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CONTINUOUS FAMILIES OF HYPERFINITE SUBFACTORS WITH THE SAME STANDARD INVARIANT

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We construct numerous continuous families of irreducible subfactors of the hyperfinite II_1 factor which are non-isomorphic, but have all the same standard invariant. In particular, we obtain 1-parameter families of irreducible, non-isomorphic subfactors of the hyperfinite II_1 factor with Jones index 6, which have all the same standard invariant with property (T). We exploit the fact that property (T) groups have uncountably many non-cocycle conjugate cocycle actions on the hyperfinite II_1 factor.

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1. Introduction

The standard invariant $\mathcal{G}_{N,M}$ of an inclusion of II_1 factors $N \subset M$ with finite Jones index is an extremely powerful invariant which leads to a complete classification of all subfactors of the hyperfinite II_1 factor R with index ≤ 4 (see for instance [13], [8], [7], [23]). It turns out that there are countably many non-isomorphic subfactors with index ≤ 4, and their (countably many distinct) standard invariants are enough to reconstruct these subfactors. However, when the Jones index becomes > 4 the standard invariant will no longer be a complete invariant for the subfactor in general. In [23] a notion of *amenability* for $\mathcal{G}_{N,M}$ was introduced, and it was shown that subfactors N of the hyperfinite II_1 factor R with amenable $\mathcal{G}_{N,R}$ are classified by this invariant. It is still open whether a converse of this result is true. In other words, it is not known whether, given a subfactor $P \subset R$ with non-amenable standard invariant $\mathcal{G}_{P,R}$, there is another subfactor $Q \subset R$ such that $\mathcal{G}_{P,R}$ and $\mathcal{G}_{Q,R}$ coincide, but the inclusions $P \subset R$ and $Q \subset R$ are not isomorphic. Compare this with the results of Ocneanu and Jones on outer actions of groups on the hyperfinite II_1 factor

([18], [11]). Namely, it is shown in [18] that an amenable group has only one outer action on the hyperfinite II_1 factor (up to outer conjugacy) whereas the result in [11] shows that non-amenable groups have always at least two.

We show in this paper that one can construct uncountably many examples of irreducible, hyperfinite subfactors with integer index which are not isomorphic, but have all the same standard invariant. The smallest Jones index for which our construction works is 6. This is a rather surprising result since the standard invariant has so far been sufficiently powerful to classify subfactors with small index. Our work shows that 6 has to be considered as a "big" index from this point of view. The construction of our exotic subfactors relies mainly on two ingredients. One is the class of subfactors introduced in [5]. Those subfactors are simple quantum dynamical systems that arise from outer actions of finite groups H and K on a II_1 factor M. The subfactor M^H is the fixed point algebra under the H action. It is contained in the crossed product algebra $M \rtimes K$. The second ingredient is a rigidity result in [25], which says that infinite discrete groups with Kazhdan's property (T) have continuously many non-cocycle conjugate cocycle actions on the hyperfinite II_1 factor. Since there are many property (T) groups which can be written as a quotient of $\mathbb{Z}_2 * \mathbb{Z}_3 = \mathrm{PSL}(2, \mathbb{Z})$ (for instance $\mathrm{SL}(2n+1, \mathbb{Z})$ for $n \geq 14$), we obtain irreducible, hyperfinite subfactors with index 6 of the form $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$, whose relative fundamental group is trivial. This means that the subfactors $pR^{\mathbb{Z}_2}p \subset p(R \rtimes \mathbb{Z}_3)p$, where $p \in R^{\mathbb{Z}_2}$ is a projection of trace t, are mutually non-isomorphic as t runs through $(0, 1]$. They have of course all the same standard invariant.

Here is a more detailed description of the sections in this article. In section 2 we collect and prove several results on cocycle actions of discrete groups on II_1 factors. In particular we identify the reduction of a crossed product subfactor by a projection with a cocycle crossed product built on the reduced factor. We give several explicit examples of groups with non-cocycle conjugate (cocycle) actions on the hyperfinite II_1 factor. In section 3 we prove the main result (theorem 3.2). We show that if G is a discrete ICC group, generated by a finite abelian group H and a cyclic group K of prime order, with an outer and ergodic action α on the hyperfinite II_1 factor R, then the relative fundamental group of the subfactor $R^H \subset R \rtimes_\alpha K$ is contained in the fundamental group of the action α. As a corollary we obtain numerous 1-parameter families of irreducible, hyperfinite subfactors of index 6 which have all the same standard invariant.

2. Preliminaries

For the convenience of the reader we collect in this section several results on crossed products by cocycle actions. Most of these are well-known to experts.

Definition 2.1. (Cocycle Actions). Let G be a discrete group and M a II_1 factor. Let $\mathrm{Aut}(M), \mathcal{U}(M)$ denote the automorphism group, respectively the unitary group of M. A *cocycle action* α of G on M is a map $\alpha : G \to \mathrm{Aut}(M)$ such that there exists a map $v : G \times G \to \mathcal{U}(M)$, with the properties:

(i) $\alpha_e = \mathrm{id}$ and $\alpha_g \alpha_h = \mathrm{Ad}\, v_{g,h} \alpha_{gh}$, for all $g,\ h \in G$,

(ii) $v_{g,h} v_{gh,k} = \alpha_g(v_{h,k}) v_{g,hk}$, for all $g,\ h,\ k \in G$.

The map v is called a *2-cocycle* for α. v is *normalized* if $v_{g,e} = v_{e,g} = 1$ for all $g \in G$, where e denotes the identity of G. Any 2-cocycle v can be normalized by replacing it, if necessary, by $v'_{g,h} = v^*_{e,e} v_{g,h}$, $g,\ h \in G$ (note that $v_{e,e}$ is a scalar since M is a factor).

