On the structure of the moduli space of commuting squares around the Fourier spin model

Shuler Hopkins
Sewanee: The University of the South (sghopkin@sewanee.edu)
and
Remus Nicoară
University of Tennessee Knoxville (rnicoara@utk.edu)

Abstract

We investigate the structure of the moduli space of non-isomorphic commuting squares around the Fourier spin model commuting square, or equivalently the structure of the moduli space of normalized complex $n \times n$ Hadamard matrices around $\sqrt{n}F_n$, where F_n denotes the Fourier matrix. If a belongs to the enveloping tangent space to the real algebraic manifold of complex Hadamard matrices at $\sqrt{n}F_n$, we find new restrictions for a to be a direction of convergence (in the sense of [12],[14]) of a sequential deformation of $\sqrt{n}F_n$ by complex Hadamard matrices.

As an application, we show that if $n = 30$ then not every norm-one element a in the enveloping tangent space at $\sqrt{n}F_n$ corresponds to a sequential family of complex Hadamard matrices converging to $\sqrt{n}F_n$ in the direction of a. It follows that for $n = 30$ the dimension of any differentiable family of complex Hadamard matrices containing $\sqrt{n}F_n$ is strictly less than the dimension of the enveloping tangent space at $\sqrt{n}F_n$ (called the defect of F_n). This is particularly surprising considering that, for every n, there exist sufficiently many 1-dimensional analytic families of complex Hadamard matrices through $\sqrt{n}F_n$ to form a basis of tangents at $\sqrt{n}F_n$ in the enveloping tangent space (as shown in [14]).

1 Introduction

Commuting squares were introduced by S. Popa in [18], as invariants and construction data in V. Jones' theory of subfactors ([7],[6]). They encode the symmetries of a subfactor, and for some large classes of subfactors they are complete invariants ([18],[19]). In particular, any finite group G can be encoded in a group commuting square:

$$\mathfrak{C}_G = \begin{pmatrix} \mathcal{D}_n & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}$$

where $\mathcal{D}_n \simeq l^\infty(G)$ is the algebra of $n \times n$ diagonal matrices with complex entries, and $\mathbb{C}[G]$ denotes the group algebra of G. The subfactor associated to $\mathfrak{C}_G$ by iterating Jones' basic construction is a cross product subfactor, hence of depth 2. Moreover, if G is abelian then $\mathfrak{C}_G$ is a spin model commuting square, and the associated subfactor is a Hadamard subfactor in the sense of [11]. When $G = \mathbb{Z}_n$ we have $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$, where $F_n = \frac{1}{\sqrt{n}}(\exp \frac{2\pi i k l}{n})_{1 \leq k,l \leq n}$ is the Fourier matrix of size n. We refer to $\mathfrak{C}_{\mathbb{Z}_n}$ as the Fourier Commuting Square, or the Standard Spin Model commuting square.

In [10],[12] the second author initiated a study of the deformations of a commuting square, in the moduli space of non-isomorphic commuting squares. It was shown that if a commuting square satisfies a certain *span condition*, then it is isolated among all non-isomorphic commuting squares. In the case of $\mathfrak{C}_G$, the span condition is $V = \mathrm{M_n}(\mathbb{C})$, where V is the subspace of $\mathrm{M_n}(\mathbb{C})$ given by:

$$V = [\mathcal{D}_n, \mathbb{C}[G]] + \mathbb{C}[G] + \mathbb{C}[G]' + \mathcal{D}_n.$$

We used the notation $[\mathcal{D}_n, \mathbb{C}[G]] = \mathrm{span}\{du - ud : d \in \mathcal{D}_n, u \in \mathbb{C}[G]\}$.

When the span condition fails, the dimension $d'(G)$ of $V^\perp = \mathrm{M_n}(\mathbb{C}) \ominus \mathrm{V}$ can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by non-isomorphic commuting squares. In [13] we computed this dimension, called *the dephased defect of the group* G. We also studied the related quantity $d(G) = \dim_{\mathbb{C}}([\mathcal{D}_n, \mathbb{C}[G]]^\perp)$, called *the undephased defect of* G (or just the defect of G), which can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by (not necessarily non-isomorphic) commuting squares. The terminology 'defect' was chosen to coincide with the notion of defect introduced independently in [22] in the study of complex Hadamard matrices (see also [1]).

From the second author's previous work in [10],[13] it follows that the defect $d(G)$ is an upper bound for the number of one-parameter sequential deformations of $\mathfrak{C}(G)$, of linearly independent directions of convergence. In fact in [14] it was shown that this bound is always reached. More precisely, a basis $\mathfrak{B}$ of $[\mathcal{D}_n, \mathbb{C}[G]]^\perp$ was constructed, such that for each $a \in \mathfrak{B}$ there exists an analytic family $(\mathfrak{C}_t)_{t \in \mathbb{R}}$ of commuting squares

$$\mathfrak{C}_t = \begin{pmatrix} \mathcal{D}_n & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_t \mathbb{C}[G] U_t^* \end{pmatrix}$$

where U_t ($t \in \mathbb{R}$) are unitaries with $U_t \neq I$ for $t \neq 0$, $U_t \to U_0 = I$ as $t \to 0$, and $a = \lim_{t \to 0} \frac{U_t - I}{\mathrm{i}||\mathrm{U_t - I}||}$. We will refer to a as the *direction of convergence* of the family $(\mathfrak{C}_t)_{t \in \mathbb{R}}$. Note that the choice of the basis $\mathfrak{B}$ was crucial to this construction; it does not follow that every (hermitian, norm-one) $a \in [\mathcal{D}_n, \mathbb{C}[G]]^\perp$ is a direction of convergence of some analytic, or even sequential, deformation of $\mathfrak{C}_G$.

If we let $G = \mathbb{Z}_n$, we have $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$ and the commuting square condition for $\mathfrak{C}_t$ is equivalent to $\sqrt{n} U_t F_n$ being a complex Hadamard matrix (i.e., all its entries are of absolute value 1 and its rows are mutually orthogonal). Thus there exist $d(\mathbb{Z}_n)$ one-parameter analytic families of complex Hadamard matrices containing $\sqrt{n} F_n$, and of linearly independent directions of convergence.

