Analytic deformations of group commuting squares and complex Hadamard matrices

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Abstract

Let G be a finite group and denote by $\mathfrak{C}_G$ the commuting square associated to G. The defect of the group G, given by the formula $d(G) = \sum_{g\in G} \frac{|G|}{order(g)}$, was introduced in [NiWh] as an upper bound for the number of linearly independent directions in which $\mathfrak{C}_G$ can be continuously deformed in the class of commuting squares. In this paper we show that this bound is actually attained, by constructing $d(G)$ analytic families of commuting squares containing $\mathfrak{C}_G$.

In the case $G = \mathbb{Z}_n$, the defect $d(\mathbb{Z}_n)$ can be interpreted as the dimension of the enveloping tangent space of the real algebraic manifold of $n \times n$ complex Hadamard matrices, at the Fourier matrix F_n (in the sense of [TaZy1], [Ba1]). The dimension of the enveloping tangent space gives a natural upper bound on the number of continuous deformations of F_n by complex Hadamard matrices, of linearly independent directions of convergence. Our result shows that this bound is reached, which is rather surprising. In particular our construction yields new analytic families of complex Hadamard matrices stemming from F_n.

In the last section of the paper we use a compactness argument to prove non-equivalence (i.e. non-isomorphism as commuting squares) for dephased versions of the families of Hadamard matrices constructed throughout the paper.

1 Introduction

Commuting squares were introduced in [Po2], as invariants and construction data in Jones' theory of subfactors ([Jo], [JS]). They encode the generalized symmetries of the subfactor, in a lot of situations being complete invariants ([Po1],[Po2]). In particular, any finite group G can be encoded in a group commuting square:

$$\mathfrak{C}_G = \begin{pmatrix} D & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}$$

where $D \simeq l^\infty(G)$ is the algebra of $n \times n$ diagonal matrices, and $\mathbb{C}[G]$ denotes the group algebra of G. It can be shown that two group commuting squares are isomorphic if and only if the corresponding groups are isomorphic. The subfactor associated to $\mathfrak{C}_G$ by iterating Jones' basic construction is a cross product subfactor, hence of depth 2. Moreover, if G is abelian then $\mathfrak{C}_G$ is a spin model commuting square, and the associated subfactor is a Hadamard subfactor in the sense of [Ni2].

In [Ni1], the first author initiated a study of the deformations of a commuting square, in the class of commuting squares. It was shown that if a commuting square satisfies a certain *span condition*, then it is isolated among all non-isomorphic commuting squares. In the case of $\mathfrak{C}_G$, the span condition is $V = \mathrm{M_n}(\mathbb{C})$, where V is the subspace of $\mathrm{M_n}(\mathbb{C})$ given by:

$$V = \mathrm{span}\{du - ud : d \in D, u \in \mathbb{C}[G]\} + \mathbb{C}[G] + \mathbb{C}[G]' + D$$

When the span condition fails, the dimension $d'(G)$ of $V^\perp = \mathrm{M_n}(\mathbb{C}) \ominus \mathrm{V}$ can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by non-isomorphic commuting squares. In [NiWh] we computed this dimension, which we called *the dephased defect of the group* G. We also studied the related quantity $d(G) = \dim_{\mathbb{C}}([D, \mathbb{C}[G]]^\perp)$, called *the undephased defect of* G (or just the defect of G), which can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by (not necessarily non-isomorphic) commuting squares. The terminologies 'dephased defect' and 'undephased defect' are based on previous work of [Ka], [TaZy1] and [Ba1], which we explain below.

The concept of defect for unitary matrices can be traced back to [Ka]. The terminology 'defect' was first explicitly introduced in [TaZy1]. The (dephased) defect of the Fourier matrix $F_n = \frac{1}{\sqrt{n}}(e^{i\frac{2\pi kl}{n}})_{1\le k,l\le n}$ was computed, and it was proved that it gives an upper bound on the number of parameters in an analytic family of complex (non-equivalent) $n \times n$ Hadamard matrices stemming from F_n. In the language of commuting squares, the matrix F_n gives rise to a spin model commuting square (in the sense of [JS]), associated to $G = \mathbb{Z}_n$. Indeed, it is easy to check that $\mathbb{C}[\mathbb{Z}_n] = F_n D F_n^*$.

In [Ba1], Banica extended the computation of the defect to generalized Fourier matrices $F_G = F_{n_1} \otimes \ldots \otimes F_{n_r}$, which correspond to abelian groups $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ (see also [Ka] for an earlier version of this result). The same formula was also recently rediscovered in [Ta]. Notice that $\mathbb{C}[G] = F_G D F_G^*$, so in our language the matrix F_G yields the spin model, group-type commuting square $\mathfrak{C}_G$. Banica introduced the notions of dephased and undephased defects for matrices F_G, and showed that they give upper bounds for the tangent spaces at F_G to the real algebraic manifold of dephased complex Hadamard matrices, respectively to the the real algebraic manifold of all complex Hadamard matrices. Our notion of defect agrees with Banica's in the case of abelian groups, and thus generalizes it.

Our main motivation for studying the defect of group commuting squares is to better understand the structure of the moduli space of non-isomorphic commuting squares around some of its 'easier' points. Even in the case of commuting squares arising from Fourier matrices (cyclic groups), this is an unsolved problem with far-reaching consequences. For example, the structure of the moduli space of non-equivalent 6×6 Hadamard matrices in a neighborhood of F_6 has applications in quantum information theory (see [We], [TaZy2]).

From our previous work in [NiWh] it follows that the defect $d(G)$ is an upper bound for the number of one-parameter continuous deformations of $\mathfrak{C}(G)$, of linearly independent directions of convergence. In this paper we show that this bound is reached. More precisely, we construct a basis $\mathfrak{B}$ of $[D, \mathbb{C}[G]]^\perp$, such that for each $a \in \mathfrak{B}$ there exists an analytic family $(\mathfrak{C}_t)_{t\in\mathbb{R}}$ of commuting squares

$$\mathfrak{C}_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n(\mathbb{C})} \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t\mathbb{C}[G]U_t^* \end{pmatrix}$$

where U_t are unitaries with $U_t \neq I$ for $t \neq 0$, $U_t \to U_0 = I$ as $t \to 0$, and $a = \lim_{t\to 0} \frac{U_t - I}{\mathrm{i}||\mathrm{U_t - I}||}$. We refer to a as the direction of convergence of the family $(\mathfrak{C}_t)_{t\in\mathbb{R}}$.

Thus we obtain $d(G)$ analytic deformation of $\mathfrak{C}_G$, of linearly independent directions of convergence. Note that the choice of the basis $\mathfrak{B}$ is crucial to the proof; it is not true in general that every (hermitian of unit length) $a \in [D, \mathbb{C}[G]]^\perp$ is a direction of convergence of some continuous deformation of $\mathfrak{C}_G$.

When $G = \mathbb{Z}_n$, we obtain $d(\mathbb{Z}_n)$ analytic deformations of the *standard spin model* commuting square $\mathfrak{C}_{\mathbb{Z}_n}$. Each of these deformations is of the form $(\mathfrak{C}_t)_{t\in\mathbb{R}}$, where:

$$\mathfrak{C}_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n(\mathbb{C})} \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_tF_nDF_n^*U_t^* \end{pmatrix} \to \mathfrak{C}_{\mathbb{Z}_n} = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n(\mathbb{C})} \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & F_nDF_n^* \end{pmatrix}$$

Equivalently, this gives an analytic family of complex Hadamard matrices $U_tF_n \to F_n$ (for details see for instance [Ni1]). Hence we obtain $d(\mathbb{Z}_n)$ analytic families of complex Hadamard matrices, of linearly independent directions of convergence.

