The defect of a group-type commuting square

Remus Nicoara*[1] and Joseph White[2]

[1]University of Tennessee Knoxville and Simion Stoilow Institute
[2]University of Tennessee Knoxville

Dedicated to Serban Stratila, on the occasion of his 70^{th} *birthday*

Abstract

We introduce the defect $d(G)$ of a finite group G. The definition of $d(G)$ is inspired by previous work of the first author ([Ni1]), and it is given in terms of the commuting square $\mathfrak{C}_G$ associated to G. We can interpret $d(G)$ as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed in the class of commuting squares. We compute $d(G)$ in terms of the orders of the elements of G, and characterize the groups of (dephased) defect 0. When G is abelian, $\mathfrak{C}_G$ is a spin model commuting square given by a (generalized) Fourier matrix F_G, and our notion of defect for G agrees with the previously existing notion of defect for the matrix F_G (see [TaZy2], [Ba]).

1 Introduction

Commuting squares were introduced in [Po1], as invariants and construction data in Jones' theory of subfactors ([Jo], [JS]). They encode the generalized symmetries of the subfactor, in a lot of situations being complete invariants ([Po2],[Po1]). In particular, any finite group G can be encoded in a group commuting square:

$$\mathfrak{C}_G = \begin{pmatrix} D & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}.$$

where $D \simeq l^\infty(G)$ is the algebra of $n \times n$ diagonal matrices, and $\mathbb{C}[G]$ denotes the group algebra of G. It can be shown that two group commuting squares are isomorphic if and only if the corresponding groups are isomorphic. The subfactor associated to $\mathfrak{C}_G$

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by iterating Jones' basic construction is a cross product subfactor, hence of depth 2. Moreover, if G is abelian then $\mathfrak{C}_G$ is a spin model commuting square, and the associated subfactor is a Hadamard subfactor in the sense of [Ni2].

In [Ni1], the first author initiated a study of the deformations of a commuting square, in the class of commuting squares. It was shown that if a commuting square satisfies a certain *span condition*, then it is isolated among all non-isomorphic commuting squares. In the case of $\mathfrak{C}_G$, the span condition asks that V be equal to $\mathrm{M_n}(\mathbb{C})$, where V is the subspace of $\mathrm{M_n}(\mathbb{C})$ given by:

$$V = \mathrm{span}\{du - ud : d \in D, u \in \mathbb{C}[G]\} + \mathbb{C}[G] + \mathbb{C}[G]' + D$$

When the span condition fails, the dimension $d'(G)$ of $V^{\perp} = \mathrm{M_n}(\mathbb{C}) \ominus \mathrm{V}$ can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by non-isomorphic commuting squares. In this paper we study this dimension, which we call *the dephased defect of the group G*. We also study a related quantity $d(G) = \dim_{\mathbb{C}}([D, \mathbb{C}[G]]^{\perp})$, called *the undephased defect of G* (or just the defect of G), which can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by (not necessarily non-isomorphic) commuting squares. The terminologies 'dephased defect' and 'undephased defect' are based on previous work of [Ka], [TaZy2] and [Ba], which we explain below.

The concept of defect for unitary matrices can be traced back to [Ka]. The terminology 'defect' was first explicitly introduced in [TaZy2]. The (dephased) defect of the Fourier matrix $F_n = \frac{1}{\sqrt{n}}(e^{i\frac{2\pi kl}{n}})_{1\leq k,l\leq n}$ was computed, and it was proved that it gives an upper bound on the number of parameters in an analytic family of complex (non-equivalent) $n \times n$ Hadamard matrices stemming from F_n. In the language of commuting squares, the matrix F_n gives rise to a spin model commuting square (in the sense of [JS]), associated to $G = \mathbb{Z}_n$. Indeed, it is easy to check that $\mathbb{C}[\mathbb{Z}_n] = FDF^*$.