Let $\mathcal{H}(\mathrm{n}) = \mathrm{M_n}(\mathbb{T}) \cap \sqrt{\mathrm{n}}\mathrm{U}(\mathrm{n})$ denote the real algebraic variety of $n \times n$ complex Hadamard matrices, where $\mathrm{U}(\mathrm{n}) \subset \mathrm{M_n}(\mathbb{C})$ denotes the set of unitary matrices and $\mathrm{M_n}(\mathbb{T})$ denotes the set of $n \times n$ matrices with entries of absolute value 1 . The defect $d(\mathbb{Z}_n)$, also called the defect of the Fourier matrix F_n, can be interpreted as the dimension of the enveloping tangent space of $\mathcal{H}(\mathrm{n})$ at the matrix $\sqrt{n}F_n$:

$$\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n}) = \mathrm{T}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathrm{M_n}(\mathbb{T}) \cap \mathrm{T}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\sqrt{\mathrm{n}}\mathrm{U}(\mathrm{n})$$

(see [22], [1], [2]). Thus the defect can be regarded as an upper bound for the dimension of the tangent space to $\mathcal{H}(n)$ at the point $\sqrt{n}F_n$, and the construction mentioned above shows that this bound is reached (see [14] for details). This is quite surprising, considering that the manifold $\mathcal{H}(n)$ is not smooth or connected for general n. Note that the general structure of $\mathcal{H}(n)$ is not known, even for n as small as 6.

In this paper we shed more light on the structure of the moduli space of non-isomorphic commuting squares (or equivalently, of non-equivalent complex Hadamard matrices) around the Fourier commuting square. We introduce third order necessary conditions for a unit vector a in the enveloping tangent space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ to be the direction of convergence of a sequential family of non-equivalent Hadamard matrices converging to $\sqrt{n}F_n$. The second author found in [12] first and second order conditions on a for general commuting squares, then showed in [15] that the second order conditions are always implied by the first order conditions in the case of the Fourier spin model, i.e. they are true for any norm-one a in $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$. We now introduce third order conditions on a, which turn out to be much more restrictive.

As an application, we show that for $n = 30$ not every norm one hermitian a in $\tilde{\mathrm{T}}_{\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ is a direction of convergence of a sequential family of complex Hadamard matrices approaching $\sqrt{n}F_n$. In particular, it follows that the dimension of any differentiable family of complex Hadamard matrices containing $\sqrt{n}F_n$ (with $n = 30$) is *strictly less* than the dimension of the tangent space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ (i.e., the defect of F_n). This is quite surprising, considering that for every n the space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ admits a basis of directions of convergence for one-parameter analytic families of complex Hadamard matrices (see [14]).

It follows that the $d(F_{30}) = 135$ independent 1-dimensional families of Hadamard deformations of $\sqrt{30}F_{30}$ found in [14] cannot be "joined" into a 135-dimensional family of Hadamard deformations of $\sqrt{30}F_{30}$. We note that 30 is the smallest integer with three distinct prime divisors. Based on numerical evidence for $n < 100$ from [3], it seems likely that the same might be true more generally for any n with three distinct prime divisors. Note that for $n = 6$, which only has two distinct prime divisors, the dephased defect of F_6 is equal to 4 and there exists a 4-dimensional smooth family of Hadamard matrices through $\sqrt{6}F_6$ (see [20]).

Our motivation for studying deformations of the Fourier matrix by Hadamard matrices is two-fold: On one hand, such deformations give insight into the classification of Hadamard matrices around $\sqrt{n}F_n$, with applications to Quantum Information Theory for $n \geq 6$ (see [24], [3], [21], [20]). On the other hand, families of complex Hadamard matrices can be used to construct families of (possibly non-isomorphic) subfactors (see [8],[6],[17],[18],[11]).

This paper is structured as follows:

In Section 2 we recall background information and introduce various notations.

In Section 3 we give k-th order necessary conditions ($k = 1, 2, 3, ...$) for $a \in \mathrm{M_n}(\mathbb{C})$ to be the direction of convergence of an *analytic* family of complex Hadamard matrices approaching the Fourier matrix.

In Section 4 we discuss in more detail the 3rd order conditions for an arbitrary $a \in \mathrm{M_n}(\mathbb{C})$ and prove that they can be written in an equivalent tracial form.

In Section 5 we show that the 3rd order conditions on a hold whenever a is the direction of convergence of a *sequential* family of complex Hadamard matrices approaching the Fourier matrix. This is where we introduce the most important technique of this paper: We generalize the notion of higher order derivatives, which can be used to easily identify coefficients of analytic deformations, to a notion of *higher order directional derivatives* for convergent sequences.

In Section 6 we construct, for $n = 30$, norm-one elements a in $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ which do not satisfy the 3rd order condition. We conclude that the dimension of any C^1 family of complex Hadamard matrices containing $\sqrt{n}F_n$ (with $n = 30$) is strictly less than the dimension of the tangent space at $\sqrt{n}F_n$ (i.e., the defect of F_n).

2 Preliminaries

In this section we recall classification results for complex Hadamard matrices, including examples connecting them to Quantum Information Theory, Harmonic Analysis, and Subfactor Theory. We also review how Hadamard matrices can be encoded in spin model commuting squares, and thus can be used as construction data in V. Jones' theory of subfactors.

2.1 Complex Hadamard Matrices

Definition 2.1. An $n \times n$ *complex Hadamard matrix* H is an $n \times n$ matrix with complex entries satisfying the following properties:

1. All entries of H have absolute value 1.

2. The rows of H are mutually orthogonal.

Note that this definition is equivalent to saying that $U = \frac{1}{\sqrt{n}}H$ is a unitary with all entries of the same absolute value.

Complex Hadamard matrices are natural generalizations of real Hadamard matrices. For more details on real Hadamard matrices and their connections to complex Hadamard matrices see [21]. For the remaining of this paper, when talk about a Hadamard matrix we mean one with complex entries.

For each $n \geq 2$ there exists at least one $n \times n$ Hadamard matrix, $\sqrt{n}F_n$, where F_n denotes the Fourier matrix of size n:

Definition 2.2. Let $n \geq 2$ and let $\varepsilon = e^{2\pi \mathrm{i}/n}$. The $n \times n$ *Fourier matrix*, denoted F_n, is the unitary matrix having $\frac{1}{\sqrt{n}}\varepsilon^{kl}$ in the (k, l)th entry for all $0 \leq k, l \leq n-1$.

Example 2.1.

$$F_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, F_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

(where $\omega = e^{2\pi i/3}$).

There are a couple of natural ways to create from a Hadamard matrix H new Hadamard matrices: If we permute the rows or columns of H, the resulting matrix is still a Hadamard matrix. Additionally, if we multiply any row or column of H by a complex number with modulus 1, the resulting matrix is still a Hadamard matrix. These constructions suggest a notion of equivalence for complex Hadamard matrices.

Definition 2.3. If H_1, H_2 are two $n \times n$ Hadamard matrices, then we say that H_1 is equivalent to H_2 if there exist unitary diagonal matrices D, D' and permutation matrices P, P' such that

$$H_2 = DPH_1P'D'.$$

One of the central problems in the theory of complex Hadamard matrices is to classify all $n \times n$ Hadamard matrices up to equivalence. Such a classification is currently not available even for n as low as 6. For $n \leq 5$, Hadamard matrices have been completely classified by Haagerup in [5]. For $n = 2, 3, 5$ the Fourier matrix F_n is the only Hadamard matrix (up to equivalence). When $n = 4$, there exists a 1-parameter analytic family of Hadamard matrices:

Example 2.2. For all $|t| = 1$, the following is a Hadamard matrix.

$$U(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & it & -1 & -it \\ 1 & -1 & 1 & -1 \\ 1 & -it & -1 & it \end{pmatrix}$$

The family of matrices described above contains all of the equivalence classes of 4×4 Hadamard matrices. Note in particular that $2U(1) = F_4$.