Let $\mathcal{C}(\mathrm{n}) = \mathrm{M_n}(\mathbb{T}) \cap \sqrt{\mathrm{n}}\mathrm{U(n)}$ denote the real algebraic manifold of $n \times n$ complex Hadamard matrices, where $\mathrm{U(n)} \subset \mathrm{M_n}(\mathbb{T})$ denotes the set of unitary matrices. The defect $d(\mathbb{Z}_n)$ can be interpreted as the dimension of the enveloping tangent space of $\mathcal{C}(\mathrm{n})$ at the matrix F_n:

$$\tilde{\mathrm{T}}_{\mathrm{F_n}}\mathcal{C}(\mathrm{n}) = \mathrm{T_{F_n}M_n}(\mathbb{T}) \cap \mathrm{T_{F_n}}\sqrt{\mathrm{n}}\mathrm{U(n)}$$

(see [TaZy1], [Ba1], [Ba2]). Thus the defect can be regarded as an upper bound for the dimension of the tangent space to $\mathcal{C}(n)$, at the point F_n. Our main result shows that this bound is reached, which is rather surprising. Note that, for general n, the manifold $\mathcal{C}(n)$ is not smooth or connected.

In the last section we use a compactness argument to prove a non-equivalence result for the dephased parametric families of complex Hadamard matrices that we construct in this paper. This sheds some light on the structure of the moduli space $\mathcal{E}(n)$ of equivalence classes of complex Hadamard matrices, around the point F_n.

2 Preliminaries

Let G be a finite group with n elements. In the following, we will use the indexes g, g', h, h' to represent group elements, while i, j will be reserved for natural numbers.

Fix an order on G. For each $g \in G$, let $e_g \in \mathbb{C}^n$ denote the column vector with a 1 in position g and 0 otherwise. Then the group algebra of G is $\mathbb{C}[G] = \mathrm{span}\{u_g : g \in G\}$ where $u_g \in \mathrm{M_n}(\mathbb{C})$ satisfies $u_g(e_h) = e_{gh}$ for all $h \in G$. In other words, $u_g = \sum_{h\in G} e_{h,g^{-1}h}$, where $e_{g,h}$ are the matrix units of $\mathrm{M_n}(\mathbb{C})$.

One associates to G the following *group-type commuting square*:

$$\mathfrak{C}_G = \begin{pmatrix} D & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}$$

where $D \simeq l^\infty(G)$ denotes the algebra of diagonal $n \times n$ matrices.

In [Ni1], the first author introduced a sufficient condition for a commuting square to be isolated in the class of all non-isomorphic commuting squares, which he called *the span condition*. In the case of $\mathfrak{C}_G$, the span condition reads

$$[D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D = \mathrm{M_n}(\mathbb{C})$$

where $\mathbb{C}[G]' = \{a \in \mathrm{M_n}(\mathbb{C}) : \mathrm{au_g = u_g a \text{ for all } g \in G}\}$ and $[D, \mathbb{C}[G]] = \mathrm{span}\{du - ud : d \in D, u \in \mathbb{C}[G]\}$

More generally, from work in [Ni1] and [Ni3] it follows that if the commuting square $\mathfrak{C}_G$ is not isolated then we have:

• All possible directions of convergence of sequences (in the sense of [Ni3]) of commuting squares converging to $\mathfrak{C}_G$ are contained in the vector space

$$\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$$

• All possible directions of convergence of sequences of *non-isomorphic* commuting squares converging to $\mathfrak{C}_G$ are contained in the vector space

$$\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D})$$

We refer the reader to [Ni3] for the definition of a direction of convergence of a sequence of commuting squares. The orthogonal complements above are considered with respect to the inner product on $\mathrm{M_n}(\mathbb{C})$ given by $< x, y >= \tau(xy^*)$, where τ is the normalized trace on $\mathrm{M_n}(\mathbb{C})$.

In [NiWh] we defined the undephased and dephased defect of a group G, as the dimensions of the two vector spaces above. The name defect comes from the terminology used for Hadamard matrices, developed in [TaZy1] (see also [TaZy2]).

Definition 2.1. *The undephased defect of a finite group G is*

$$d(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]])$$

The dephased defect of G is

$$d'(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D)$$

Remark 2.2. *In [NiWh] we showed that*

$$d(G) = \sum_{g \in G} \frac{|G|}{ord(g)}$$

and

$$d'(G) = d(G) - 3n + 1 + cl(G)$$

where $ord(g)$ denotes the order of the element g, and $cl(G)$ denotes the class number of the group G.

Remark 2.3. *The span condition is equivalent to $d'(G) = 0$. Thus the main result in [Ni1] can be interpreted as follows: if $d'(G) = 0$ then $\mathfrak{C}_G$ is isolated in the class of all non-isomorphic commuting squares.*

Remark 2.4. *If G is abelian, $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$, then $\mathfrak{C}_G$ is a spin model commuting square (i.e. given by a Hadamard matrix). Indeed, this is because $\mathbb{C}[G] = F_G D F_G^*$ where $F_G = F_{n_1} \otimes \ldots \otimes F_{n_r}$ is the (generalized) Fourier matrix associated to G. In this case $cl(G) = n$ and the dephased defect is $d'(G) = n^2 - \dim_{\mathbb{C}}([D, \mathbb{C}[G]]) + 2n - 1$, which can be computed to be the same as the defect of the Hadamard matrix F_G, as introduced in [TaZy1]. Also, $d(G)$ equals the undephased defect of F_G, as defined in [Ba1].*

If G is abelian, the defect $d(G)$ has a very nice interpretation as the number of entries equal to 1 of the matrix F_G (see [Ka]). For general G finite we give the following interpretation of the defect:

Proposition 2.5. *$d(G)$ equals the number of times (counted with geometric multiplicity) that 1 shows up as an eigenvalue in the matrices $\{u_g : g \in G\}$.*

Proof. 1 is an eigenvalue for u_g if there exists a non-zero vector $\mathbf{v} = \sum_{h \in G} c_h u_h$ (with $c_h \in \mathbb{C}$) satisfying $u_g \mathbf{v} = \mathbf{v}$. Equivalently $\sum_{h \in G} c_h u_{gh} = \sum_{h \in G} c_h u_h$. This means $c_{g^{-1}h} = c_h$ for all $h \in G$. It follows $c_h = c_{g^k h}$ for all $0 \leq k \leq ord(g) - 1$. Thus the dimension of the eigenspace of 1 for u_g is $\frac{|G|}{ord(g)}$. The conclusion follows from the formula for the defect established in [NiWh]: $d(G) = \sum_{g \in G} \frac{|G|}{ord(g)}$ ∎

3 Analytic families of commuting squares

In this section we construct parametric families of commuting squares of the form

$$\mathfrak{C}_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t\mathbb{C}[G]U_t^* \end{pmatrix}$$

where $t \in \mathbb{R}$, $(U_t)_{t\in\mathbb{R}}$ is a family of unitaries analytic in t, $U_t \neq I$ for $t \neq 0$, and $U_t \to U_0 = I$ as $t \to 0$. (where I denotes the $n \times n$ identity matrix). We show that there exist $d(G)$ such families which are independent in the following sense: their directions of convergence $a = \lim_{t\to 0} \frac{U_t - I}{\mathrm{i}||\mathrm{U_t - I}||}$ (for each family) exist and are linearly independent (in fact they form a basis of $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$).