All 2-cocycles considered from now on will be normalized. All (cocycle) actions considered in this paper will be assumed *properly outer*, i.e. α_g cannot be implemented by unitary elements in M, for all $g \neq e$. Also, we will usually denote a cocycle action as a pair (α, v).

The next lemma shows that the cocycle v is unique up to a perturbation by a *scalar 2-cocycle* μ.

Lemma 2.1. *If v, v' are normalized 2-cocycles for the cocycle action α of G on M then $v = \mu v'$ for some normalized scalar 2-cocycle μ (i.e. $\mu : G \times G \to \mathbb{T}$ satisfying $\mu_{e,e} = 1$ and $\mu_{g,h}\mu_{gh,k} = \mu_{h,k}\mu_{g,hk}$, for all $g,\ h,\ k \in G$).*

Proof. $\mathrm{Ad}\, v_{g,h} = \mathrm{Ad}\, v'_{g,h}$, for all $g,\ h \in G$ implies $v^*_{g,h} v'_{g,h} \in \mathcal{Z}(M) = \mathbb{C}$, so there exists $\mu : G \times G \to \mathbb{T}$ such that $v = \mu v'$. Since v, v' are normalized we have $\mu_{e,e} = 1$. Using $v' = \mu v$ in the relation $v'_{g,h} v'_{gh,k} = \alpha_g(v'_{h,k}) v'_{g,hk}$ it follows

$$\mu_{g,h}\mu_{gh,k} v_{g,h} v_{gh,k} = \mu_{h,k}\mu_{g,hk} \alpha_g(v_{h,k}) v_{g,hk}$$

so μ satisfies the 2-cocycle relation $\mu_{g,h}\mu_{gh,k} = \mu_{h,k}\mu_{g,hk}$, for all $g,\ h,\ k \in G$. □

A 2-cocycle v for the action α is called a *coboundary* (or a *trivial cocycle*) if there exists a map $w : G \to \mathcal{U}(M)$ such that $w_e = 1$ and $v_{g,h} = \alpha_g(w^*_h) w^*_g w_{gh}$, for all g, $h \in G$.

Definition 2.2. (Conjugacy of actions). We say that two cocycle actions (α_1, v^1), (α_2, v^2) of the groups G_1 resp. G_2 on the II_1 factors M_1, M_2 are *cocycle conjugate* if there exists a *-isomorphism $\Phi : M_1 \to M_2$ (onto), a group isomorphism $\gamma : G_1 \to G_2$ and $w_g \in \mathcal{U}(M_2)$ such that:

(i) $\Phi \alpha^1_g \Phi^{-1} = \mathrm{Ad}\, w_g \circ \alpha^2_{\gamma(g)}$, for all $g \in G_1$,

(ii) $\Phi(v^1_{g,h}) = w_g \alpha^2_{\gamma(g)}(w_h) v^2_{\gamma(g),\gamma(h)} w^*_{gh}$, for all $g,\ h \in G_1$.

The cocycle actions (α_1, v^1), (α_2, v^2) of G_1 resp. G_2 are said to be *outer conjugate* (or *weakly cocycle conjugate*) if condition (i) holds. If α_1, α_2 are properly outer, (i) is equivalent to saying that the images of G_i under α^i in $\text{Out}(M_i) \stackrel{\text{def}}{=} \text{Aut}(M_i)/\text{Int}(M_i)$, $i = 1, 2$, are conjugate by a *-isomorphism $\Phi : M_1 \to M_2$.

Indeed, if $\Phi\alpha^1(G_1)\Phi^{-1} = \alpha^2(G_2)$ in $\text{Out}(M_2)$, there exists a bijection $\gamma : G_1 \to G_2$ and unitaries $w_g \in \mathcal{U}(M_2)$, such that $\Phi\alpha^1_g\Phi^{-1} = \text{Ad}\, w_g \circ \alpha^2_{\gamma(g)}$, for all $g \in G_1$. Since $g \to \Phi\alpha^1_g\Phi^{-1} = \alpha^2_{\gamma(g)} \in \text{Out}(M_2)$ is a group morphism and α^2 is properly outer, it follows that γ is a group morphism.

The cocycle actions α_1, α_2 are called *conjugate* if both conditions (i), (ii) are satisfied with $w = 1$.

Jones proved that any two outer actions of a finite group on the hyperfinite II_1 factor R are conjugate ([10]). In fact, any two outer actions of an amenable group on R are cocycle conjugate ([18]). The situation is very different when the group is not amenable. Any non-amenable group has at least two outer actions on R which are not outer conjugate ([11]). If the group is rigid, a much stronger result is true. Following [25] we call a group G *weakly rigid* (or *w-rigid*) if it has an infinite normal subgroup such that the pair (G, H) has the Kazhdan-Margulis relative property (T) ([14], [16]). It is shown in [25] that if G is w-rigid, then there exists a continuous family of non-outer conjugate cocycle actions of G on R. We will use this fact in the next section.

The next lemma shows that perturbing a cocycle action α by unitaries of M gives a cocycle action that is cocycle conjugate to α.