For $n = 6$, the classification of Hadamard matrices remains an open problem, with potential applications in quantum information theory [24]. Only some partial classification results are available so far: In [4] all 6×6 self-adjoint Hadamard matrices were classified as a 1-dimensional smooth family; in [9] a 3-dimensional smooth family was constructed; in [20] a 4-dimensional smooth family through $\sqrt{6}F_6$ was constructed; in [23] the so-called Tao matrix was constructed, which is isolated among the 6×6 non-equivalent matrices.

Example 2.3. The following complex Hadamard matrix was discovered by T. Tao ([23]), as a counterexample to Fuglede's conjecture in harmonic analysis. For $\omega = e^{2\pi i/3}$ let

$$T_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \end{pmatrix}$$

For general n, it is known that there exist smooth families of Hadamard matrices through $\sqrt{n}F_n$ (and not equivalent to $\sqrt{n}F_n$) if and only if n is not prime. When n is prime, a theorem of Petrescu ([16]) shows that $\sqrt{n}F_n$ is isolated among all complex Hadamard matrices, up to equivalence. Nevertheless, for certain prime values of n it is still possible to construct other smooth parametric families of Hadamard matrices. The following example was found in [16].

Example 2.4. Let $\omega = e^{2\pi i/6}$. Petrescu's family of Hadamard matrices, depending on a parameter $|t| = 1$, is defined as follows:

$$P(t) = \begin{pmatrix} \omega t & \omega^4 t & \omega^5 & \omega^3 & \omega^3 & \omega & 1 \\ \omega^4 t & \omega t & \omega^3 & \omega^5 & \omega^3 & \omega & 1 \\ \omega^5 & \omega^3 & \omega\bar{t} & \omega^4\bar{t} & \omega & \omega^3 & 1 \\ \omega^3 & \omega^5 & \omega^4\bar{t} & \omega\bar{t} & \omega & \omega^3 & 1 \\ \omega^3 & \omega^3 & \omega & \omega & \omega^4 & \omega^5 & 1 \\ \omega & \omega & \omega^3 & \omega^3 & \omega^5 & \omega^4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2.2 Commuting Squares

Commuting squares were introduced by S. Popa in [18], as invariants and construction data in V. Jones' theory of subfactors ([7]). They encode the symmetries of a subfactor, and for some large classes of subfactors they are complete invariants ([18],[19]).

Definition 2.4. A commuting square of finite dimensional von Neumann Algebras is a square of inclusions:

$$\begin{pmatrix} P_{-1} & \subset & P_0 & & \\ \cup & & \cup & , & \tau \\ Q_{-1} & \subset & Q_0 & & \end{pmatrix}$$

where P_0, P_{-1}, Q_0, Q_{-1} are finite dimensional *-algebras (i.e., matrix algebras) and τ is a positive, faithful trace on P_0 such that

$$Q_0 \ominus Q_{-1} \perp P_{-1} \ominus Q_{-1}$$

Here the inner product on P_0 given by $\langle x, y\rangle = \tau(xy^*)$, and the symbol "$\ominus$" is defined by $A \ominus B := (B \cap A^{\perp})$.

Before giving examples of commuting squares, we introduce some notations for matrix units that will be used throughout this paper.

Definition 2.5. The matrix unit $e_{i,j}$ for $i, j \in S$, where S is a finite indexing set, is the matrix in $M_{|S|}(\mathbb{C})$ which has 1 in the $(i, j)^{\text{th}}$ entry, and 0 elsewhere. These matrices satisfy the following multiplicative relation for $i, j, k, l \in S$:

$$e_{i,j}e_{k,l} = \begin{cases} e_{i,l} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

The most basic examples of commuting squares arise from finite groups.

Example 2.5. Let G be a finite group of order n, let $\mathcal{D}_n$ denote the $n \times n$ diagonal matrices, and set $u_g = \sum_{h \in G} e_{h,g^{-1}h}$ for all $g \in G$. If $\mathbb{C}[G] := \text{span}\{u_g : g \in G\}$, then the following is a commuting square:

$$\mathfrak{C}_G := \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & & \\ \cup & & \cup & , & \text{Tr} \\ \mathbb{C} & \subset & \mathbb{C}[G] & & \end{pmatrix}.$$

A particularly interesting special case arises for $G = \mathbb{Z}_n$. One can check that the matrices in $\mathbb{C}[\mathbb{Z}_n]$ are those which are constant on each diagonal. Such matrices are know as *circulant matrices*.

Definition 2.6. The $n \times n$ *circulant matrices*, denoted $\mathcal{C}_n$, are the collection of $n \times n$ matrices $C = (c_{i,j})_{0 \leq i,j \leq n-1}$ which satisfy $c_{i,j} = c_{k,l}$ if $i - j \equiv k - l \mod n$.

Example 2.6. The 4×4 circulant matrices are given by

$$\mathcal{C}_4 = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{pmatrix} : x_0, x_1, x_2, x_3 \in M_n(\mathbb{C}) \right\}$$

Since $\mathbb{Z}_n$ is an abelian group, $\mathbb{C}[\mathbb{Z}_n] = \mathcal{C}_n$ is a maximal abelian *-subalgebra (MASA) of the matrices, and thus isomorphic to $\mathcal{D}_n$. This isomorphism is realized by conjugating the diagonal matrices with the Fourier matrix, F_n.

Proposition 2.1.

$$\mathcal{C}_n = \mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*.$$

Proof. We show that $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$. We use the Fourier basis of $\mathcal{D}_n$ given by $q_k := \sum_{i \in \mathbb{Z}_n} \varepsilon^{ik} e_{i,i}$ for each $k \in \mathbb{Z}_n$ and check that

$$\frac{1}{n} F_n q_{-k} F_n^* = \sum_{i \in \mathbb{Z}_n} e_{i,i-k} = u_k.$$

Since F_n is unitary and $\mathcal{D}_n$ and $\mathbb{C}[\mathbb{Z}_n]$ both have dimension n it follows that $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$. ■

With this particular example in mind it is natural to ask which MASA's, when placed in the bottom right corner, form a commuting square. That is to say, we are looking for unitaries U for which the following is a commuting square:

$$\mathfrak{C}(U) = \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & & \\ \cup & & \cup & , & \mathrm{Tr} \\ \mathbb{C} & \subset & U \mathcal{D}_n U^* & & \end{pmatrix}$$

Such commuting squares are known as *spin model* commuting squares (see [6] for details). The name comes from connections to Statistical Mechanics.