The vector space $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$ can be thought of as the space of all possible directions of convergence of sequences of commuting squares converging to $\mathfrak{C}_G$ (see [Ni1]). Recall that its dimension is the undephased defect $d(G)$. One consequence of our construction is that the defect is not just an upper bound for the number of directions of convergence, but it is in fact attained.

We start by introducing (canonical) bases for $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$.

Theorem 3.1. *For every $g, h \in G$ let $a^{h,g} \in M_n(\mathbb{C})$ be the matrix having the entry on position (p,q) given by*

$$a^{h,g}_{p,q} = \begin{cases} 1 & \text{if } p = h^k g \text{ and } q = h^{k+1}g \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for all $p, q \in G$. For each $h \in G$, let $g_1^h, ...g_{n(h)}^h$ be a choice of representatives of the right cosets of $G/ < h >$, where $n(h) = |G|/ord(h)$ is the number of elements of $G/ < h >$. Then the matrices $\{a^{h,g_k^h} : h \in G, 1 \leq k \leq n(h)\}$ form a basis for $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$.

Proof. We first show that the matrices $\{a^{h,g_k^h} : h \in G, 1 \leq k \leq n(h)\}$ are linearly independent. This follows from the stronger fact that no two of them have non-zero entries on the same position. To check this, observe that if $a^{h,g}_{p,q} = a^{h',g'}_{p,q}$ for some $h, g, h', g' \in G$, then $p = h^k g = (h')^l g'$ and $q = h^{k+1}g = (h')^{l+1}g'$ for some k, l positive integers. It follows that $h = qp^{-1} = h'$. This together with $h^k g = (h')^l g'$ implies that $g' \in < h > g$. Since this is not the case for any pair of distinct matrices in the set $\{a^{h,g_k^h} : h \in G, 1 \leq k \leq n(h)\}$, we obtain that no two of them have non-zero entries on the same position, hence they are linearly independent.

We now show that $\{a^{h,g_k^h} : h \in G, 1 \leq k \leq n(h)\}$ span $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$. An $n \times n$ matrix a is in $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$ if and only if it $\tau(a[d_g, u_h]) = 0$ for all $g, h \in G$ (we used here that $[D, \mathbb{C}[G]]$ is $*$-closed). Thus $\tau(a(e_{g,h^{-1}g} - e_{hg,g})) = 0$, or equivalently $a_{h^{-1}g,g} = a_{g,hg}$ for all $g, h \in G$.

By replacing g by $h^k g$ for $k = 1, 2, ..., ord(h)$, it follows that:

$$a_{g,hg} = a_{hg,h^2g} = \ldots = a_{h^{ord(h)-1}g,g} \text{ for all } h, g \in G$$

Conversely, any matrix a satisfying the relation above must satisfy $a_{h^{-1}g,g} = a_{g,hg}$ for all $h, g \in G$. Thus any such a can be written as a span of matrices of the form a^{h,g_k^h}:

$$a = \sum_{h \in G, 1 \le k \le n(h)} a_{g_k^h, hg_k^h} \cdot a^{h,g_k^h}$$

■

The following lemma establishes a formula that we will need, for the powers of $a = a^{h,g}$ and $a^* = a^{h^{-1},g}$.

Lemma 3.2. *Fix g, $h \in G$ and let $a = a^{h,g}$. Then for $m \in \mathbb{N}$, a^m is the matrix with entries $(a^m)_{h^k g, h^{k+m} g} = 1$ for $k = 1, \ldots, |h|$ and 0 otherwise. Furthermore, a is a partial isometry, and for all $m, n \in \mathbb{N}$, we have*

$$a^m a^{*n} = \begin{cases} a^{m-n} & \text{if } m \ge n \\ a^{*(n-m)} & \text{if } n > m \end{cases}$$

where we define a^0 to be the projection matrix with entries $(a^0)_{h^k g, h^k g} = 1$ for $k = 1, \ldots, |h|$ and 0 otherwise.

Proof. We induct on m for the first part of the claim. The result is trivial when $m = 1$. Assume for some $m \in \mathbb{N}$, a^m is as claimed. Let $S_h = \langle h \rangle g$ (which is a subset of G). Fix $g', h' \in G$. Clearly if $g' \notin S_h$, we have $\left(a^{m+1}\right)_{g',h'} = 0$. For $g' \in S_h$, we have $\left(a^m\right)_{g',\tilde{h}} = \delta_{\tilde{h}}^{h^m g'}$ (for all $\tilde{h} \in G$). Hence, for $g' \in S_h$, $0 \neq \left(a^{m+1}\right)_{g',h'} = \sum_{\tilde{h} \in G} a^m_{g',\tilde{h}} a_{\tilde{h},h'} \Leftrightarrow h' = h^{m+1} g$.

The second part of the claim follows from the fact that $aa^* = a^* a = a^0$, which can be easily checked.

■

We are now ready to prove the main result of this paper. We construct continuous deformations of the commuting square $\mathfrak{C}_G$, through parametric families of unitaries given as exponentials of hermitians constructed from the matrices $a^{h,g}$.

Theorem 3.3. *Fix $k, l \in G$ and let $a = a^{l,k}$. For $t \in \mathbb{R}$, let $U_t = e^{\mathrm{it}(\mathrm{a}+\mathrm{a}^*)}$ and $V_t = e^{\mathrm{it}\left(\frac{\mathrm{a}-\mathrm{a}^*}{\mathrm{i}}\right)}$. Then the following are commuting squares:*

$$\mathfrak{C}^1_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} I_n & \subset & U_t \mathbb{C}[G] U_t^* \end{pmatrix}, \mathfrak{C}^2_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} I_n & \subset & V_t \mathbb{C}[G] V_t^* \end{pmatrix}$$

Proof. Since a and a^* commute, we have:

$$U_t = I + \sum_{p \ge 1} \frac{(\mathrm{it})^{\mathrm{p}}}{p!} \sum_{q=0}^{p} \binom{p}{q} a^q (a^*)^{p-q}$$

and

$$V_t = I + \sum_{p\geq 1} \frac{t^p}{p!} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} a^q (a^*)^{p-q}.$$

To show that $\mathfrak{C}_t^1$ is a commuting square, we need to show that for each elements of the bases $d_g \in \mathrm{D}$ and $u_h \in \mathbb{C}[G]$ $(g, h \in G)$ we have:

$$\tau(d_g u_h) = \tau(d_g U_t u_h U_t^*)$$

Since $\tau(d_g u_h) = \tau(d_g u_h U_t U_t^*)$, it is sufficient to show that

$$\tau(d_g u_h U_t U_t^*) = \tau(d_g U_t u_h U_t^*)$$

By using the formula for U_t, V_t and the previous lemma, it follows that U_t, V_t are in the linear span of I, a^n $(n \geq 0)$, $(a^*)^n$ $(n \geq 1)$. Recall that $a^0 \neq I$, according to our convention for a^0 in lemma 3.2. Thus we must show that for each g, $h \in G$, we have

$$\tau(d_g u_h xy) = \tau(d_g x u_h y)$$

for any x and y which are powers of a or a^*, or are equal to I.

