xy^{-1},xbx^{-1})} = z_{(y,ab^{-1}),(xy^{-1},xax^{-1})}$$

for all $x, y \in G$ and $a, b \in F$.

Using the change of variables $g = x$, $f = ab^{-1}$, $g' = xy^{-1}$, and $f' = xbx^{-1}$ yields

$$z_{(g,f),(g',f')} = z_{(g'^{-1}g,f),(g',(gfg^{-1})f')}$$

for all $g, g' \in G$ and $f, f' \in F$.

Iterating this relationship gives

$$z_{(g,f),(g',f')} = z_{(g'^{-n}g,f),(g',(g'^{-(n-1)}gfg^{-1}g'^{n-1})\cdots(gfg^{-1})f')}$$

for all n positive integers.

Note that $(g'^{-(n-1)}gfg^{-1}g'^{n-1})\cdots(gfg^{-1}) = g'^{-n} \cdot (g'gfg^{-1})^n$. Since we are under the assumption that G is abelian, we have

$$(g'gfg^{-1})^n = (gg'fg^{-1})^n = g(g'f)^n g^{-1}.$$

Thus we obtain

$$z_{(g,f),(g',f')} = z_{(g'^{-n}g,f),(g',g'^{-n}g(g'f)^n g^{-1}f')}$$

for all n positive integers.

It follows that for given g, f, g', f' there are exactly $lcm(|g'|, |g'f|)$ entries of z that must be equal to $z_{(g,f),(g',f')}$. All these entries have the same f, g' indices. Thus for every f, g' fixed we have $\frac{|F||G|}{lcm(|g'|,|g'f|)}$ linearly independent vectors in the space V of matrices z which are solutions to our system. We obtain

$$d(\mathbb{C}^G \# \mathbb{C}[F]) = \dim(V) = \sum_{f \in F, g' \in G} \frac{|L|}{lcm(|g'|, |g'f|)}.$$

Note however that when $L = FG$ is an exact factorization with F normal in L, it is true that $|a|$ divides $|ab|$ for any $a \in G$ and $b \in F$. Indeed, let $k = |ab|$. We have $(ab)^k = e$, which implies $(aba^{-1})(a^2ba^{-2})\cdots(a^kba^{-k}) = (ab)^k a^{-k} = a^{-k}$. Since the left side of the equality is in F, it follows that $a^{-k} \in F$. However, a is in G and $F \cap G = \{e\}$. It follows that $a^{-k} = e$ and thus $|a|$ divides k.

Applying this to the formula above for $a = g'$ and $b = f$, we actually have $lcm(|g'|, |g'f|) = |g'f|$. It follows that

$$d(\mathbb{C}^G \# \mathbb{C}[F]) = \sum_{f \in F, g' \in G} \frac{|L|}{|g'f|} = \sum_{l \in L} \frac{|L|}{|l|} = d(L).$$

□

We now give a concrete example of an exact factorization arising from a semidirect product $L = F \rtimes G$, with G abelian and F not abelian. The computations follow from the previous result together with the group defect formula, or they can easily be verified using Mathematica.

Example 4. *Let $L = S_4$ be the group of permutations of 4 elements. Consider its subgroups $F = A_4$ (the alternating group with 12 elements), and $G \simeq \mathbb{Z}_2$ generated by the transposition (12). Note that $L = F \rtimes G$, with G abelian and F non-abelian. The dimension of the space of solutions $z_{(g,f),(g',f')}$ (for $g, g' \in \mathbb{Z}_2$ and $f, f' \in A_4$) of the equation*

$$\sum_{\substack{g,g' \in \mathbb{Z}_2 \\ f,f' \in A_4}} z_{(g,f),(g',f')}[u_{(g,f)}, v_{(f',g')}] = 0.$$

is

$$d(\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]}) = 232 = d(S_4).$$

Remark 5. *While we could not find a closed formula for $d(\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]})$ in the case of exact factorizations arising from general semidirect products $L = F \rtimes G$ (without the assumption that G is abelian), such a formula is also possible when F is abelian. Indeed, let $L = F \rtimes G$ be an exact factorization of L with F abelian and normal. In this case the bismash product Hopf algebra $\mathbb{C}^F \# \mathbb{C}[G]$ is co-commutative and thus it is a group Hopf algebra (see for instance Remark 5.3 of [2]). It follows that the defect of the commuting square is equal to the defect of the corresponding group.*

5.3 Further defect computations

We end with an example of an internal Zappa-Szép product which is not a semidirect product.

Example 5. *Let $L = A_5$ be the alternating group with 60 elements. Embed $F = A_4$ (the alternating group with 12 elements) in L, by defining F as the set of all permutations in L that leave 5 invariant. Consider also the subgroup $G = \mathbb{Z}_5$ generated by the 5-cycle $\langle (12345) \rangle$. Note that $L = FG$ is not a semidirect product, since A_5 is simple. By employing Mathematica to find the dimension of the space of solutions $z_{(g,f),(g',f')}$ (for $g, g' \in \mathbb{Z}_5$ and $f, f' \in A_4$) of the equation*

$$\sum_{\substack{g,g' \in \mathbb{Z}_5 \\ f,f' \in A_4}} z_{(g,f),(g',f')}[u_{(g,f)}, v_{(f',g')}] = 0.$$

we get that

$$d(\mathcal{C}^L_{\mathbb{C}^G \# \mathbb{C}[F]}) = 1168.$$

Note that this agrees with $d(L) = d(A_5) = 1168$.

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