It turns out that the commuting square condition for a spin model is equivalent to U being a (complex) Hadamard matrix.

Proposition 2.2. For a unitary matrix U, the following are equivalent

1. $\mathfrak{C}(U)$ is a commuting square.
2. $\sqrt{n} U$ is a Hadamard matrix.
3. For all $d, d' \in \mathcal{D}_n$, $\tau(U d U^* d') = \tau(d) \tau(d')$.

Proof. We start with a trace computation that will be used throughout the proof. For $d, d' \in \mathcal{D}_n$ with $d = \sum_{k \in \mathbb{Z}_n} d_k e_{k,k}$ and $d' = \sum_{k' \in \mathbb{Z}_n} d'_k e_{k',k'}$ we have

$$\begin{aligned} UdU^*d' &= \left(\sum_{i,j \in \mathbb{Z}_n} u_{i,j} e_{i,j}\right) \cdot \left(\sum_{k \in \mathbb{Z}_n} d_k e_{k,k}\right) \cdot \left(\sum_{i',j' \in \mathbb{Z}_n} \overline{u_{i',j'}} e_{j',i'}\right) \cdot \left(\sum_{k' \in \mathbb{Z}_n} d'_{k'} e_{k',k'}\right) \\ &= \sum_{i,j,i' \in \mathbb{Z}_n} u_{i,j} d_j \overline{u_{i',j}} d'_{i'} e_{i,i'} \end{aligned}$$

so taking the trace,

$$\begin{aligned} \tau(UdU^*d') &= \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} u_{i,j} d_j \overline{u_{i,j}} d'_i \\ &= \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} |u_{i,j}|^2 d_j d'_i. \end{aligned}$$

(1) $\Rightarrow$ (2). Suppose $\mathfrak{C}(U)$ is a commuting square. To proceed we will need to use the Fourier basis for the diagonal matrices $q_k := \sum_{i \in \mathbb{Z}_n} \varepsilon^{ik} e_{i,i}$. Note that $q_k \in \mathcal{D}_n \ominus \mathbb{C}$ and $Uq_kU^* \in UDU^* \ominus \mathbb{C}$, for all $k \neq 0$. So for all $k \neq 0$ and $l \neq 0$ we have, using the commuting square relation, that

$$0 = \tau(Uq_kU^*q_{-l}) = \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} \varepsilon^{jk} |u_{i,j}|^2 \varepsilon^{-il}.$$

Note also that for $k = 0$ and $l \neq 0$ (and vice versa) we have $q_k = I$ so

$$\tau(Uq_kU^*q_{-l}) = \tau(UU^*q_{-l}) = \tau(q_l) = 0$$

if $k = 0$ and $l = 0$ we have

$$\tau(Uq_kU^*q_l) = \tau(I) = 1.$$

Now if we denote $V = (|u_{i,j}|^2)_{i,j \in \mathbb{Z}_N}$ note that we have

$$\tau(Uq_kU^*q_{-l}) = (F_n V F_n^*)_{k,l}$$

so the above computations are equivalent to

$$e_{0,0} = F_n V F_n^*, \text{ or equivalently, } F_n^* e_{0,0} F_n = V.$$

Thus we have for each $i, j \in \mathbb{Z}_n$

$$V_{i,j} = (F_n^* e_{0,0} F_n)_{i,j} = \frac{1}{n}$$

so we have $|u_{i,j}| = \frac{1}{\sqrt{n}}$ for all $i, j \in \mathbb{Z}_n$.

(2) ⇒ (3) This follows immediately from the computation done at the beginning of the proof:

$$\tau(UdU^*d') = \frac{1}{n}\sum_{i,j\in\mathbb{Z}_n} |u_{i,j}|^2 d_i d'_j = \left(\frac{1}{n}\sum_{i\in\mathbb{Z}_n} d_i\right)\left(\frac{1}{n}\sum_{j\in\mathbb{Z}_n} d'_j\right) = \tau(d)\tau(d').$$

(3) ⇒ (1) If $UdU^* \in U\mathcal{D}_nU^* \ominus \mathbb{C}$ and $d' \in \mathcal{D}_n \ominus \mathbb{C}$ then we have

$$\tau(UdU^*d') = \tau(d)\tau(d') = 0.$$

Thus $U\mathcal{D}_nU^* \ominus \mathbb{C} \perp \mathcal{D}_n \ominus \mathbb{C}$. ■

The goal of this paper is to study sequential deformations of the Fourier spin model $\mathfrak{C}_{F_n}$ in the moduli space of commuting squares. We say that the sequence

$$\mathfrak{C}_j = \begin{pmatrix} \mathcal{D}_n & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & U_j\mathbb{C}[\mathbb{Z}_n]U_j^* \end{pmatrix} \text{ converges to } \mathfrak{C}(F_n) = \begin{pmatrix} \mathcal{D}_n & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[\mathbb{Z}_n] \end{pmatrix}$$

if $U_j \to I_n$ as $j \to \infty$. Here $(U_j)_{j\geq 1}$ are unitary matrices such that $\mathfrak{C}_j$ are commuting squares.

Equivalently, we can work exclusively with Hadamard matrices: $\sqrt{n}U_jF_n$ is a sequence of Hadamard matrices converging to $\sqrt{n}F_n$. We will also want the matrices $\sqrt{n}U_jF_n$ to not be equivalent to $\sqrt{n}F_n$. In the language of commuting squares, this says that $\mathfrak{C}_j$ is not isomorphic to $\mathfrak{C}(F_n)$ (see for instance [10]).

Our motivation for studying deformations of the Fourier matrix by Hadamard matrices is two-fold: On one hand, such deformations shed further light on the classification of Hadamard matrices, with applications to Quantum Information Theory (see [24], [3], [21], [20]). On the other hand, families of complex Hadamard matrices can be used to construct families of (possibly non-isomorphic) subfactors (see [8],[6],[17],[18],[11]).

3 Analytic deformations of the Fourier matrix

In this section we find necessary conditions for the existence of analytic 1-parameter families of $n \times n$ Hadamard matrices containing $\sqrt{n}F_n$, where F_n denotes the Fourier matrix. Concrete examples of such families were given in Section 2.1: Example 2.2 for $n = 4$ and Example 2.4 for $n = 7$.

We introduce higher order relations that a tangent vector to an analytic family, at the Fourier matrix, must satisfy. Due to the analyticity of the family, this is easy to do by identifying the coefficients of the Taylor series. In a latter section we will prove the surprising result that the first three of these relations still hold true for directions of convergence of sequential (rather than analytic) families approaching $\sqrt{n}F_n$.

Let $a_0 = I_n$, $a_j \in \mathrm{M_n}(\mathbb{C})$ for $j \geq 1$, and assume that $U_t = \sum_{j=0}^{\infty} a_j t^j$ converges for all t in a real neighborhood of 0. Let $H_t = \sqrt{n}U_tF_n$, and further assume that H_t are complex Hadamard matrices and $H_t \neq \sqrt{n}F_n$ when $t \neq 0$. Note $H_t \to \sqrt{n}F_n$ as $t \to 0$.