Fix $p, q \in \mathbb{N}$, g, $h \in G$, and let $S_l = \langle l \rangle k$ (as a subset of G). We first check the result for $y = I$. We have:

$$\begin{aligned} \tau(d_g u_h a^p) &= \sum_{h' \in S_l} a^p_{h^{-1}g, h'} I_{h', g} \\ &= \delta^{l^{-p}g}_{h^{-1}g} \left| S_l \cap \{g\} \right| \\ &= \delta^{l^p}_h \left| S_l \cap \{g\} \right| \end{aligned}$$

and

$$\begin{aligned} \tau(d_g a^p u_h) &= \sum_{h' \in S_l} a^p_{g, h'} I_{h^{-1}h', g} \\ &= \delta^{l^{-p}hg}_{g} \left| S_l \cap \{hg\} \right| \\ &= \delta^{l^p}_h \left| S_l \cap \{hg\} \right|. \end{aligned}$$

Observe that for $h = l^p$ we have $|S_l \cap \{g\}| \neq \emptyset$ if and only if $hg \in S_l$. This shows that $\tau(d_g u_h a^p) = \tau(d_g a^p u_h)$. A similar argument shows that $\tau(d_g u_h a^{*p}) = \tau(d_g a^{*p} u_h)$.

Next, we check the result when $x = a^p$ and $y = a^q$. Indeed:

$$
\begin{aligned}
\tau(d_g u_h a^p a^q) &= \sum_{h' \in S_l} a^p_{h^{-1}g,h'} a^q_{h',g} \\
&= \sum_{h' \in S_l} \delta_g^{l^q h'} a^p_{h^{-1}l^q h',h'} \\
&= \sum_{h' \in S_l} \delta_g^{l^q h'} \delta_{h^{-1}l^q h'}^{l^{-p} h'} \\
&= \delta_h^{l^{q+p}} \left| S_l \cap \{g\} \right|
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\tau(d_g a^p u_h a^q) &= \sum_{h' \in S_l} a^p_{g,h'} a^q_{h^{-1}h',g} \\
&= \sum_{h' \in S_l} \delta_g^{l^{-p} h'} \delta_{h^{-1}h'}^{l^{-q} l^{-p} h'} \\
&= \delta_h^{l^{q+p}} \left| S_l \cap \{g\} \right|.
\end{aligned}
$$

Thus $\tau(d_g u_h a^p a^q) = \tau(d_g a^p u_h a^q)$. A similar argument establishes that

$$
\tau(d_g u_h a^{*p} a^{*q}) = \delta_{h^{-1}}^{l^{p+q}} \left| S_l \cap \{g\} \right| = \tau(d_g a^{*p} u_h a^{*q})
$$

We now check $\tau(d_g u_h a^p a^{*q}) = \tau(d_g a^p u_h a^{*q})$.

$$
\begin{aligned}
\tau(d_g u_h a^p a^{*q}) &= \sum_{h' \in S_l} a^p_{h^{-1}g,h'} a^{*q}_{h',g} \\
&= \sum_{h' \in S_l} \delta_g^{l^{-q} h'} a^p_{h^{-1}l^{-q} h',h'} \\
&= \sum_{h' \in S_l} \delta_g^{l^{-q} h'} \delta_{h^{-1}l^{-q} h'}^{l^{-p} h'} \\
&= \delta_h^{l^{p-q}} \left| S_l \cap \{g\} \right|
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\tau(d_g a^p u_h a^{*q}) &= \sum_{h' \in S_l} a^p_{g,h'} a^{*q}_{h^{-1}h',g} \\
&= \sum_{h' \in S_l} \delta_g^{l^{-p} h'} a^{*q}_{h^{-1}h',l^{-p} h'} \\
&= \sum_{h' \in S_l} \delta_g^{l^{-p} h'} \delta_{h^{-1}h'}^{l^q l^{-p} h'} \\
&= \delta_h^{l^{p-q}} \left| S_l \cap \{g\} \right|.
\end{aligned}
$$

An almost identical argument shows that

$$\tau(d_g u_h a^{*p} a^q) = \delta_h^{lq-p} \, |S_l \cap \{g\}| = \tau(d_g a^{*p} u_h a^q)$$

■

Remark 3.4. *If $a = a^{h,g}$ then $a^* = a^{h^{-1},g}$. It follows that $a = a^*$ if and only if $ord(h) \leq 2$. We construct a basis of hermitians for $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$ as follows: Start with a basis from theorem 3.1. For each $a = a^{h,g}$ in this basis, keep a if $ord(h) \leq 2$. If $ord(h) > 2$ then remove a and a^* from the basis and replace them by the self-adjoint elements $a + a^*$ and $\frac{a-a^*}{\mathrm{i}}$. Since $a = \frac{1}{2}(a + a^*) + \frac{\mathrm{i}}{2}\left(\frac{a-a^*}{\mathrm{i}}\right)$, these new elements still span $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$. Linear independence also follows from $span\{a, a^*\} = span\{a + a^*, \frac{a-a^*}{\mathrm{i}}\}$. Thus, Theorem 3.3 shows that there exist $d(G)$ analytic deformations of the commuting square $\mathfrak{C}_G$, whose directions are the $d(G)$ hermitians forming a basis for $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$.*

Remark 3.5. *If $G = \mathbb{Z}_n$ we have $\mathbb{C}[G] = F_n D F_n^*$, where F_n is the Fourier matrix of size n. In this case the commuting square $\mathfrak{C}_G$ is a spin model, and the analytic deformations from Theorem 3.3 give analytic 1-parameter families of Hadamard matrices: $U_t F_n$ and $V_t F_n$.*

Remark 3.6. *Theorem 3.3 shows that $d(G)$ is the best possible bound for the number of independent directions of convergence, in the following sense: there exists a basis for $[\mathrm{D}, \mathbb{C}[\mathrm{G}]]^\perp$, such that for every a in the basis there is an analytic family of commuting squares containing $\mathfrak{C}_G$ and of direction a. However, it is not true in general that every (hermitian of unit length) $a \in \mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$ is a direction of convergence. This is shown by the following example.*

Example 3.7. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If*

$$a = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

then a is a hermitian in $[\mathrm{D}, \mathbb{C}[\mathrm{G}]]^\perp$, but there do not exist unitary matrices $U_t \to I, U_t \neq I$ such that

$$\begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t \mathbb{C}[G] U_t^* \end{pmatrix}$$

are commuting squares for all t and

$$\lim_{t \to 0} \frac{U_t - I}{\mathrm{i}||\mathrm{U_t - I}||} = a$$

Proof. By [Ni3], if a is a direction of convergence then there must exist a matrix b such that

$$\tau(b[d_g, u_h]) = \tau(d_g u_h a^2) - \tau(d_g a u_h a) \text{ for all } g, h \in G$$

This gives a linear system whose variables are the entries of b. We checked that this system has no solutions for the given matrix a, by using Mathematica. ■

4 Dephased analytic families of commuting squares

The families constructed in Theorem 3.3 may contain mutually isomorphic commuting squares. Indeed, from results in [Ni1] it follows that there exist at most

$$d'(G) = \dim(\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D}))$$

families of mutually non-isomorphic commuting squares containing $\mathfrak{C}_G$, whose directions of convergence are linearly independent. Observe that the dephased defect $d'(G)$ is significantly smaller than the defect $d(G)$. The bound $d'(G)$ follows from the fact that any family is isomorphic to a family whose direction of convergence is *dephased*, i.e. it is orthogonal to $[D, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + D$ (for a proof see Lemma 1.8 in [Ni1]).

Thus in order to be able to argue non-isomorphism for some of the families that we constructed, we will first refine our construction to just $d'(G)$ dephased families. The non-isomorphism question will be addressed in the next section.