In [Ba], Banica extended the computation of the defect to generalized Fourier matrices $F_G = F_{n_1} \otimes \ldots \otimes F_{n_r}$, which correspond to abelian groups $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ (see also [Ka] for an earlier version of this result). The same formula was very recently rediscovered in [Ta]. Notice that $\mathbb{C}[G] = F_G D F_G^*$, so in our language the matrix F_G yields the spin model, group-type commuting square $\mathfrak{C}_G$. Banica introduced the notions of dephased and undephased defects for matrices F_G, and showed that they give upper bounds for the tangent spaces at F_G to the real algebraic manifold of dephased complex Hadamard matrices, respectively to the the real algebraic manifold of all complex Hadamard matrices.

In this paper we compute $d(G)$ and $d'(G)$ for any finite group G. When particularized to the case of abelian groups, our computations agree with Banica's computations for the Fourier matrix of G, and yield a different proof of this result.

We also investigate when $d'(G) = 0$, or equivalently when does the span condition hold for a group commuting square $\mathfrak{C}_G$. This turns out to happen if and only if $G \simeq \mathbb{Z}_p$ with p prime. Since $\mathbb{C}[\mathbb{Z}_p] = F_p D F_p^*$, a consequence of this result together with the isolation result from [Ni1] is Petrescu's result, that the Fourier matrix F_p is isolated among all dephased Hadamard matrices when p is prime.

Our definition of the defect can be easily extended to any (not necessarily group-type) commuting square. Our main motivation for studying the defect of group commuting squares is to better understand the structure of the moduli space of non-isomorphic commuting squares around some of its 'easier' points. Even in the case of commuting squares arising from Fourier matrices, this is an unsolved problem with far-reaching consequences. For example, the structure of the moduli space of non-equivalent 6×6 Hadamard matrices in a neighborhood of F_6 has applications in quantum information theory (see [We], [TaZy1]).

In a related paper ([NiWh]) we show that $d(G)$ is the best possible bound for the number of independent directions of convergence, in the following sense: there exists a basis for $[D, \mathbb{C}[G]]^\perp$, such that for every a in the basis there is an analytic family of commuting squares containing $\mathfrak{C}_G$ and of direction a. However, it is not true in general that every (hermitian of unit length) $a \in [D, \mathbb{C}[G]]^\perp$ is a direction of convergence.

2 The Defect of a Group

Let G be a finite group with n elements. In the following, we will use the indexes g, g', h, h' to represent group elements, while i, k will be reserved for natural numbers.

Fix some order on G. For each $g \in G$, let $e_g \in \mathbb{C}^n$ denote the column vector with a 1 in position g and 0 otherwise. Then the group algebra of G is $\mathbb{C}[G] = \text{span}\{u_g : g \in G\}$ where $u_g \in \mathrm{M_n}(\mathbb{C})$ satisfies $u_g(e_h) = e_{gh}$ for all $h \in G$. In other words, $u_g = \sum_{h \in G} e_{h,g^{-1}h}$, where $e_{g,h}$ are the matrix units of $\mathrm{M_n}(\mathbb{C})$.

One associates to G the commuting square

$$\mathfrak{C}_G = \begin{pmatrix} D & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}.$$

where $D \simeq l^\infty(G)$ denotes the algebra of diagonal $n \times n$ matrices.

In [Ni1], the first author introduced a sufficient condition for a commuting square to be isolated in the class of all non-isomorphic commuting squares, which he called *the span condition*. In the case of $\mathfrak{C}_G$, the span condition reads

$$[D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D = \mathrm{M_n}(\mathbb{C})$$

where $\mathbb{C}[G]' = \{a \in \mathrm{M_n}(\mathbb{C}) \, : \, \mathrm{au_g = u_g a} \, \text{ for all } \mathrm{g \in G}\}$ and $[D, \mathbb{C}[G]] = \text{span}\{du - ud : d \in D, u \in \mathbb{C}[G]\}$

More generally, from work in [Ni1] and [Ni3] it follows that if the commuting square $\mathbb{C}_G$ is not isolated then we have:

• All possible directions of convergence of sequences of commuting squares converging to $\mathfrak{C}_G$ are contained in the vector space

$$\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$$

• All possible directions of convergence of sequences of *non-isomorphic* commuting squares converging to $\mathfrak{C}_G$ are contained in the vector space

$$\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D})$$

The orthogonal complements above are considered with respect to the inner product on $\mathrm{M_n}(\mathbb{C})$ given by $< x, y >= \tau(xy^*)$, where τ is the normalized trace on $\mathrm{M_n}(\mathbb{C})$.