Lemma 2.2. *Let (α, v) be a cocycle action of G on M and let w_g be unitaries in M for all $g \in G$. Then $\beta_g = Ad\, w_g\alpha_g$ is a cocycle action of G on M with cocycle v', where $v'_{g,h} = w_g\alpha_g(w_h)v_{g,h}w^*_{gh}$ for all g, $h \in G$. Note that β is (trivially) cocycle conjugate to α.*

Proof. We show that (β, v') satisfies conditions 2.1 (i) and 2.1 (ii). We compute

$$\begin{aligned}
\beta_g\beta_h &= \text{Ad}\, w_g\alpha_g \text{Ad}\, w_h\alpha_h \\
&= \text{Ad}\,(w_g\alpha_g(w_h))\alpha_g\alpha_h \\
&= \text{Ad}\,(w_g\alpha_g(w_h))\text{Ad}\, v_{g,h}\alpha_{gh} \\
&= \text{Ad}\,(w_g\alpha_g(w_h)v_{g,h}w^*_{gh})\text{Ad}\, w_{gh}\alpha_{gh} \\
&= \text{Ad}\,(v'_{g,h})\beta_{gh}
\end{aligned}$$

for all g, $h \in G$, which proves 2.1 (i). We check 2.1 (ii).

$$\begin{aligned}
\beta_g(v'_{h,k})v'_{g,hk} &= (\text{Ad}\, w_g\alpha_g)(v'_{h,k})v'_{g,hk} \\
&= w_g\alpha_g(w_h\alpha_h(w_k)v_{h,k}w^*_{hk})w^*_g w_g\alpha_g(w_{hk})v_{g,hk}w^*_{ghk} \\
&= w_g\alpha_g(w_h)\text{Ad}\, v_{g,h}(\alpha_{gh}(w_k))\alpha_g(v_{h,k})\alpha_g(w^*_{hk})\alpha_g(w_{hk})v_{g,hk}w^*_{ghk} \\
&= w_g\alpha_g(w_h)v_{g,h}\alpha_{gh}(w_k)(v^*_{g,h}\alpha_g(v_{h,k})v_{g,hk})w^*_{ghk} \\
&= w_g\alpha_g(w_h)v_{g,h}w^*_{gh}w_{gh}\alpha_{gh}(w_k)v_{gh,k}w^*_{ghk} \\
&= v'_{g,h}v'_{gh,k}.
\end{aligned}$$

□

The next lemma shows that if two cocycle actions (α^1, v^1), (α^2, v^2) are outer conjugate then there exists a scalar 2-cocycle μ such that (α^1, v^1), $(\alpha^2, \mu v^2)$ are cocycle conjugate.

Lemma 2.3. *If the cocycle actions (α_1, v^1), (α_2, v^2) of G_1, G_2 on M_1, M_2 are outer conjugate by $\Phi : M_1 \to M_2$, then there exists a group isomorphism $\gamma : G_1 \to G_2$, unitaries $w_g \in \mathcal{U}(M_2)$, for all $g \in G_1$ and a scalar 2-cocycle $\mu : G_2 \times G_2 \to \mathbb{T}$ such that*

(i) $\Phi\alpha^1_g\Phi^{-1} = Ad\, w_g \circ \alpha^2_{\gamma(g)}$, for all $g \in G_1$.

(ii) $\Phi(v^1_{g,h}) = \mu_{\gamma(g),\gamma(h)} w_g \alpha^2_{\gamma(g)}(w_h) v^2_{\gamma(g),\gamma(h)} w^*_{gh}$, for all g, $h \in G_1$.

Proof. Since α^1, α^2 are outer conjugate, there exist $w_g \in \mathcal{U}(M_2)$, $g \in G_1$, and an isomorphism $\gamma : G_1 \to G_2$ such that $\Phi\alpha^1_g\Phi^{-1} = \mathrm{Ad}\, w_g\ \alpha^2_{\gamma(g)}$, for all $g \in G_1$.

Since v^1 is a 2-cocycle for α^1, $\Phi(v^1)$ is a 2-cocycle for the cocycle action $\Phi\alpha^1\Phi^{-1}$. On the other hand, lemma 2.2 implies that $v'_{g,h} = w_g\alpha^2_{\gamma(g)}(w_h)v^2_{\gamma(g),\gamma(h)}w^*_{gh}$ is a 2-cocycle for the cocycle action $g \to \mathrm{Ad}\, w_g \circ \alpha^2_{\gamma(g)} = \Phi\alpha^1_g\Phi^{-1}$.

Since by lemma 2.1 any two 2-cocycles of the same cocycle action differ by a scalar 2-cocycle, there exists μ such that $\Phi(v^1_{g,h}) = \mu_{\gamma(g),\gamma(h)} w_g \alpha^2_{\gamma(g)}(w_h) v^2_{\gamma(g),\gamma(h)} w^*_{gh}$, for all g, $h \in G_1$. □

Definition 2.3. (Crossed Products by Cocycle Actions). Let M be a II_1 factor and τ its unique normalized faithful trace. Let (α, v) be a cocycle action of the discrete group G on M.

The *crossed product algebra* $(M \rtimes_{\alpha,v} G, \tau)$ is defined as the von Neumann subalgebra of $B(l^2(G, L^2(M,\tau)))$ generated by unitaries $u_g \in B(l^2(G, L^2(M,\tau)))$, $g \in G$, where $u_g(f)(h) = v_{g,g^{-1}h}f(g^{-1}h)$, for all $f \in l^2(G, L^2(M,\tau))$, g, $h \in G$, and by a copy of the algebra M given by $(x \cdot f)(g) = \alpha_g^{-1}(x)f(g)$, for all $x \in M$, $f \in l^2(G, L^2(M,\tau))$, $g \in G$. In this paper we will most of the time drop the cocycle from the notation and simply write $(M \rtimes_\alpha G, \tau)$. The formula $\tau(X) = \langle X\delta_e, \delta_e\rangle$, for all $X \in M \rtimes_\alpha G$, where $\delta_e \in l^2(G, L^2(M,\tau))$ is the $L^2(M,\tau)$-valued function on G that takes value 1 at e and 0 elsewhere, defines a trace τ on $M \rtimes_\alpha G$. See for instance [29], [30] for more details.