In this section we prove that when G is abelian there exist $d'(G)$ analytic families of commuting squares whose directions of convergence form a basis for

$$\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D})$$

Let $\mathcal{B} = \{a^{h,g_k^h} : h \in G, 1 \le k \le n(h)\}$ be a basis for $\mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$ as defined in Theorem 3.1. For each $a^{h,g} \in \mathcal{B}$ with $h, g \neq e$, denote $\alpha^{h,g} = a^{h,e} - a^{h,g}$. Let $\mathcal{B}'$ be the set of all such $\alpha^{h,g}$. Clearly $\mathcal{B}' \subset \mathrm{M_n}(\mathbb{C}) \ominus [\mathrm{D}, \mathbb{C}[\mathrm{G}]]$. Then next lemma shows that the elements of $\mathcal{B}'$ are orthogonal to $\mathbb{C}[G]$.

Lemma 4.1. *Let $h, g_1, g_2 \in G$. Then $a^{h,g_1} - a^{h,g_2}$ is orthogonal to $\mathbb{C}[G]$.*

Proof. Note that for a matrix $a \in \mathrm{M_n}(\mathbb{C})$ we have:

$$\begin{aligned} \tau(au_g) &= \sum_{h,h'} a_{h,h'} \, (u_g)_{h',h} \\ &= \sum_h a_{h,gh}. \end{aligned}$$

Now let $k,\ l \in G$, $a = a^{l,k}$ and $S_l = \langle l \rangle k$. It follows that $\tau(au_g) = \sum_h a_{h,gh} = \sum_{h \in S_l} a_{h,gh}$ is non-zero if and only if $gh = lh$. Thus $\tau(a^{l,k}u_g) = 0$ if $l \neq g$ and $\tau(a^{l,k}u_g) = ord(g)/n$ if $l = g$. It follows that $\tau((a^{h,g_1} - a^{h,g_2})u_g) = 0$.

■

Proposition 4.2. *If G is abelian then $\mathcal{B}'$ is a basis for the vector space* $\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D})$.

Proof. Since G is abelian, we have $\mathbb{C}[G] \subset \mathbb{C}[G]'$. Using $\dim(\mathbb{C}[G]) = \dim(\mathbb{C}[G]') = n$ it follows that $\mathbb{C}[G] = \mathbb{C}[G]'$. Thus

$$\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathbb{C}[\mathrm{G}]' + \mathrm{D}) = \mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathbb{C}[\mathrm{G}] + \mathrm{D})$$

It is easy to see that $[D, \mathbb{C}[G]]$ is orthogonal to $\mathbb{C}[G]+D$, and that $\mathbb{C}[G]\cap D = \mathbb{C}$ (see [NiWh] for these computations). Thus the dimension $d'(G)$ of $\mathrm{M_n(\mathbb{C})} \ominus (\mathrm{[D, \mathbb{C}[G]] + \mathbb{C}[G] + D})$ is equal to $d(G) - 2n + 1$.

Observe that $\mathcal{B}'$ has exactly $d(G)-2n+1$ elements, since $\mathcal{B}$ has cardinality $d(G)$ and we ask that $h, g \neq e$ for each $\alpha^{h,g} \in \mathcal{B}'$. Thus to show that $\mathcal{B}'$ is a basis it is sufficient to show that its elements are linearly independent, and that they are contained in the vector space $\mathrm{M_n(\mathbb{C})} \ominus (\mathrm{[D, \mathbb{C}[G]] + \mathbb{C}[G] + D})$.

We first show that $\mathcal{B}' \subset \mathrm{M_n(\mathbb{C})} \ominus (\mathrm{[D, \mathbb{C}[G]] + \mathbb{C}[G] + D})$. Indeed, from the previous lemma we know that $\mathcal{B}' \perp \mathbb{C}[G]$. Also $\mathcal{B}' \perp D$, since each matrix $a^{h,g}$ has 0 on the diagonal if $h \neq e$.

Finally, the elements of $\mathcal{B}'$ are linearly independent. This is because the nonzero entries of $a^{h,g}$ are zero entries for $a^{l,k}$, whenever $h \neq l$ or $h = l$ and $k \notin \langle h \rangle g$, as proven in Theorem 3.3.

■

Lemma 4.3. *Let $h \in G$ and $g \notin \langle h \rangle$. Then $a^{h,e}a^{h,g} = 0 = a^{h,g}a^{h,e}$ and $a^{h,e}(a^{h,g})^* = 0 = (a^{h,g})^* a^{h,e}$.*

Proof. We check $a^{h,e}a^{h,g} = 0$. The other equalities follow in a similar fashion. We have $(a^{h,e}a^{h,g})_{g',h'} \neq 0 \Leftrightarrow g' \in \langle h \rangle$, $hg' \in \langle h \rangle g$ and $h' = h^2 g'$. But $hg' \in \langle h \rangle g$ implies $g' \in \langle h \rangle g$. In this case we can't have $g' \in < h >$ since $g \notin < h >$.

■

Corollary 4.4. *Let $h \in G$ and $g \notin \langle h \rangle$. For $m \in \mathbb{N}$, we have*

$$(\alpha^{h,g})^m = (a^{h,e})^m + (-1)^m (a^{h,g})^m.$$

Theorem 4.5. *Let $h, g \in G$ such that $g \notin \langle h \rangle$ and $h, g \neq e$. Let $\alpha^{h,g} = a^{h,e} - a^{h,g}$. For $t \in \mathbb{R}$, let $U_t = e^{\mathrm{it}(\alpha^{\mathrm{h,g}} + (\alpha^{\mathrm{h,g}})^*)}$ and $V_t = e^{\mathrm{it}\frac{\alpha^{\mathrm{h,g}} - (\alpha^{\mathrm{h,g}})^*}{\mathrm{i}}}$. Then the following are commuting squares:*

$$\mathfrak{C}_t^1 = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n(\mathbb{C})} \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t\mathbb{C}[G]U_t^* \end{pmatrix}, \mathfrak{C}_t^2 = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n(\mathbb{C})} \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & V_t\mathbb{C}[G]V_t^* \end{pmatrix}$$

Proof. We check that $\mathfrak{C}_t^1$ is a commuting square for all $t \in \mathbb{R}$. The computations for $\mathfrak{C}_t^2$ follow similarly. From Lemma 4.3 we see that $\alpha^{h,g}(\alpha^{h,g})^* = (\alpha^{h,g})^*\alpha^{h,g}$. By using this together with the previous corollary, it follows that U_t can be expanded as

$$U_t = I + \sum_{p \geq 1} \frac{(\mathrm{it})^{\mathrm{p}}}{p!} \sum_{q=0}^{p} \binom{p}{q} \left((a^{h,e})^q (a^{h,e})^{*(p-q)} + (-1)^n (a^{h,g})^q (a^{h,g})^{*(p-q)} \right).$$

Thus, to show that $\mathfrak{C}_t^1$ is a commuting square, it is sufficient to check that

$$\tau(d_{g'}u_{h'}xy) = \tau(d_{g'}xu_{h'}y)$$

for any $g', h' \in G$ and any x, y which are powers of $a^{h,e}$, $(a^{h,e})^*$, $a^{h,g}$ or $(a^{h,g})^*$, or are equal to the identity. By Theorem 3.3 we already know that this statement is true when x, y are powers of $a^{h,e}$, $(a^{h,e})^*$, respectively when x, y are powers of $a^{h,g}$ or $(a^{h,g})^*$, or the identity. So it suffices to check that $\tau(d_{g'} x u_{h'} y) = 0$ whenever x is a power of $a^{h,e}$ or $(a^{h,e})^*$ and y is a power of $a^{h,g}$ or $(a^{h,g})^*$ and vice-versa (as Lemma 4.3 shows that $\tau(d_{g'} u_{h'} xy) = 0$). To that end, let $l, m \in \mathbb{N}$. Fix $h', g' \in G$. We show $\tau\left(d_{g'}\left(a^{h,e}\right)^l u_{h'}\left(a^{h,g}\right)^m\right) = 0$. Indeed,

$$\begin{aligned}\tau(d_{g'}(a^{h,e})^l u_{h'}(a^{h,g})^m) &= \sum_k \left((a^{h,e})^l\right)_{g',k}\left((a^{h,g})^m\right)_{(h')^{-1}k,g'} \\ &= 0\end{aligned}$$

The last equality follows from the fact that we can not simultaneously have $g' \in \langle h \rangle$ and $g' \in \langle h \rangle g$.