We refer the reader to [Ni3] for the definition of a direction of convergence of a sequence of commuting squares.

This inspires the following definitions of the undephased and dephased defect for $\mathfrak{C}_G$, or equivalently for the group G. The name defect comes from the terminology used for Hadamard matrices, developed in [TaZy2] (see also [TaZy1]). The dephased and undephased defect were introduced, for Hadamard matrices, in [Ba].

Definition 2.1. *The undephased defect of a finite group G is*

$$d(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]])$$

The dephased defect of G is

$$d'(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D)$$

Remark 2.2. *The span condition is equivalent to $d'(G) = 0$, in which case $\mathfrak{C}_G$ is isolated in the class of all non-isomorphic commuting squares.*

In order to better relate the quantities $d(G)$ and $d'(G)$, we will need the dimension of $\mathbb{C}[G]'$ and $\mathbb{C}[G] \cap \mathbb{C}[G]'$.

Proposition 2.3. *A matrix $a \in \mathrm{M_n}(\mathbb{C})$ is in $\mathbb{C}[G]'$ if and only if*

$$a_{g^{-1}g',h} = a_{g',gh}$$

for all g, g', $h \in G$. Furthermore, $\dim_{\mathbb{C}} \mathbb{C}[G]' = n$ and $\mathbb{C}[G]' \cap D = \mathbb{C}I_n$.

Proof. Fix $g \in G$. Routine calculations show that

$$(u_g a)_{g',h} = \sum_{h'} (u_g)_{g',h'} a_{h',h} = a_{g^{-1}g',h}$$

and

$$(a u_g)_{g',h} = \sum_{h'} a_{g',h'} (u_g)_{h',h} = a_{g',gh}$$

Thus, it follows that a commutes with all u_g's if and only if $a_{g^{-1}g',h} = a_{g',gh}$ for any g, g', $h \in G$.

If we make $h = e$ we obtain $a_{g',g} = a_{g^{-1}g',e}$. This shows that all entries of a depend on its first column. Conversely, if we fix any $(c_g)_{g\in G} \in \mathbb{C}$, we can consider the matrix a given by $a_{g',g} = c_{g^{-1}g'}$. This matrix will have the first column given by the c_g's, and it is easy to check that it satisfies $a_{g^{-1}g',h} = a_{g',gh}$ for any g, g', $h \in G$. Consequently, $\dim_{\mathbb{C}} \mathbb{C}[G]' = n$.

The claim $\mathbb{C}[G]' \cap D = \mathbb{C}I_n$ easily follows from the above.

■

Let $cl(G)$ denote the class number of G; i.e. $cl(G)$ is the number of distinct conjugacy classes of G. We have:

Proposition 2.4. *A matrix $a = \sum_g c_g u_g \in \mathbb{C}[G]$ is also in $\mathbb{C}[G]'$ if and only if*

$$c_g = c_{hgh^{-1}}$$

for all $g, h \in G$. Thus $\dim_{\mathbb{C}}(\mathbb{C}[G] \cap \mathbb{C}[G]') = cl(G)$.