Alternatively, $(M \rtimes_\alpha G, \tau)$ can be viewed in the following way: Consider the Hilbert algebra $\mathcal{M}$ of finite formal sums $\mathcal{M} = \{\sum_{g\in G} x_g u_g, x_g \in M\}$, with multiplication rules

$$u_g u_h = v_{g,h} u_{gh}, \quad u_g x = \alpha_g(x) u_g, \quad x = x u_e = 1x,$$

for all g, $h \in G$, $x \in M$, and $*$-operation $(u_g x)^* = u_{g^{-1}}\alpha_g(x^*)$. The trace is given by $\tau(\sum_{g\in G} x_g u_g) = \tau(x_e)$. Then $M \rtimes_\alpha G$ is defined as the closure of $\mathcal{M}$ in norm $\| \|_{2,\tau}$ on bounded sequences.

$(M \rtimes_\alpha G, \tau)$ is a finite von Neumann algebra with normal faitful trace τ. If the cocycle action α is outer then $M' \cap M \rtimes_\alpha G = \mathbb{C}$. In particular, if α is outer then $(M \rtimes_\alpha G, \tau)$ is a II_1 factor.

For the convenience of the reader we include a proof of the well-known result that the isomorphism class of the *inclusion* $(M \subset M \rtimes_{\alpha,v} G)$ is determined by the cocycle conjugacy class of the (cocycle) action (α, v) ([10]).

Proposition 2.1. *Let α^1, α^2 be cocycle actions of the discrete groups G_1, G_2 on the II_1 factors M_1, M_2, with 2-cocycles v^1, v^2. If there exists a surjective $*$-isomorphism $\Phi : M_1 \rtimes_{\alpha^1} G_1 \to M_2 \rtimes_{\alpha^2} G_2$ such that $\Phi(M_1) = M_2$, then α^1 and α^2 are cocycle conjugate. More precisely, there exists a group isomorphism $\gamma : G_1 \to G_2$, and unitaries $w_g \in \mathcal{U}(M_2)$, for all $g \in G_1$, such that:*

(i) $\Phi\alpha^1_g\Phi^{-1} = Ad\, w_g\ \alpha^2_{\gamma(g)}$, for all $g \in G_1$,

(ii) $\Phi(v^1_{g,h}) = w_g\alpha^2_{\gamma(g)}(w_h)v^2_{\gamma(g),\gamma(h)}w^*_{gh}$, for all $g, h \in G_1$.

Conversely, if $\Phi : M_1 \to M_2$ is a $*$-isomorphism (onto), $\gamma : G_1 \to G_2$ is a group isomorphism, and there exist unitaries $w_g \in \mathcal{U}(M_2)$ for all $g \in G_1$ such that (i), (ii) are satisfied, then Φ can be extended to an isomorphism $M_1 \rtimes_{\alpha^1} G_1 \simeq M_2 \rtimes_{\alpha^2} G_2$ (hence Φ is an isomorphism of the associated inclusions).

Proof. For $i = 1, 2$ let u^i_g denote the unitaries implementing the action α^i on M_i, i.e. $\alpha^i_g = \mathrm{Ad}\, u^i_g, u^i_g u^i_h = v^i_{g,h}u^i_{gh}$, for all g, $h \in G$. Let $\mathcal{N}_{M_1 \rtimes_{\alpha^1} G_1}(M_1) = \{u \in \mathcal{U}(M_1 \rtimes_{\alpha^1} G_1) | uM_1u^* = M_1\}$ be the normalizer of M_1 in $M_1 \rtimes_{\alpha^1} G_1$.

There exists an isomorphism $\mathcal{N}_{M_1 \rtimes_{\alpha^1} G_1}(M_1)/\mathcal{U}(M_1) \simeq G_1$ taking u^1_g to g, and similarly $\mathcal{N}_{M_2 \rtimes_{\alpha^2} G_2}(M_2)/\mathcal{U}(M_2) \simeq G_2$. Since Φ induces an isomorphism from $\mathcal{N}_{M_1 \rtimes_{\alpha^1} G_1}(M_1)$ to $\mathcal{N}_{M_2 \rtimes_{\alpha^2} G_2}(M_2)$, there exists a group isomorphism $\gamma : G_1 \to G_2$ such that

$$\Phi(u^1_g) = u^2_{\gamma(g)}(\mathrm{mod}\ \mathcal{U}(M_2))$$

for all $g \in G_1$.

Thus $\Phi(u^1_g) = w_g u^2_{\gamma(g)}$, for some unitaries $w_g \in \mathcal{U}(M_2)$, $g \in G_1$. So $\Phi\alpha^1_g\Phi^{-1} = \mathrm{Ad}\,\Phi(u^1_g) = \mathrm{Ad}\,(w_g u^2_{\gamma(g)}) = \mathrm{Ad}\, w_g\ \alpha^2_{\gamma(g)}$, for all $g \in G_1$, which proves (i).

Let g, $h \in G_1$. Then $\Phi(v^1_{g,h}) = \Phi(u^1_g)\Phi(u^1_h)\Phi(u^1_{gh})^* = w_g u^2_{\gamma(g)} w_h u^2_{\gamma(h)}(u^2_{\gamma(gh)})^*$ $w^*_{gh} = w_g(\mathrm{Ad}\, u^2_{\gamma(g)})(w_h)u^2_{\gamma(g)}u^2_{\gamma(h)}(u^2_{\gamma(gh)})^* w^*_{gh} = w_g\alpha^2_{\gamma(g)}(w_h)v^2_{\gamma(g),\gamma(h)}w^*_{gh}$, which proves (ii).