Since H_t is Hadamard, by Proposition 2.2 we have:

$$\tau(U_t p U_t^* q) = \frac{1}{n}\tau(H_t F_n^* p F_n H_t^* q) = \frac{1}{n}\tau(H_t F_n^* p F_n H_t^*)\tau(q) = \tau(p)\tau(q) = \tau(pq).$$

for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$. Here we used the fact that $F_n p F_n^* \in \mathcal{D}_n$ and $\mathcal{C}_n = F_n \mathcal{D}_n F_n^*$.

In light of these two facts, we introduce the following bilinear continuous functions on $\mathrm{M_n}(\mathbb{C}) \times \mathrm{M_n}(\mathbb{C})$:

$$f_0(x, y) = xy$$

$$f^{p,q}(x, y) = \tau(xpyq) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Proposition 3.1. If $\mathcal{F} = \{f^{p,q} : p \in \mathcal{C}_n, q \in \mathcal{D}_n\} \cup \{f_0\}$, then $H_t = \sqrt{n}U_tF_n$ are Hadamard matrices if and only if $f(U_t, U_t^*) = f(I, I)$ for all $f \in \mathcal{F}$.

Proof. The forward direction of this proof follows from the computations above. Indeed, for $f = f^{p,q}$ the condition $f(U_t, U_t^*) = f(I, I)$ is the same as $\tau(U_t p U_t^* q) = \tau(pq)$, which is true for H_t Hadamard. Note that for $f = f_0$ the condition $f(U_t, U_t^*) = f(I, I)$ holds true because H_t, hence also U_t are unitary matrices.

For the reverse direction, first note that U_t unitary follows from $U_t U_t^* = f_0(U_t, U_t^*) = f_0(I, I) = I$. Since U_t, F_n are unitary, we have $H_t^* H_t = nI_n$ for all t. Additionally, we have for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$ that

$$\tau(U_t p U_t^* q) = f^{p,q}(U_t, U_t^*) = f^{p,q}(I, I) = \tau(pq).$$

Using this fact, for all $q, q' \in \mathcal{D}_n$ we have

$$\tau(H_t q H_t^* q') = n\tau(U_t F_n q F_n^* U_t^* q) = n\tau(F_n q F_n^* q) = n\tau(F_n q F_n^*)\tau(q) = \tau(H_t q H_t^*)\tau(q).$$

We used the fact that $F_n q F_n^* \in \mathcal{C}_n$, and in the last equality we introduced $U_t U_t^* = I$ into the first trace and used the property of the trace to rearrange. ■

We now find relations between the coefficients a_j of the Taylor series of U_t.

Proposition 3.2. If H_t are Hadamard for all t in a neighborhood of 0, then we have:

$$f(a_j, I) + f(I, a_j^*) = -\sum_{k=1}^{j-1} f(a_{j-k}, a_k^*)$$

for all $j \geq 1$.

Proof. We have $f(U_t, U_t^*) = f(I, I)$ for all $f \in \mathcal{F}$. This gives the following chain of equivalent equalities:

$$f(U_t, U_t^*) = f(I, I).$$

$$f\left(\sum_{l=0}^{\infty} a_l t^l, \sum_{k=0}^{\infty} a_k^* t^k\right) = f(I, I).$$

$$\sum_{j=1}^{\infty}\left(\sum_{l+k=j} f(a_l, a_k^*)\right) t^j = 0.$$

$$\sum_{j=1}^{\infty}\left(\sum_{k=0}^{j} f(a_{j-k}, a_k^*)\right) t^j = 0.$$

$$\sum_{k=0}^{j} f(a_{j-k}, a_k^*) = 0 \text{ for all } j \geq 1.$$

$$f(a_j, I) + f(I, a_j^*) = -\sum_{k=1}^{j-1} f(a_{j-k}, a_k^*) \text{ for all } j \geq 1.$$

■

We now analyze in more detail these equalities. Applying Proposition 3.2 to $f = f_0$ gives for all j that

$$a_j + a_j^* = -\sum_{k=1}^{j-1} a_{j-k} a_k^*.$$

Solving for a_j^* and plugging into the relation given by Proposition 3.2 for $f \in \mathcal{F}'$, we obtain

$$f(a_j, I) - f(I, a_j) = \sum_{k=1}^{j-1} f(I, a_{j-k} a_k^*) - f(a_{j-k}, a_k^*).$$

Equivalently:

$$\tau(a_j[p, q]) = \sum_{k=1}^{j-1} \tau([p, a_{j-k}] a_k^* q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

When $j = 1$, it follows that a_1 must satisfy $a_1 + a_1^* = 0$ and $\tau(a_1[p, q]) = 0$ for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. In other words, $a_1 \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ and a_1 is skew-adjoint: $a_1^* = -a_1$. We introduce a notation for the set of $n \times n$ matrices satisfying these conditions.

Definition 3.1. Let $\mathcal{A}_n = \{a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp : a^* = -a\}$. We say that an $n \times n$ matrix a satisfies the *first order relations* if and only if $a \in \mathcal{A}_n$.

This definition is justified by the fact that the elements of $\mathcal{A}_n$ are the only candidates for tangents to the analytic family H_t of Hadamard matrices at F_n, i.e. for the linear coefficient a_1 in the Taylor series of H_t. We can interpret the rest of the relations from Proposition 3.2 as further restrictions on a_1. Note that $\mathcal{A}_n$ is only a *real* vector space.

Let us recall a different terminology for $\mathcal{A}_n$, which was introduced in [1], [2]. Let $\mathcal{H}(\mathrm{n}) = \mathrm{M_n}(\mathbb{T}) \cap \sqrt{\mathrm{n}}\mathrm{U}(\mathrm{n})$ denote the real algebraic variety of $n \times n$ complex Hadamard matrices, where $\mathrm{U}(\mathrm{n}) \subset \mathrm{M_n}(\mathbb{C})$ denotes the set of unitary matrices and $\mathrm{M_n}(\mathbb{T})$ denotes the set of $n \times n$ matrices with entries of absolute value 1. Let $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n}) = \mathrm{T}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathrm{M_n}(\mathbb{T}) \cap \mathrm{T}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\sqrt{\mathrm{n}}\mathrm{U}(\mathrm{n})$ denote the enveloping tangent space of $\mathcal{H}(n)$ at the matrix $\sqrt{n}F_n$. With these notations, we have $\mathcal{A}_n = \tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$. The dimension of this real vector space is called the defect of the Fourier matrix F_n, and it is denoted $d(F_n)$.