The fact that the other mixed powers have trace 0 follows similarly.

■

Remark 4.6. *We can construct bases of hermitians for* $\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathrm{D} + \mathbb{C}[\mathrm{G}])$ *as follows: Start with the basis $\mathcal{B}'$ obtained from $\mathcal{B}$. For each $\alpha = \alpha^{h,g}$ in this basis, keep α if $\alpha = \alpha^*$. If not, then remove α and α^* from the basis and replace them by the self-adjoint elements $\alpha + \alpha^*$ and $\frac{\alpha - \alpha^*}{\mathrm{i}}$. Since span$\{\alpha, \alpha^*\}$ = span$\{\alpha + \alpha^*, \frac{\alpha-\alpha^*}{\mathrm{i}}\}$, it follows that the new elements form a basis $\mathcal{B}''$. Thus Theorem 4.5 gives $\mathrm{d}'(G)$ analytic families of commuting squares through $\mathfrak{C}_G$, and whose directions of convergence form a basis for* $\mathrm{M_n}(\mathbb{C}) \ominus ([\mathrm{D}, \mathbb{C}[\mathrm{G}]] + \mathrm{D} + \mathbb{C}[\mathrm{G}])$

5 A non-equivalence result for continuous families of complex Hadamard matrices

In this section we use the results from the previous section to construct new (dephased) families of complex Hadamard matrices, and we prove a non-equivalence result for many of the matrices in these families.

For the rest of the section we will assume that $G = \mathbb{Z}_n$. In this case $\mathfrak{C}_G$ is the so-called *standard spin model* commuting square:

$$\mathfrak{C}_{\mathbb{Z}_n} = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & F_n D F_n^* \end{pmatrix}$$

where F_n is the Fourier matrix of size n. Moreover, if

$$\mathfrak{C}_t = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t F_n D F_n^* U_t^* \end{pmatrix}$$

is a parametric family of (spin model) commuting squares, where $U_t \to I$ are unitaries, then $U_tF_n \to F_n$ are complex Hadamard matrices (see for instance [Ni1]). In particular, the constructions from Theorem 3.3 and Theorem 4.5 yield analytic families of complex Hadamard matrices:

Corollary 5.1. *Let $g, h \in \mathbb{Z}_n$ and let $a = a^{h,g}$ be defined as in Theorem 3.1. For $t \in \mathbb{R}$, let*

$$U_t = e^{\mathrm{it}(\mathrm{a}+\mathrm{a}^*)} \text{ and } V_t = e^{\mathrm{it}\left(\frac{\mathrm{a}-\mathrm{a}^*}{\mathrm{i}}\right)}.$$

Then U_tF_n and V_tF_n are complex Hadamard matrices for all $t \in \mathbb{R}$.

Corollary 5.2. *Let $h, g \in \mathbb{Z}_n$, such that $g \notin \langle h \rangle$ and $h, g \neq 0$. Let $\alpha^{h,g} = a^{h,0} - a^{h,g}$, where $a^{h,g}$ are defined as in Theorem 3.1. For $t \in \mathbb{R}$, let*

$$U_t = e^{\mathrm{it}(\alpha^{\mathrm{h,g}}+(\alpha^{\mathrm{h,g}})^*)} \text{ and } V_t = e^{\mathrm{it}\frac{\alpha^{\mathrm{h,g}}-(\alpha^{\mathrm{h,g}})^*}{\mathrm{i}}}.$$

Then U_tF_n and V_tF_n are complex Hadamard matrices for all $t \in \mathbb{R}$.

Consider now two sequences of complex Hadamard matrices $(U_1^kF_n)_{k\geq 1}$ and $(U_2^kF_n)_{k\geq 1}$, where $U_1^k \neq I$ and $U_2^k \neq I$ $(k \geq 1)$ are unitaries which converge to I as $k \to \infty$. We encode these sequences in two sequences of commuting squares $(\mathfrak{C}_k^1)_{k\geq 1}$ and $(\mathfrak{C}_k^2)_{k\geq 1}$, where

$$\mathfrak{C}_k^i = \begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_i^kF_nDF_n^*(U_i^k)^* \end{pmatrix}$$

for $i = 1, 2$ and $k \geq 1$.

Our goal is to show that there exists a neighborhood of F_n in which many of the complex Hadamard matrices constructed in Corollary 5.2 are not isomorphic (see Theorem 5.5 for the precise statement). To prove this, we will assume by contradiction that there exist infinitely many pairs of such equivalent matrices $U_1^kF_n, U_2^kF_n$, with $U_i^k \to I$ and $U_i^k \neq I$ $(i = 1, 2)$. In other words, there must exist d_1^k, d_2^k unitary diagonals and p_1^k, p_2^k permutation matrices such that

$$U_2^k = p_1^kd_1^kU_1^kF_np_2^kd_2^kF_n^*$$

The idea of the proof is to take a directional derivative of the relation $U_2^k = p_1^kd_1^kU_1^kF_np_2^kd_2^kF_n^*$ as $k \to \infty$, and thus obtain a new relation which leads to a contradiction.

We start with a couple of lemmas. The first lemma gives a normalization for the direction of convergence of a sequence of unitaries. The second lemma analyses the limit of (a subsequence of) the equalities $U_2^k = p_1^kd_1^kU_1^kF_np_2^kd_2^kF_n^*$.

Lemma 5.3. *Let $x, x_1, x_2, x_3, ... \in \mathrm{M_n}(\mathbb{C})$ be unitaries satisfying $x_k \to x$ as $k \to \infty$. Assume that there exists $X \in \mathrm{M_n}(\mathbb{C})$ such that $\frac{x_k - x}{i||x_k - x||} \to X$. Then, after replacing $(x_k)_{k\geq 1}$ by one of its subsequences, there exists a sequence of complex numbers $\{\lambda_k\}_{k\geq 1}$ such that: $|\lambda_k| = 1$, $\lambda_kx_k \to x$ and $\frac{\lambda_kx_k - x}{i||\lambda_kx_k - x||} \to \widetilde{X}$ with $\tau(\widetilde{X}x^*) = 0$.*

Proof. Since $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is compact, it follows that for each k there exists λ_k such that

$$||\lambda_k x_k - x||_2 = \inf_{|\lambda|=1}\{||\lambda x_k - x||_2\}.$$

Note that $||\lambda_k x_k - x||_2 \leq ||x_k - x||_2 \to 0$. Therefore, $\lambda_k x_k \to x$.