The converse follows easily by noticing that $\pi : L^2(M_1 \rtimes_{\alpha^1} G_1) \simeq L^2(M_2 \rtimes_{\alpha^2} G_2)$, defined by $\pi(\sum_{g\in G} x_g u^1_g) = \sum_{g\in G} \Phi(x_g)w_g u^2_{\gamma(g)}$, for all $x_g \in M_1, g \in G$, is a Hilbert space isomorphism intertwining the $M_1 \rtimes_{\alpha^1} G_1$ resp. $M_2 \rtimes_{\alpha^2} G_2$-actions (and hence the M_1 and M_2-actions) on $L^2(M \rtimes_{\alpha^1} G_1)$ resp. $L^2(M \rtimes_{\alpha^2} G_2)$. □

Recall the simple fact that if the subfactors $N \subset M$ and $\tilde{N} \subset \tilde{M}$ are isomorphic, then the basic constructions M_1 and $\tilde{M}_1$ are isomorphic as well (see e.g. [13]).

Corollary 2.1. *Let (α^1, v^1), (α^2, v^2) be cocycle actions of the finite groups G_1, G_2 on the II_1 factors M_1, M_2. For $i = 1, 2$ let $M_i^{G_i} = \{x \in M_i | \alpha^i_g(x) = x$, for all $g \in G_i\}$. If there exists an isomorphism of inclusions*

$$\Phi : (M_1^{G_1} \subset M_1) \to (M_2^{G_2} \subset M_2)$$

then α^1 *and* α^2 *are outer conjugate via* Φ.

Proof. Since $M_i^{G_i} \subset M_i \subset M_i \rtimes_{\alpha^i} G_i$, $i = 1, 2$, is the basic construction ([13]), the isomorphism Φ can be extended to $M_1 \rtimes_{\alpha^1} G_1 \simeq M_2 \rtimes_{\alpha^2} G_2$. Proposition 2.1 implies then that α^1 and α^2 are outer conjugate via Φ. □

The next lemma shows that the "restriction" of an action of a group G on M to the reduced algebra pMp, p a projection in M, gives rise to a cocycle action of G on pMp.

Lemma 2.4. *Let G be a discrete group, α an action of G on the II_1 factor M, and p a non-zero projection in M. Then there exists a cocycle action β of G on pMp such that:*

$$(pMp \subset p(M \rtimes_\alpha G)p) \simeq (pMp \subset pMp \rtimes_\beta G)$$

Proof. Let τ be the unique faithful normalized trace of M. Since $\tau(\alpha_g(p)) = \tau(p)$, for all $g \in G$, there exist unitaries $w_g \in \mathcal{U}(M)$ such that $\alpha_g(p) = w_g^* p w_g$, for all $g \in G$. Let $\beta(g) = \mathrm{Ad}\, w_g \alpha_g$. Denote by u_g the unitary that implements α_g on M, for all $g \in G$. Thus β_g is implemented by $w_g u_g$. By lemma 1.4, β is a cocycle action of G on M with cocycle $v_{g,h} = w_g \alpha_g(w_h) w_{gh}^*$. Note that this is of course a coboundary for the G-action on M, but it may not be a coboundary when restricted to pMp.

We show that β is a cocycle action of G on pMp with cocycle vp. β_g is an automorphism of pMp, since $\beta_g(p) = w_g \alpha_g(p) w_g^* = p$, for all $g \in G$. Applying to p the relation $\beta_g \beta_h = \mathrm{Ad}\, v_{g,h} \beta_{gh}$ implies that $p = \mathrm{Ad}\, v_{g,h}(p)$. Thus p commutes with $v_{g,h}$, for all $g, h \in G$, so $v_{g,h}p$ is a cocycle for the cocycle action β restricted to pMp. We have $v_{g,h} p v_{gh,k} p = \beta_g(v_{h,k}p) v_{g,hk} p$, for all g, h, $k \in G$.

Since α, β are cocycle conjugate, the inclusions $(M \subset M \rtimes_\alpha G)$ and $(M \subset M \rtimes_\beta G)$ are isomorphic, through an isomorphism that can be assumed to be the identity on M (lemma 2.2, proposition 2.1). Hence we can identify $M \rtimes_\alpha G$ with $M \rtimes_\beta G$ (as von Neumann algebras), the unitaries implementing β being identified with $w_g u_g \in M \rtimes_\alpha G$. The inclusions $(pMp \subset p(M \rtimes_\alpha G)p)$ and $(pMp \subset p(M \rtimes_\beta G)p)$ are isomorphic, so we only have to show that $p(M \rtimes_\beta G)p = pMp \rtimes_\beta G$ (we identify the abstract crossed product $pMp \rtimes_\beta G$ with its realization inside $M \rtimes_\beta G$).

The von Neumann algebra $p(M \rtimes_\beta G)p$ is generated by $pxu_g p$ $(x \in M, g \in G)$. Since $pxu_g p = (pxw_g^* p)(w_g u_g)$, it follows that $(pxu_g p)_{x \in M, g \in G}$ generate $pMp \rtimes_\beta G$ as a von Neumann algebra, so $p(M \rtimes_\beta G)p = pMp \rtimes_\beta G$. □

Remark 2.1. It is easy to see that the cocycle conjugacy class of the cocycle action β on pMp depends only on $t = \tau(p)$ ([25]). We will denote the cocycle conjugacy class of (β, v, pMp) by (α^t, v^t, M^t), and call it *the amplification of* α *by* t. For values of t greater than 1, define α^t to be the t/n-amplification of the action $id \otimes \alpha$ of G on $M_n(\mathbb{C}) \otimes M$, for some $n \geq t$. Note that α^t is a properly outer cocycle action when α is properly outer ([25]).