Definition 3.2. Let $j \geq 2$, let $a \in \mathcal{A}_n$, and let $a_1 = a$. If there exist $a_2, \ldots, a_j \in \mathrm{M_n}(\mathbb{C})$ satisfying for all $2 \leq l \leq j$

$$a_l + a_l^* = -\sum_{k=1}^{l-1} a_{l-k} a_k^*$$

and

$$\tau(a_l[p, q]) = \sum_{k=1}^{l-1} \tau([p, a_{l-k}] a_k^* q)$$

for every $p \in \mathcal{C}_n, q \in \mathcal{D}_n$, then we say a satisfies the *jth order relations*.

We can now restate Proposition 3.2 as follows:

Proposition 3.3. If $H_t = (I + a_1 t + a_2 t^2 + a_3 t^3 + \ldots.)(\sqrt{n}F_n)$ are Hadamard for all t in a neighborhood of 0, then a_1 satisfies the j-th order relations for all $j \geq 1$.

4 Third order trace relations

In the previous section, motivated by the study of analytic deformations of $\sqrt{n}F_n$ by complex Hadamard matrices, we defined j-th order relations ($j \geq 1$) on $n \times n$ matrices a (Definitions 3.1 and 3.2). In [15] it was shown that every a satisfying the first order relations must also satisfy the second order relations. Thus the second order relations provide no further restrictions on a. However, in subsequent sections of this paper we will show that the third order relations provide significant additional restrictions on a.

In this section we discuss in more detail the third order relations, with two goals in mind: First, we show that if the third order relations hold for *some* solution a_1, a_2 to the second order relations (with $a = a_1$), then they hold for *any* solution a_1, a_2 to the second order relations (with $a = a_1$). Second, we show that the third order relations can be simplified to just keeping the second (trace-based) equality from Definition 3.2 for $j = 3$.

We start by defining, for $j \geq 2$, *trace relations* of order j. These are just the second equation of Definition 3.2.

Definition 4.1. For $a_1 \in \mathcal{A}_n$ (i.e., satisfying the first order relations), if there exist $a_2, \ldots, a_j \in \mathrm{M_n}(\mathbb{C})$ satisfying for all $2 \leq l \leq j$

$$\tau(a_l[p,q]) = \sum_{k=1}^{l-1} \tau([p, a_{l-k}]a_k^* q)$$

then we say that a_1 satisfies the *jth order trace relations*.

Applying this definition for $j = 3$ and $a_1 = a \in \mathcal{A}_n$, we get the *3rd order trace relations* for a:

There exist $a_2, a_3 \in \mathrm{M_n}(\mathbb{C})$ such that

$$\tau(a_2[p,q]) = \tau([p,a]a^*q)$$

$$\tau(a_3[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$.

We now present two lemmas that allow us to easily do computations in $[\mathcal{C}_n, \mathcal{D}_n]$, by establishing elegant bases for $\mathcal{C}_n$, $\mathcal{D}_n$, $[\mathcal{C}_n, \mathcal{D}_n]$ and $[\mathcal{C}_n, \mathcal{D}_n]^\perp$.

Lemma 4.1. Let $\varepsilon = e^{2\pi \mathrm{i}/n}$ be the primitive root of order n of unity, let $p_i = \sum_{k \in \mathbb{Z}_n} e_{k,k-i}$ be the canonical basis of the algebra of circulant matrices ($i \in \mathbb{Z}_n$), and let $q_j = \sum_{k \in \mathbb{Z}_n} \varepsilon^{kj} e_{k,k}$ be the Fourier basis of $\mathcal{D}_n$ ($j \in \mathbb{Z}_n$). For all $i, j \in \mathbb{Z}_n$ we have:

$$q_j p_i = \varepsilon^{ij} p_i q_j.$$

Proof. For each $i, j \in \mathbb{Z}_n$, we have that

$$p_i = \sum_{k \in \mathbb{Z}_n} e_{k,k-i}, \quad q_j = \sum_{k \in \mathbb{Z}_n} \varepsilon^{kj} e_{k,k}.$$

Thus it follows

$$\begin{aligned} q_j p_i &= \sum_{k,l\in\mathbb{Z}_n} \varepsilon^{jk} e_{k,k} e_{l,l-i} \\ &= \sum_{k\in\mathbb{Z}_n} \varepsilon^{jk} e_{k,k-i} \end{aligned}$$

and

$$\begin{aligned} p_i q_j &= \sum_{k,l\in\mathbb{Z}_n} \varepsilon^{jl} e_{k,k-i} e_{l,l} \\ &= \sum_{k\in\mathbb{Z}_n} \varepsilon^{j(k-i)} e_{k,k-i} \\ &= \varepsilon^{-ij} \sum_{k\in\mathbb{Z}_n} \varepsilon^{jk} e_{k,k-i} \\ &= \varepsilon^{-ij} q_j p_i. \end{aligned}$$

Thus

$$q_j p_i = \varepsilon^{ij} p_i q_j.$$

■

Lemma 4.2.

- The set $\mathcal{X} = \{q_x p_{x'} : x, x' \in \mathbb{Z}_n\}$ is an orthonormal basis of $M_n(\mathbb{C})$.
- The set $\mathcal{X}' = \{q_x p_{x'} : xx' \neq 0 \mod n\}$ is an orthonormal basis of $[\mathcal{C}_n, \mathcal{D}_n]$.
- The set $\mathcal{X}'' = \{q_x p_{x'} : xx' = 0 \mod n\}$ is an orthonormal basis of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$.

Proof. For $x, x', y, y' \in \mathbb{Z}_n$ see that

$$\begin{aligned} \tau(q_x p_{x'} (q_y p_{y'})^*) &= \tau(q_x p_{x'} p_{-y'} q_{-y}) \\ &= \tau(q_{x-y} p_{x'-y'}) \\ &= \tau(q_{x-y}) \tau(p_{x'-y'}) \\ &= \begin{cases} 1 & \text{if } x = y \text{ and } x' = y' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $a_x^{x'} \in \mathcal{X}'$, then we have (since $xx' \neq 0 \mod n$) that

$$q_x p_{x'} = \frac{[p_{x'}, q_x]}{\varepsilon^{-xx'} - 1} \in [\mathcal{C}_n, \mathcal{D}_n].$$

Conversely, if $x \in [\mathcal{C}_n, \mathcal{D}_n]$, we use the fact that $[p_i, q_j] = 0$ if and only if $ij = 0 \mod n$ to write

$$\begin{aligned} x &= \sum_{x,x' \in \mathbb{Z}_n} \alpha_{x,x'}[p_{x'}, q_x] \\ &= \sum_{x,x' \in \mathbb{Z}_n} (\varepsilon^{-xx'} - 1) q_x p_{x'} \\ &= \sum_{xx' \neq 0 \mod n} (\varepsilon^{-xx'} - 1) q_x p_{x'} \in \operatorname{span}\{\mathcal{X}'\}. \end{aligned}$$

This shows that $\mathcal{X}'$ is a basis for $[\mathcal{C}_n, \mathcal{D}_n]$. The fact that $\mathcal{X}'' = \mathcal{X} \setminus \mathcal{X}'$ completes the proof. ■

We now give a lemma about zero products in $\mathbb{Z}_n$, which will be needed in the subsequent proposition.