Proof. Fix $h \in G$. It is easy to see that $au_h = \sum_g c_g u_{gh}$ and $u_h a = \sum_g c_g u_{hg}$. Relabeling, from $au_h = u_h a$ it follows that

$$\sum_{g'} c_{g'h^{-1}} u_{g'} = \sum_{g'} c_{h^{-1}g'} u_{g'}.$$

We conclude that $c_{g'h^{-1}} = c_{h^{-1}g'}$ for all $g', h \in G$. Setting $g = h^{-1}g'$, this is equivalent to

$$c_{hgh^{-1}} = c_g.$$

■

Theorem 2.5. *The dephased and undephased defect of a finite group G are related as follows:*

$$d(G) = d'(G) + 3n - 1 - cl(G)$$

Proof. We need to relate $d'(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D)$ and $d(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]])$. It is easy to check that $D \perp \mathbb{C}[G] \ominus \mathbb{C}I_n$, which just says that $\mathfrak{C}_G$ is a commuting square. Using this, it follows that $[D, \mathbb{C}[G]]$ is orthogonal to $\mathbb{C}[G]$, $\mathbb{C}[G]'$ and D. Indeed, let's check for instance that $[D, \mathbb{C}[G]] \perp \mathbb{C}[G]$ (the other two follow similarly). For $d \in D$ and $a, b \in \mathbb{C}[G]$ we have:

$$\tau([d,a]b^*) = \tau(dab^* - adb^*) = \tau(dab^* - db^*a) = \tau(d[a,b^*]) = 0$$

since $[a, b^*] \in \mathbb{C}[G] \ominus \mathbb{C}$.

Also notice that D is orthogonal to $\mathbb{C}[G]' \ominus \mathbb{C}I_n$. Indeed, from the previous proposition we know that and $a \in \mathbb{C}[G]'$ is of the form $a_{g',g} = c_{g^{-1}g'}$. In particular, all the diagonal entries of a are equal to c_e. If a is orthogonal onto $\mathbb{C}I_n$, then $\tau(a) = 0$ so $c_e = 0$. It follows that the projection of a onto D, which is the diagonal of a, is 0. Thus a is orthogonal to D.

Since the intersection of the algebras $\mathbb{C}[G]$ and $\mathbb{C}[G]'$ has dimension $cl(G)$, we obtain:

$$\dim_{\mathbb{C}}([D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D) = \dim_{\mathbb{C}}([D, \mathbb{C}[G]]) + n + (n - cl(G)) + (n - 1))$$

which shows that $d(G) = d'(G) + 3n - 1 - cl(G)$.

■

Remark 2.6. *If G is abelian, $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$, then $\mathfrak{C}_G$ is a spin model commuting square (i.e. given by a Hadamard matrix). Indeed, this is because $\mathbb{C}[G] = F_G D F_G^*$ where $F_G = F_{n_1} \otimes \ldots \otimes F_{n_r}$ is the (generalized) Fourier matrix associated to G. In this case $cl(G) = n$ and the dephased defect is $d'(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]]) + 2n - 1$, which can be computed to be the same as the defect of the Hadamard matrix F_G, as introduced in [TaZy2]. Also, $d(G)$ equals the undephased defect of F_G, as defined in [Ba].*

Remark 2.7. *If G is abelian, the defect $d(G)$ has a very nice interpretation as the number of entries equal to 1 in the matrix F_G (see [Ka]). This raises the following question, for which we don't have an answer:*

For a finite group G, can $d(G)$ be interpreted as the number of $'1'$ entries of some matrix naturally associated to G?

We now compute $d(G)$ for any finite group G, in terms of the orders of its elements.

Theorem 2.8.
$$d(G) = \sum_{g \in G} \frac{|G|}{ord(g)}.$$

Proof. Let $d_g = e_{g,g}$ denote the diagonal matrix with a 1 on position g, g and 0 elsewhere. Let
$$W = [D, \mathbb{C}[G]] = \mathrm{span}\{[d_g, u_h] : g, h \in G\})$$

We want to compute $d(G) = n^2 - \dim(W)$. This number is the same as the dimension of the space
$$V = \{(c_{g,h})_{g,h \in G} \in \mathrm{M_n}(\mathbb{C}) : \sum_{\mathrm{g,h}} \mathrm{c_{g,h}}[\mathrm{d_g}, \mathrm{u_h}] = 0\}$$

Indeed, if s denotes the $n^2 \times n^2$ matrix, indexed by $G \times G$, which has as its $(g, h)^{\text{th}}$ column the "row by row" column vector form of $[d_g, u_h]$, then $\dim(V)$ is the nullity of s and $\dim(W)$ is the rank of s.