It is easy to check that for u unitary we have $||u-I||_2^2 = 2-2\Re\tau(u)$ (where $\Re z$ denotes the real part of the complex number z). Thus, the previous relation implies that $\Re\tau(\lambda_k x_k x^*) \geq \Re\tau(\lambda x_k x^*)$ for all $|\lambda| = 1$. Hence,

$$\Re\tau(\lambda_k(e^{\mathrm{it}} - 1)x_k x^*) \leq 0$$

for all $t \in \mathbb{R}$. Now we divide by t for $t > 0$ and take the limit as t approaches 0, to obtain $\Re\tau(\mathrm{i}\lambda_\mathrm{k}\mathrm{x}_\mathrm{k}\mathrm{x}^*) \leq 0$; doing the same for $t < 0$, we obtain $\Re\tau(\mathrm{i}\lambda_\mathrm{k}\mathrm{x}_\mathrm{k}\mathrm{x}^*) \geq 0$. Thus,

$$\Re\tau(\mathrm{i}\lambda_\mathrm{k}\mathrm{x}_\mathrm{k}\mathrm{x}^*) = 0.$$

Equivalently, $\Re\tau(\mathrm{i}(\lambda_\mathrm{k}\mathrm{x}_\mathrm{k}\mathrm{x}^* - \mathrm{I})) = 0$. Let $\widetilde{X} = \lim_{k\to\infty} \frac{\lambda_k x_k x^* - I}{\mathrm{i}||\lambda_\mathrm{k}\mathrm{x}_\mathrm{k}\mathrm{x}^* - \mathrm{I}||}$, after passing to a subsequence if needed. Dividing the previous equality by $||\lambda_k x_k x^* - I||$ and taking the limit, we obtain $\Re\tau(\mathrm{i}(\mathrm{i}\widetilde{\mathrm{X}})) = 0$. Since $\widetilde{X}$ is hermitian, we have $\tau(\widetilde{X}) = \Re\tau(\widetilde{X}) = 0$.

■

Lemma 5.4. *Let σ_1, σ_2 be permutations of $\mathbb{Z}_n$ and let $p_1, p_2 \in \mathrm{M_n}(\mathbb{C})$ be the permutation matrices associated to $\sigma_1, \sigma_2 \in S_n$. Let $d_1, d_2 \in \mathrm{M_n}(\mathbb{C})$ be diagonal matrices. Assume that:*

$$p_1 d_1 F_n d_2 p_2 F_n^* = I.$$

Then there exists $b \in \{0, 1, ..., n-1\}$ such that $(b, n) = 1$ and

$$\sigma_1(k) = \sigma_1(0) - bk, \sigma_2^{-1}(k) = \sigma_2^{-1}(0) - b^{-1}k$$

for all $k \in \mathbb{Z}_n$. Furthermore,

$$d_{1,\sigma_1(k)} = d_{1,0}\varepsilon^{bk\sigma_2^{-1}(0)} \text{ and } d_{2,\sigma_2^{-1}(k)} = d_{2,0}\varepsilon^{b^{-1}k\sigma_1(0)}$$

for all $k \in \mathbb{Z}_n$, where $\varepsilon = e^{\frac{2\pi i}{n}}$ and $d_{i,k}$ denotes the k^{th} diagonal entry of the matrix d_i ($i = 1, 2$).

Proof. Let $\varepsilon = e^{\frac{2\pi i}{n}}$. The relation $p_1 d_1 F_n d_2 p_2 F_n^* = I$ is equivalent to:

$$\varepsilon^{kl} = \varepsilon^{\sigma_1^{-1}(k)\sigma_2^{-1}(l)} d_{1,\sigma_1(k)} d_{2,\sigma_1^{-1}(l)}.$$

Let $x_{k,l} = \varepsilon^{kl-\sigma_1(k)\sigma_2^{-1}(l)}$. For simplicity let's denote $d_k = d_{1,\sigma_1(k)}$ and $d'_l = d_{2,\sigma_1^{-1}(l)}$. We have:

$$d_k d'_l = x_{kl}.$$

It easily follows that for all $k, k', l, l' \in \mathbb{Z}_n$ we must have

$$\frac{x_{kl}}{x_{k'l}} = \frac{x_{kl'}}{x_{k'l'}}.$$

Note that this set of conditions on (x_{kl}) is also sufficient for the existence of $(d_k), (d'_l)$ satisfying $d_k d'_l = x_{kl}$. Indeed, choose $d_0 d'_0 = x_{00}$ and set $d_k = d_0 \frac{x_{k0}}{x_{00}}$ and $d'_l = d'_0 \frac{x_{0l}}{x_{00}}$. Then

$$d_k d'_l = d_0 \frac{x_{k0}}{x_{00}} d'_0 \frac{x_{0l}}{x_{00}} = \frac{x_{k0} x_{0l}}{x_{00}} = \frac{x_{kl} x_{00}}{x_{00}} = x_{kl}.$$

From $\frac{x_{kl}}{x_{k'l}} = \frac{x_{kl'}}{x_{k'l'}}$ it follows that σ_1 and σ_2 must satisfy

$$kl - \sigma_1(k)\sigma_2^{-1}(l) = \sigma_1(0)\sigma_2^{-1}(0) - \sigma_1(0)\sigma_2^{-1}(l) - \sigma_1(k)\sigma_2^{-1}(l) \text{ for } 0 \leq k, l \leq n-1$$

or equivalently

$$kl = (\sigma_1(0) - \sigma_1(k)) \left(\sigma_2^{-1}(0) - \sigma_2^{-1}(l)\right) \text{ for } 0 \leq k, l \leq n-1.$$

Recall that these equalities are all modulo n, since work with elements of $\mathbb{Z}_n$. Choose a such that $1 = \sigma_2^{-1}(0) - \sigma_2^{-1}(a)$. Then, we have $ka = \sigma_1(0) - \sigma_1(k)$ for all k. It follows that $(a, n) = 1$. Similarly, we get that $\sigma_2^{-1}(0) - \sigma_2^{-1}(k) = bk$ with $(b, n) = 1$. Since $kl = abkl$ for all k, l, we must have $b = a^{-1}$. Thus, there exists b with $(b, n) = 1$ and

$$\sigma_1(k) = \sigma_1(0) - bk$$

and

$$\sigma_2^{-1}(k) = \sigma_2^{-1}(0) - b^{-1}k.$$

■

We are now ready to prove the main result of this section, which shows non-equivalence for many of the complex Hadamard matrices constructed in Corollary 5.2.

Theorem 5.5. *For any $g, h \in \mathbb{Z}_n$ let $\alpha^{h,g} = a^{h,0} - a^{h,g}$, where $a^{h,g}$ are defined as in Theorem 3.1. Let $g_1, g_2, h_1, h_2 \in \mathbb{Z}_n$ such that $g_1 \notin \langle h_1 \rangle$, $g_2 \notin \langle h_2 \rangle$ and $|h_1| \neq |h_2|$. Then there exists $\delta > 0$ such that for every $t, s \in (-\delta, \delta) \setminus \{0\}$ the complex Hadamard matrices $e^{it(\alpha^{h_1,g} + (\alpha^{h_1,g})^*)}$ and $e^{is(\alpha^{h_2,g} + (\alpha^{h_2,g})^*)}$ are not equivalent.*

Proof. Assume by contradiction that the statement does not hold. Then there exist two sequences $(s_k)_{k \geq 1}$, $(t_k)_{k \geq 1}$ of real non-zero numbers, converging to 0, such that the complex Hadamard matrices U_k^1 and U_k^2 are equivalent for all $k = 1, 2, 3, ...$, where:

$$U_k^1 = e^{it_k(\alpha^{h_1,g} + (\alpha^{h_1,g})^*)} \text{ and } U_k^2 = e^{is_k(\alpha^{h_2,g} + (\alpha^{h_2,g})^*)}$$