Lemma 4.3. Let $x, x', y, y' \in \mathbb{Z}_n$. If $xx' = yy' = (x+y)(x'+y') = 0 \mod n$, then $xy' = x'y = 0 \mod n$.

Proof. Let $x, x', y, y' \in \mathbb{Z}_n$ and suppose $xx' = yy' = (x+y)(x'+y') = 0 \mod n$. Let p^m be the largest power of a given prime p that divides n. We must have that $p^m | xx'$. Let t be the largest nonnegative integer less than or equal to m such that $p^t | x'$. Thus $p^{m-t} | x$. Similarly, let v be the largest nonnegative integer, at most m, such that $p^v | y'$. We have $p^{m-v} | y$. To summarize:

$$\begin{aligned} p^t | x', \quad p^{m-t} | x \\ p^v | y', \quad p^{m-v} | y \end{aligned}$$

These together imply the following:

$$p^{m-v+t} | x'y \quad \text{and} \quad p^{m-t+v} | xy'.$$

Either $v \geq t$ or $t \geq v$, so, without loss of generality, suppose $t \geq v$. Thus we have that $p^m | x'y$. However, since

$$(x+y)(x'+y') = xy' + x'y = 0 \mod p^m.$$

and since $p^m | x'y$, we must also have that $p^m | xy'$. Since this holds for all primes p that divide n, we must have that both xy' and $x'y$ are 0 mod n. ■

The following proposition is key to establishing the main results of this section, and will also be used to prove the main theorem of the next section.

Proposition 4.1. There exists a $\mathbb{C}$-bilinear map

$$\varphi_2 : [\mathcal{C}_n, \mathcal{D}_n]^\perp \times [\mathcal{C}_n, \mathcal{D}_n]^\perp \to [\mathcal{C}_n, \mathcal{D}_n]$$

satisfying

$$\tau(\varphi_2(a', a'')[p, q]) = \tau\left([p, a']a''q\right)$$

for all $a', a'' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, $p \in \mathcal{C}_n, q \in \mathcal{D}_n$.

Proof. It suffices to define φ_2 on pairs of elements in a basis of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$, then extend it to $[\mathcal{C}_n, \mathcal{D}_n]^\perp$ as a bilinear map. Let $q_x p_{x'}$ and $q_y p_{y'}$ be basis elements of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$, with $xx' = yy' = 0 \mod n$ (as explained in Lemma 4.2). We first show that there exists a unique element $b_{x,x',y,y'} \in [\mathcal{C}_n, \mathcal{D}_n]$ such that

$$\tau(b_{x,x',y,y'}[p,q]) = \tau([p, q_x p_{x'}] q_y p_{y'} q)$$

for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$.

Consider the map $f : [\mathcal{C}_n, \mathcal{D}_n] \to \mathbb{C}$ given by

$$f(\sum_{i,j \in \mathbb{Z}_n} c_{i,j}[p_i, q_j]) = \sum_{i,j \in \mathbb{Z}_n} c_{i,j} \tau([p_i, q_x p_{x'}] q_y p_{y'} q_j)$$

for any $c_{i,j} \in \mathbb{C}$. Note that f depends on x, x', y, y'. We next check that f is well-defined.

Recall that $[p_i, q_j] = 0$ when $ij = 0 \mod n$, and that the elements $[p_i, q_j]$ with $ij \neq 0 \mod n$ form an orthonormal basis for $[\mathcal{C}_n, \mathcal{D}_n]$ (Lemma 4.2). Thus, in order to show that f is well-defined, it suffices to check that $\tau([p_i, q_x p_{x'}] q_y p_{y'} q_j) = 0$ whenever $ij = 0 \mod n$. Indeed:

$$\begin{aligned}
\tau([p_i, q_x p_{x'}] q_y p_{y'} q_j) &= \tau(p_i q_x p_{x'} q_y p_{y'} q_j) - \tau(q_x p_{x'} p_i q_y p_{y'} q_j) \\
&= \varepsilon^{xy'} \tau(p_i q_x q_y p_{x'} p_{y'} q_j) - \tau(q_x p_{x'} p_i q_y p_{y'} q_j) \\
&= \varepsilon^{xy'} \tau(q_{x+y} p_{x'+y'} q_j p_i) - \tau(p_{x'+i} q_y p_{y'} q_{j+x}) \\
&= \varepsilon^{xy'} \tau(p_{x'+y'+i} q_{x+y+j}) - \tau(p_{x'+y'+i} q_{x+y+j}) \\
&= (\varepsilon^{xy'} - 1) \tau(q_{x+y+j} p_{x'+y'+i}) \\
&= (\varepsilon^{xy'} - 1) \tau(q_{x+y+j}) \tau(p_{x'+y'+i}).
\end{aligned}$$

In the above computation we used commutation relations implied by $[p_{x'}, q_x] = 0$, $[p_{y'}, q_y] = 0$, and $[p_i, q_j] = 0$. These follow from $x'x = y'y = ij = 0 \bmod n$.

If $x + y + j \neq 0 \mod n$ or $x' + y' + i \neq 0 \mod n$, then this product is 0 as desired. Otherwise, assume $x + y + j = 0 \mod n$ and $x' + y' + i = 0 \mod n$. Rearranging, this implies

$$(x + y)(x' + y') = (-j)(-i) = 0 \mod n.$$

We now apply Lemma 4.3 to see that $xy' = 0 \mod n$, hence $(\varepsilon^{xy'} - 1) = 0$ and the desired trace is 0 in this case as well.

This shows that the map f is well defined. Clearly f is $\mathbb{C}$-bilinear on $[\mathcal{C}_n, \mathcal{D}_n]$. Recall also that f depends on x, x', y, y'. By the Riesz representation theorem, there exists a unique $b_{x,x',y,y'} \in [\mathcal{C}_n, \mathcal{D}_n]$ such that $f(z) = \tau(b_{x,x',y,y'} z)$ for any $z \in [\mathcal{C}_n, \mathcal{D}_n]$. Define $\varphi_2(q_x p_{x'}, q_y p_{y'}) = b_{x,x',y,y'}$. Hence, for $z = [p, q]$ with $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$, we have

$$\tau(\varphi_2(q_x p_{x'}, q_y p_{y'})[p,q]) = \tau(b_{x,x',y,y'}[p,q]) = f([p,q]) = \tau([p, q_x p_{x'}] q_y p_{y'} q).$$

In other words, $\tau(\varphi_2(a', a'')[p,q]) = \tau([p, a'] a'' q)$ for any $(a', a'') = (q_x p_{x'}, q_y p_{y'})$. This defines φ_2 on a basis of $[\mathcal{C}_n, \mathcal{D}_n]^\perp \times [\mathcal{C}_n, \mathcal{D}_n]^\perp$, hence we can extend φ_2 uniquely to a $\mathbb{C}$-bilinear map on $[\mathcal{C}_n, \mathcal{D}_n]^\perp \times [\mathcal{C}_n, \mathcal{D}_n]^\perp$. Since the map $(a', a'') \to \tau([p, a']a''q)$ is also $\mathbb{C}$-bilinear, we have

$$\tau(\varphi_2(a', a'')[p, q]) = \tau([p, a']a''q)$$

for all $a', a'' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, $p \in \mathcal{C}_n, q \in \mathcal{D}_n$.