Routine calculations show that $d_g u_h = e_{g,h^{-1}g}$ and $u_h d_g = e_{hg,g}$. Thus,
$$0 = \sum_{g,h} c_{g,h}[d_g, u_h] = \sum_{g,h} c_{g,h} e_{g,h^{-1}g} - \sum_{g,h} c_{g,h} e_{hg,g}$$

After changing variables $g' = g, h' = h^{-1}g$ in the first sum and $g' = hg, h' = g$ in the second sum, we obtain:
$$\sum_{g',h'} (c_{g',g'h'^{-1}} - c_{h',g'h'^{-1}}) e_{g',h'} = 0$$

which is equivalent to $c_{g',g'h'^{-1}} = c_{h',g'h'^{-1}}$ for all $g', h' \in G$. Changing variable again by $g = h', h = g'h'^{-1}$, we obtain
$$c_{hg,h} = c_{g,h} \text{ for all } g, h \in G$$

It follows that for all $g' \in \langle h \rangle g$ we must have $c_{g',h} = c_{g,h}$. It follows that we have $[G : \langle h \rangle]$ choices to make for the column associated to h. Thus
$$d(G) = \sum_{h \in G} \frac{|G|}{\mathrm{ord}(h)}$$

. ∎

Remark 2.9. *The formula that we obtained for the undephased defect $d(G)$ coincides with the formula obtained by Banica in [Ba], for the undephased defect of the generalized Fourier matrix $F_G = F_{n_1} \otimes \ldots \otimes F_{n_r}$ associated to the abelian group $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$. Thus, if we particularize our result to abelian groups we obtain a different proof of Banica's result.*

Remark 2.10. *When the group $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ is abelian, a more explicit formula for $\sum_{g \in G} \frac{|G|}{ord(g)}$ can be given (see [Ba]).*

Corollary 2.11. *If G is a finite group, we have*

$$d'(G) = \sum_{g \in G} \frac{|G|}{ord(g)} - 3n + 1 + cl(G)$$

Remark 2.12. *If for fixed $g, h \in G$ we define $c(h,g) \in M_n(\mathbb{C})$ by*

$$(c(h,g))_{p,q} = \begin{cases} 1 & \text{if } p = h^k g \text{ and } q = h \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

then the distinct $c(h,g)$ form a basis for $\{(c_{g,h})_{g,h \in G} \in \mathrm{M_n(\mathbb{C})} : \sum_{\mathrm{g,h}} \mathrm{c_{g,h}[d_g, u_h]} = 0\}$.

We now give a basis for the $W^{\perp}$. The interest in this space is justified by a result of [Ni1]: any direction of convergence of a sequence of commuting squares approaching $\mathfrak{C}_G$ must belong to $W^{\perp}$.

Theorem 2.13. *For every $g, h \in G$ let $a(h,g) \in M_n(\mathbb{C})$ be the matrix*

$$(a(h,g))_{p,q} = \begin{cases} 1 & \text{if } p = h^k g \text{ and } q = h^{k+1} g \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

For each $h \in G$, let $g_1^h, \ldots g_{n(h)}^h$ be a choice of representatives of the right cosets of $G/ < h >$, where $n(h) = |G|/ord(h)$ is the number of elements of $G/ < h >$. Then the matrices $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$ form a basis for $W^{\perp}$.

Proof. We first show that the matrices $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$ are linearly independent. This follows from the stronger fact that no two of them have non-zero entries on the same position. To check this, observe that if $a(h,g)_{p,q} = a(h',g')_{p,q}$ for some $h, g, h', g' \in G$, then $p = h^k g = (h')^l g'$ and $q = h^{k+1} g = (h')^{l+1} g'$ for some k, l positive integers. It follows that $h = qp^{-1} = h'$. This together with $h^k g = (h')^l g'$ implies that $g' \in < h > g$. Since this is not the case for any two matrices in the set $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$, it follows that no two of them have non-zero entries on the same position, so in particular they are linearly independent.