Definition 2.4. ([25]). Let G be a discrete group with (cocycle) action α on the II_1 factor M. The fundamental group of the action α is

$$\mathcal{F}(\alpha) = \{t > 0 \,|\, \alpha^t \text{ is outer conjugate to } \alpha\}.$$

Similarly we define

$$\mathcal{F}^c(\alpha) = \{t > 0 \,|\, \alpha^t \text{ is cocycle conjugate to } \alpha\}.$$

$\mathcal{F}(\alpha)$ is an outer conjugacy invariant of α, and $\mathcal{F}^c(\alpha)$ is a cocycle conjugacy invariant of α. Note that $\mathcal{F}^c(\alpha) \subset \mathcal{F}(\alpha)$ (see definition 2.2).

Let G be an infinite discrete group, τ_0 the normalized trace on $M_2(\mathbb{C})$, and let $(R, \tau) = \overline{\otimes_{g\in G}(M_2(\mathbb{C}), \tau_0)}^w$ be a copy of the hyperfinite II_1 factor. The *(non-commutative) Bernoulli G-action* on R is the action $\sigma : G \to \mathrm{Aut}(R)$ defined as $\sigma_g(\otimes_{h\in G} x_h) = \otimes_{h\in G} x'_h$, where $x'_h = x_{g^{-1}h}$, and $\{x_h\}_{h\in G}$ is such that all but finitely many x_h are equal to 1. It is easy to see (and well-known) that σ is a properly outer, ergodic action.

The following rigidity theorem from [25] will provide the main examples to which we will apply our construction in the next section.

Theorem 2.1. *Let G be a w-rigid group and σ the Bernoulli G-action on R. Then $\mathcal{F}(\sigma) = \{1\}$.*

More generally, any of the Connes-Størmer Bernoulli G-actions with countable spectrum considered in [25] have countable fundamental group.

We will recall next the notion of *relative fundamental group*. Let $N \subset M$ be an inclusion of II_1 factors. The *relative fundamental group* $\mathcal{F}(N \subset M)$ is defined as

$$\mathcal{F}(N \subset M) = \{t > 0 \,|\, (N \subset M)^t \text{ is isomorphic to } N \subset M\}$$

$(N \subset M)^t$ denotes as usual the t-amplification of $N \subset M$ (see [20], [21]). Observe that $\mathcal{F}(N \subset M)$ is clearly a multiplicative subgroup of $\mathbb{R}_+^*$. If $N \subset M$ is stable, i.e. splits a common copy of the hyperfinite II_1 factor R, then clearly $\mathcal{F}(N \subset M) = \mathbb{R}_+^*$. This happens for instance if the subfactor $N \subset M$ is constructed from an initial commuting square by iterating the basic construction. See [2], [3] for more on this.

If $\mathcal{F}(N \subset M)$ is at most countable, then at most countably many of the inclusions $pNp \subset pMp$, where p runs through the set of inequivalent projections in N, are isomorphic (as inclusions). Observe that the subfactors $pNp \subset pMp$ have all the same standard invariant.

Note that if G is a discrete group with cocylc action (α, v) on the II_1 factor M, then $\mathcal{F}^c(\alpha) = \mathcal{F}(M \subset M \rtimes_{\alpha,v} G)$ (this follows simply from the definitions).

The following rigidity result ([26], [17]) provides further examples of actions having a fundamental group which is at most countable.

Theorem 2.2. *Let M be a separable II_1 factor. Assume there exists a diffuse von Neumann subalgebra B such that $B \subset M$ is a rigid inclusion (in the sense of* [26]*) and $B' \cap M \subset B$. Then the fundamental group of M is at most countable.*

Remark 2.2. Let G be a countable discrete ICC (infinite conjugacy classes) group with property (T) and let α be an outer and ergodic action of G on the hyperfinite II_1 factor R. For instance, let α be the Bernoulli G-action on R described above. Let $M = R \rtimes_\alpha G$ and let u_g, $g \in G$, be the unitaries in M implementing α. Set $B = \{u_g \,|\, g \in G\}''$, and note that $B \simeq L(G)$. Then $B \subset M$ is rigid and $L(G)' \cap M = R^G = \mathbb{C}$. Thus, by the above theorem, we deduce that the fundamental group of $R \rtimes_\alpha G$ is at most countable. More generally, the same is true if G is w-rigid and the action α is mixing. In particular, the relative fundamental group $\mathcal{F}(R \subset R \rtimes_\alpha G) = \mathcal{F}^c(\alpha)$ is at most countable. If in addition $H^2(G, \mathbb{T})$ is countable, the next lemma implies that $\mathcal{F}(\alpha)$ is at most countable.

Lemma 2.5. *Let G be a discrete group with countable second cohomology group $H^2(G, \mathbb{T})$. Let M be a II_1 factor, $I \subset \mathbb{R}$ an uncountable set and $(\alpha^i, v^i)_{i \in I}$ non cocycle conjugate cocycle actions of G on M. Then $(\alpha^i, v^i)_{i \in I}$ are non outer conjugate modulo a countable set, i.e. $I(i_0) = \{i \in I, (\alpha^i, v^i)$ outer conjugate to $(\alpha^{i_0}, v^{i_0})\}$ is at most countable for each $i_0 \in I$.*

In particular, given uncountably many conjugate actions of G on M, uncountably many of these are actually cocycle conjugate actions.