It follows that there exists permutation matrices p_i^k and unitary diagonal matrices d_i^k (for $i = 1, 2$) satisfying

$$p_1^k d_1^k U_1^k F p_2^k d_2^k F^* = U_2^k$$

for all k. By passing to subsequences and using that the set of $n \times n$ permutations matrices is finite, we may assume that there exist matrices d_i and p_i $(i = 1, 3)$ such that $d_i^k \to d_i$ and $p_i^k = p_i$. Note that d_i are diagonal unitaries and p_i are permutation matrices, for $i = 1, 2$. Taking the limit of the relation $p_1^k d_1^k U_1^k F p_2^k d_2^k F^* = U_2^k$ as $k \to \infty$, we obtain:

$$p_1 d_1 F d_2 p_2 F^* = I.$$

Applying Lemma 5.4, we have there exists b with $(b, n) = 1$ such that

$$\sigma_1(l) = \sigma_1(0) + bl,$$
$$\sigma_2^{-1}(l) = \sigma_2^{-1}(0) + b^{-1}l,$$

and

$$d_{1,\sigma_1(l)} = d_{1,0}\varepsilon^{bl\sigma_2^{-1}(0)} \text{ for all } l.$$

By passing to subsequences if needed, we may assume that $\lim_k \frac{d_i^k - d_i}{||d_i^k - d_i||} = D_i$, where D_i is a diagonal matrix for $i = 1, 2$. By replacing d_1^k with $\lambda_k d_1^k$ and d_2^k with $\overline{\lambda_k} d_2^k$ as in Lemma 5.3, we may assume that $\tau(D_1 d_1^*) = 0$. Note that this does not change the relation $p_1 d_1^k U_1^k F d_2^k p_2 F^* = U_2^k$.

Let

$$r_k = \max\{||d_1^k - d_1||, ||d_2^k - d_2||, ||U_1^k - I||, ||U_2^k - I||\}.$$

By passing again to subsequences if necessary, we may assume that there exist complex constants δ_i and α_i for $i = 1, 2$ with $\alpha_i = \lim_k \frac{||U_i^k - I||}{r_k}$ and $\delta_i = \lim_k \frac{||d_i^k - d_i||}{r_k}$ for $i = 1, 2$. It follows:

$$\lim_n \frac{d_i^k - d_i}{r_k} = \delta_i D_i \text{ and } \lim_k \frac{U_i^k - I}{k_n} = \alpha_i \alpha^{h_i, g_i} \text{ for } i = 1, 2.$$

Furthermore, $0 \leq \alpha_1, \alpha_2, \delta_1, \delta_2 \leq 1$ and at least one of α_1, α_2, δ_1, and δ_2 are nonzero. Indeed, if all four constants would equal zero, then for large n we have

$$\frac{||U_i^k - I||}{r_k} < \frac{1}{2} \text{ and } \frac{||d_i^k - d_i||}{r_k} < \frac{1}{2} \text{ for } i = 1, 2$$

which is a contradiction as at least one of these quantities is 1 for each n.

Since $p_1 d_1 F d_2 p_2 F^* = I$, we have

$$U_2^k - I = p_1(d_1^k - d_1)U_1^k F d_2^k p_2 F^* + p_1 d_1 (U_1^k - I) F d_2^k p_2 F^* + p_1 d_1 F (d_2^k - d_2) p_2 F^*.$$

Dividing by r_k and taking the limit, we have

$$\begin{aligned}\alpha_2 \left(\alpha^{h_2,g_2} + (\alpha^{h_2,g_2})^*\right) &= p_1(\delta_1 D1) F d_2 p_2 F^* + p_1 d_1 \alpha_1 \left(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*\right) F d_2 p_2 F^* + p_1 d_1 F (\delta_2 D_2) p_2 F^* \\ &= \delta_1 p_1 D_1 d_1^* p_1^* + \alpha_1 p_1 d_1 \left(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*\right) d_1^* p_1^* + \delta_2 F p_2^* d_2^* D_2 p_2 F^*.\end{aligned}$$

Note that $p_1D_1d_1^*p_1^*$ and $p_2^*d_2^*D_2p_2$ are both diagonal matrices.

We have that $d_{1,\sigma_1(p)} = d_0\varepsilon^{-\sigma_2^{-1}(0)bp}$ for any p. Therefore,

$$(p_1d_1\alpha^{h_1,g_1}d_1^*p_1^*)_{k,l} = \varepsilon^{-\sigma_2^{-1}(0)b(k-l)}\left(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*\right)_{\sigma_1(k),\sigma_1(l)}.$$

For any $g \in G$, $(a^{h_1,g} + \left(a^{h_1,g}\right)^*)_{\sigma_1(k),\sigma_1(l)} \neq 0$ if and only if $\sigma_1(k) = h_1p + g$ and $\sigma_1(l) = h_1(p \pm 1) + g$ for some $1 \leq p \leq |h_1|$, which is equivalent to $k = b^{-1}h_1p + b^{-1}(g - \sigma_1(0))$ and $b(l-k) = \pm h_1$ for some $1 \leq p \leq |h_1|$. Note that since $(b,n) = 1$, $|b^{-1}h_1| = |h_1|$. Letting $a = a^{b^{-1}h_1,-b^{-1}\sigma_1(0)} - a^{b^{-1}h_1,b^{-1}(g-\sigma_1(0))}$, we then have that

$$p_1d_1\left(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*\right)d_1^*p_1^* = \varepsilon^{-h_1\sigma_1(0)}a + \left(\varepsilon^{-h_1\sigma_1(0)}a\right)^*.$$

By Remark 4.1, it follows that $p_1d_1(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*)d_1^*p_1^*$ is orthogonal to D and FDF^*. Hence, we must have $\alpha_2\left(\alpha^{h_2,g_2} + (\alpha^{h_2,g_2})^*\right) - \alpha_1p_1d_1\left(\alpha^{h_1,g_1} + (\alpha^{h_1,g_1})^*\right)d_1^*p_1^* = 0$ which can only happen if $\alpha_1 = \alpha_2 = 0$ since $|h_1| \neq |h_2|$. We conclude that

$$0 = \delta_1p_1D_1d_1^*p_1^* + \delta_2Fp_2^*d_2^*D_2p_2F^*$$

with both δ_1 and δ_2 non-zero. This implies that $D_i = \beta_id_i$ $(i = 1,2)$ for some complex numbers β_1 and β_2 (since for a diagonal d, FdF^* is circulant). It follows that $\lim_n \dfrac{d_1^n - d_1}{||d_1^n - d_1||} = \beta_1d_1$ with $0 = \tau(\beta_1d_1d_1^*)$ and hence, $\beta_1 = 0$. This contradicts $||\beta_1d_1|| = 1$.

■

Remark 5.6. *Under the same hypothesis, a similar proof gives non-equivalence for pairs of matrices of the form* $e^{it(\alpha^{h_1,g}+(\alpha^{h_1,g})^*)}$ *and* $e^{s(\alpha^{h_2,g}-(\alpha^{h_2,g})^*)}$, *and also for pairs of matrices of the form* $e^{t(\alpha^{h_1,g}-(\alpha^{h_1,g})^*)}$ *and* $e^{s(\alpha^{h_2,g}-(\alpha^{h_2,g})^*)}$, *with* t,s *small and non-zero. This covers all the types of matrices constructed in Corollary 5.2.*

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