We now show that $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$ span $W^{\perp}$. An $n \times n$ matrix a is in $W^{\perp}$ if and only if it $\tau(a[d_g, u_h]) = 0$ for all $g, h \in G$ (we used here that W is $*$-closed). Thus $\tau(a(e_{g,h^{-1}g} - e_{hg,g})) = 0$, or equivalently $a_{h^{-1}g,g} = a_{g,hg}$ for all $g, h \in G$.

By replacing g by $h^k g$ for $k = 1, 2, \ldots, ord(h)$, it follows that:

$$a_{g,hg} = a_{hg,h^2 g} = \ldots = a_{h^{ord(h)-1} g, g} \text{ for all } h, g \in G$$

Conversely, any matrix a satisfying the relation above must satisfy $a_{h^{-1}g,g} = a_{g,hg}$ for all $h, g \in G$, which shows that $a \in W^{\perp}$. And any such a can be written as a span of matrices of the form $a(h, g_k^h)$:

$$a = \sum_{h \in H, 1 \leq k \leq n(h)} a_{g_k^h, hg_k^h} \cdot a(h, g_k^h)$$

This shows that $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$ is a basis of $W^{\perp}$.

■

Remark 2.14. *The cardinality of the basis* $\{a(h, g_k^h) : h \in G, 1 \leq k \leq n(h)\}$ *is* $d(G) = \dim(W^{\perp}) = \sum_{h \in G} n(h) = \sum_{h \in G} \frac{|G|}{ord(h)}$*, which gives a somewhat different proof of Theorem 2.8.*

We now give an example of a computation of the defect for the smallest non-abelian group, $G = S_3$.

Example 2.15. *If* $G = S_3$*, it is easy to see that* $d(G) = 6 + 3 + 3 + 3 + 2 + 2 = 19$*. Thus,* $d'(G) = d(G) - ((2n - 1) + (n - cl(G)) = 19 - (11 + 3) = 5$.

We now describe the groups G which satisfy the span condition, i.e. have $d'(G) = 0$.

Theorem 2.16. *Let* G *be a finite group with at least* 2 *elements. Then* $d'(G) = 0$ *if and only if* $G \simeq \mathbb{Z}_p$ *with* p *prime.*

Proof. Let G be a group with n elements with $d'(G) = 0$. We have

$$d(G) = 3n - 1 - cl(G)$$

On the other hand, in the previous theorem we showed that

$$d(G) = \sum_{g \in G} \frac{|G|}{\operatorname{ord}(g)}$$

If G is not cyclic, then for every $g \in G$ we have $\frac{|G|}{\operatorname{ord}(g)} \geq 2$. Note that for $g = e$ we have $\frac{|G|}{\operatorname{ord}(e)} = n$. Thus, $d(G) \geq n + 2(n - 1) = 3n - 2$. However, since $d(G) = 3n - 1 - cl(G)$, it follows that $cl(G) = 1$, which is impossible if G has at least 2 elements.

Thus, G must be cyclic. In this case $d(G) = 3n - 1 - cl(G) = 2n - 1$. But $\frac{|G|}{\operatorname{ord}(g)} \geq 1$ for all $g \in G$, and $\frac{|G|}{\operatorname{ord}(e)} = n$, which imply

$$d(G) = \sum_{g \in G} \frac{|G|}{\operatorname{ord}(g)} \geq 2n - 1$$

We must thus have equality in all above inequalities, meaning that $ord(g) = |G|$ for all $g \neq e$. Thus $G \simeq Z_p$ with p prime.

■

Remark 2.17. *Since $d'(G) = 0$ is equivalent to $\mathfrak{C}_G$ satisfying the span condition, in this case $\mathfrak{C}_G$ is isolated ([Ni1]). Combining this with the 'if' implication of the preceding corollary, we recover Petrescu's result ([Pe]) that the Fourier matrix of prime order is isolated among all Hadamard matrices. Indeed, this follows from $\mathbb{C}[\mathbb{Z}_p] = F_p D F_p^*$, where F_p is the Fourier matrix of order p.*

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