Proof. Assume by contradiction that $I(i_0)$ is uncountable for some i_0. According to lemma 2.1, for every i there exists μ^i scalar 2-cocycle such that the actions (α^i, v^i) and $(\alpha^{i_0}, \mu_i v^{i_0})$ are cocycle conjugate. Since $H^2(G, \mathbb{T})$ is countable and $I(i_0)$ is uncountable, there exist $j_1, j_2 \in I(i_0)$ such that $\mu^{j_1} \bar{\mu^{j_2}}$ is a coboundary. But (α^{j_1}, v^{j_1}) is cocycle conjugate to $(\alpha^{i_0}, \mu^{j_1} v^{i_0})$, which is cocycle conjugate to $(\alpha^{j_2}, \mu^{j_1} \bar{\mu^{j_2}} v^{j_2})$. Thus α^{j_1} and α^{j_2} are cocycle conjugate, which is a contradiction.□

3. The Construction

We consider in this section the class of subfactors introduced in [5]. Let M be a II_1 factor and let G be a countable discrete group with an outer action α on M. Suppose $G = \langle H, K \rangle$ is generated by two finite groups H and K. The subfactor $M^H \subset M \rtimes_\alpha K$ has index $|H| \cdot |K|$ and is irreducible if and only if $H \cap K = \{e\}$, where e denotes the identity in G. Note that we could start with a cocycle action of G and M. By [29], [30] we can modify the induced cocycle actions of the *finite* groups H and K to actual actions.

It is shown in [5], [6] that many analytical and algebraic properties of the subfactor $M^H \subset M \rtimes K$ are reflected by properties of the group G. For instance, the following result is shown in [6].

Theorem 3.1. *Let H and K be two finite groups with outer actions σ resp. ρ on the II_1 factor M. Then the standard invariant of $M^H \subset M \rtimes K$ has property (T) (*[24]*)*

if and only if the group G generated by $\sigma(H)$ and $\rho(K)$ in the outer autmorphism group of M has Kazhdan's property (T).

We will see below that the next theorem can be used to construct continuous families of non-isomorphic, irreducible, finite index subfactors of the hyperfinite II_1 factor all having the same standard invariant. The construction can be carried out in such a way that this standard invariant will have property (T).

The main result of this article is the following theorem.

Theorem 3.2. *Let H be a finite abelian group and let $K = \mathbb{Z}_q$ be a cyclic group, where q is a prime number. Let $G = \langle H, K \rangle$ be an infinite ICC group generated by H and K. Let α be a properly outer and ergodic action of G on R.*

Then $\mathcal{F}(R^H \subset R \rtimes_\alpha K) \subseteq \mathcal{F}(\alpha)$. Hence, if $\mathcal{F}(\alpha)$ is countable (resp. trivial), one obtains uncountably many (resp. a 1-parameter family of) irreducible subfactors of the hyperfinite II_1 factor R, which are non-isomorphic, but have all the same standard invariant.

Before we prove this theorem, let us give several examples of groups and actions satisfying the hypothesis.

If G has property (T) with $H^2(G, \mathbb{T})$ at most countable, and α is any properly outer and ergodic action of G on R, then we established that $\mathcal{F}(\alpha)$ is at most countable in remark 2.2 and lemma 2.5. For instance, the groups $G_n = SL(2n+1, \mathbb{Z})$ have Kazhdan's property (T) by [14] and are ICC (see also [9]). They are (2,3)-generated for $n \geq 14$ by [31] (see also [32]), i.e. G_n is a quotient of the free product of $H = \mathbb{Z}_2$ and $K = \mathbb{Z}_3$ (this free product is of course just $PSL(2, \mathbb{Z})$). It follows from results of Steinberg ([27], [28], see also [15]) that the second cohomology group $H^2(SL(n, \mathbb{Z}), \mathbb{T})$ is a finite group (in fact it is equal to $\mathbb{Z}_2$) for $n \geq 5$. These groups provide therefore (countably many) examples of groups satisfying the hypothesis of our theorem. We would like to thank Marsden Conder for pointing out reference [32] and Pierre de la Harpe for the references [27], [28], [15].

Recall that, by theorem 2.1, if G is any w-rigid group and σ the Bernoulli G-action on R, then $\mathcal{F}(\sigma) = \{1\}$. Thus if $G = SL(2n+1, \mathbb{Z}), n \geq 14$, and $\alpha = \sigma$, we obtain one-parameter families of non-isomorphic, irreducible, index 6 hyperfinite subfactors having the same standard invariant. This standard invariant has property (T) (theorem 3.1).

A much larger class of examples can be obtained as follows: Let G_1 be the free product of any finite abelian group and a cyclic group of prime order. If $G_1 \neq \mathbb{Z}_2 * \mathbb{Z}_2$ then G_1 is a hyperbolic group. Let G_2 be any hyperbolic property (T) group. By results of Olshanskii (see for instance [19], [1]), there exists an infinite hyperbolic group G which is a common quotient of G_1 and G_2. In particular, G has property (T) and is generated by a finite abelian subgroup and a subgroup of prime order. Note that G is ICC, since any non-elementary hyperbolic group is ICC. Thus, G together with the Bernoulli G-action (or more generally any of the Connes-Størmer Bernoulli G-actions with countable spectrum considered in [25]) satisfies the hypothesis of our theorem. We would like to thank Mark Sapir for pointing out this class of examples.

Note that similar results as in theorem 3.2 can be obtained by using the rigidity results in [20] rather than theorems 2.1 and 2.2 quoted above.

We proceed with the proof of theorem 3.2. We start with some lemmas.

Lemma 3.1. *Let H be a finite abelian group, let $K = \mathbb{Z}_q$ be a cyclic group, where q is a prime number. Suppose that $G = \langle H, \mathbb{Z}_q \rangle \neq H \cdot \mathbb{Z}_q$. Let α be an outer cocycle action of G on the hyperfinite II_1 factor R and let Q be the von Neumann algebra generated by the normalizer of R^H in $R \rtimes_\alpha K$ (notation: $Q = \mathcal{N}_{R \rtimes K}(R^H)''$). Then $Q = R$.*

Proof. We have by definition that $Q = \{u \in \mathcal{U}(R \rtimes K) \,|\, uR^H u^* = R^H\}''$. Since H is abelian, we conclude that $(R^H \subset R) \cong (R_0 \subset R_0 \rtimes H)$, for some $R_0 \cong R$. Hence $R = \mathcal{N}_R(R^H)''$ and we obtain therefore the chain of inclusions $R^H \subset R \subset Q \subset R \rtimes K$. Since $R \subset R \rtimes K$ has no intermediate subfactors by [Bi3, Theorem 3.2] we must have either $Q = R$ or $Q = R \rtimes K$. If $Q = R \rtimes K$, then $(R^H \subset R \rtimes K) \cong (R \subset R \rtimes (\mathcal{N}(R^H)/\mathcal{U}(R^H)))$ ([10], [12]). Hence $R^H \subset R \rtimes K$ would have depth 2, contradicting the fact that $G \neq H \cdot \mathbb{Z}_q$ ([5]). Thus indeed $Q = R$. □

Corollary 3.1. *Let H be a finite abelian group and $K = \mathbb{Z}_q$, q a prime number. Suppose that $G = \langle H, K \rangle$ is an infinite group and let α^i be outer cocycle actions of G on the hyperfinite II_1 factor R_i, $i = 1, 2$. Suppose that there is a surjective $*$-isomorphism $\Phi : R_1 \rtimes_{\alpha^1} K \to R_2 \rtimes_{\alpha^2} K$ such that $\Phi(R_1^H) = R_2^H$. Then $\Phi(R_1) = R_2$. In particular we have $(R_1^H \subset R_1) \overset{\Phi}{\cong} (R_2^H \subset R_2)$ and $(R_1 \subset R_1 \rtimes_{\alpha^1} K) \overset{\Phi}{\cong} (R_2 \subset R_2 \rtimes_{\alpha^2} K)$.*

Proof. We have seen in lemma 3.1 that $\mathcal{N}_{R_i^H}(R_i \rtimes_{\alpha^i} K)'' = R_i$, $i = 1, 2$. But every (surjective) $*$-isomorphism takes normalizers to normalizers. □

Proposition 3.1. *Let H be a finite abelian group and $K = \mathbb{Z}_q$, q a prime number. Suppose that $G = \langle H, K \rangle$ is an infinite group and let α^i be outer cocycle actions of G on the hyperfinite II_1 factor R_i, $i = 1, 2$. Suppose that there is a surjective $*$-isomorphism $\Phi : R_1 \rtimes_{\alpha^1} K \to R_2 \rtimes_{\alpha^2} K$ such that $\Phi(R_1^H) = R_2^H$. Then the cocycle actions α^1, α^2 of G are outer conjugate by Φ.*

Proof. It follows from corollary 3.1 and proposition 2.1 that $\Phi\alpha^1(K)\Phi^{-1} = \alpha^2(K)$ in $\mathrm{Out}(R_2)$. From corollary 3.1 and corollary 2.1 we deduce that $\Phi\alpha^1(H)\Phi^{-1} = \alpha^2(H)$ in $\mathrm{Out}(R_2)$. Since K and H generate G, this implies $\Phi\alpha^1(G)\Phi^{-1} = \alpha^2(G)$ in $\mathrm{Out}(R_2)$. □

We give now the proof of theorem 3.2.

Proof. Let $G = \langle H, K \rangle$ be a quotient of the free product $H * K$ as in the theorem. Since G is infinite, we have $G \neq H \cdot K$. Since K is of prime order and $K \not\subset H$, it follows that $H \cap K = \{e\}$.

Let α be an outer and ergodic action of G on the hyperfinite II_1 factor R. Let $t \in \mathcal{F}(R^H \subset R \rtimes_\alpha K)$, $0 < t < 1$ (which is sufficient since $\mathcal{F}(R^H \subset R \rtimes_\alpha K)$ is a group). We will show that $t \in \mathcal{F}(\alpha)$.

Let p be a projection in R^H such that $\tau(p) = t$ (τ denotes as usual the normalized trace of R). Thus, the inclusions $R^H \subset R \rtimes_\alpha K$ and $pR^Hp \subset p(R \rtimes_\alpha K)p$ are isomorphic.

By lemma 2.4, there exists a cocycle action (β, v) of G on pRp and an isomorphism $\Phi : (pRp \subset p(R \rtimes_\alpha G)p) \simeq (pRp \subset pRp \rtimes_\beta G)$, which is the identity on pRp. Moreover, from the construction of Φ (see lemma 2.4) it follows that Φ takes $p(R \rtimes_\alpha K)p$ onto $pRp \rtimes_\beta K$. Since $p \in R^H$, the actions α, β coincide on H so the fixed point algebra R^H is the same for both actions. Hence Φ takes $pR^Hp \subset pRp \subset p(R \times_\alpha K)p$ onto $pR^Hp \subset pRp \subset pRp \rtimes_\beta K$. Since $\beta_g(p) = p$, for all $g \in G$, we have $pR^Hp = (pRp)^H$ as subalgebras of $pRp \rtimes_\beta G$. This yields

$$(R^H \subset R \rtimes_\alpha K) \simeq (pR^Hp \subset p(R \rtimes_\alpha K)p) \simeq ((pRp)^H \subset pRp \rtimes_\beta K)$$

Thus there exists an isomorphism $(R^H \subset R \rtimes_\alpha K) \simeq ((pRp)^H \subset pRp \rtimes_\beta K)$. Proposition 3.1 implies that the cocycle actions α, β of G are outer conjugate. Since β is cocycle conjugate to α^t, this implies $t \in \mathcal{F}(\alpha)$ which ends the proof. $\square$

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