■

We now show that if a matrix a satisfies the third order trace relations, then for *any* choice of a_2 for which the second order trace relations $\tau(a_2[p, q]) = \tau([p, a]a^*q)$ hold there exists an a_3 for which the third order trace relations hold. In particular, we may use $a_2 = \varphi_2(a, a)$, with φ_2 as defined in the previous lemma, to check if a satisfies the third order trace relations.

Proposition 4.2. If $a \in \mathcal{A}_n$ satisfies the third order trace relations then for any choice of $a_2 \in \mathrm{M_n}(\mathbb{C})$ satisfying:

$$\tau(a_2[p, q]) = \tau([p, a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n$$

there exists $a_3 \in \mathrm{M_n}(\mathbb{C})$ satisfying:

$$\tau(a_3[p, q]) = \tau([p, a]a_2^*q) + \tau([p, a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Proof. Since a satisfies the third order trace relations, there exist a_2' and a_3' satisfying:

$$\tau(a_2'[p, q]) = \tau([p, a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n$$

and

$$\tau(a_3'[p, q]) = \tau([p, a]a_2'^*q) + \tau([p, a_2']a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

In particular we have $\tau(a_2'[p, q]) = \tau([p, a]a^*q) = \tau(a_2[p, q])$ so $\tau((a_2 - a_2')[p, q]) = 0$. Thus $a_2 - a_2' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$. Let $y \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ be such that $a_2' = a_2 + y$. Since $a \in \mathcal{A}_n$, in particular we have $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$. Using the definition of φ_2 from the previous proposition, we obtain:

$$\begin{aligned}
\tau(a_3'[p, q]) &= \tau([p, a]a_2'^*q) + \tau([p, a_2']a^*q) \\
&= \tau([p, a]a_2^*q) + \tau([p, a_2]a^*q) + \tau([p, a]y^*q) + \tau([p, y]a^*q) \\
&= \tau([p, a]a_2^*q) + \tau([p, a_2]a^*q) + \tau(\varphi_2(a, y^*)[p, q]) + \tau(\varphi_2(y, a^*)[p, q]).
\end{aligned}$$

By letting $a_3 = a_3' - \varphi_2(a, y^*) - \varphi_2(y, a^*)$, the conclusion follows:

$$\tau(a_3[p, q]) = \tau([p, a]a_2^*q) + \tau([p, a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

■

We end this section by proving that the third order trace relations are equivalent to the (a priori more general) third order relations.

Proposition 4.3. Let $a \in \mathcal{A}_n$. Then a satisfies the third order relations if and only if a satisfies the third order trace relations.

Proof. If a satisfies the third order relations then clearly a must satisfy the less restrictive third order trace relations.

Now suppose a satisfies the third order trace relations. We must show that a satisfies the remaining third order relations:

$$a_2 + a_2^* = -aa^* \text{ and } a_3 + a_3^* = -aa_2^* - a_2 a^*.$$

Since $a \in \mathcal{A}_n$ (i.e. a satisfies the first order relations), there exists a_2 such that a satisfies the second order relations together with a_2 (this is the main result of [15]). That is to say:

$$a_2 + a_2^* = -aa^* \text{ and } \tau(a_2[p,q]) = \tau([p,a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

From the previous Proposition 4.2, there exists a_3' such that

$$\tau(a_3'[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

After taking the adjoint of this relation, replacing p with p^* and q with q^*in the new relation, and adding this new relation to the one above, we obtain:

$$\begin{aligned}
\tau((a_3' + a_3'^*)[p,q]) &= \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) + \tau(a_2[p,a^*]q) + \tau(a[p,a_2^*]q) \\
&= \tau(paa_2^*q) - \tau(apa_2^*q) + \tau(pa_2a^*q) - \tau(a_2pa^*q) + \tau(a_2pa^*q) - \tau(a_2a^*pq) \\
&+ \tau(apa_2^*q) - \tau(aa_2^*pq) \\
&= \tau(aa_2^*qp) - \tau(aa_2^*pq) + \tau(a_2a^*qp) - \tau(a_2a^*pq) \\
&= -\tau(aa_2^*[p,q]) - \tau(a_2a^*[p,q]).
\end{aligned}$$

Rearranging this equality, we have:

$$\tau((a_3' + a_3'^* + aa_2^* + a_2a^*)[p,q]) = 0 \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Let $a_3 = a_3' - \frac{a_3' + a_3'^* + aa_2^* + a_2a^*}{2}$. It follows

$$\tau(a_3[p,q]) = \tau(a_3'[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

and

$$\begin{aligned}
a_3 + a_3^* &= \left(a_3' - \frac{a_3' + a_3'^* + aa_2^* + a_2a^*}{2}\right) + \left(a_3'^* - \frac{a_3'^* + a_3' + a_2a^* + aa_2^*}{2}\right) \\
&= -aa_2^* - a_2a^*.
\end{aligned}$$

Thus a satisfies the third order relations. ∎

5 Sequential deformations of the Fourier matrix

In Section 3 we introduced k-th order relations necessary for the existence of *analytic* families of $n \times n$ complex Hadamard matrices containing $\sqrt{n}F_n$, where F_n is the Fourier matrix. In this section we prove that, for $k \leq 3$, these relations are still necessary for the existence of *sequential* families of $n \times n$ complex Hadamard matrices converging to $\sqrt{n}F_n$. The necessity of the first and second order relations was already proved in previous papers of the second author ([10], [13]). We now show that the third order relations hold, and in the process of doing so we obtain easier proofs for the necessity of the first and second order relations.

The main technique of this section consists in generalizing the notion of higher order derivatives, which were implicitly used in Section 3 to identify the coefficients of analytic deformations, to a notion of higher order directional derivatives for convergent sequences.

Let $(H_k)_{k\geq 0}$, be a sequence of $n \times n$ complex Hadamard matrices such that $H_k \to \sqrt{n}F_n$ as $k \to \infty$, and $H_k \neq \sqrt{n}F_n$ for all k. Let $U_k = \frac{1}{\sqrt{k}} H_k F_n^*$. U_k are unitaries, $U_k \neq I$ and $U_k \to I$.

We recall the bilinear continuous functions on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ defined in Section 3:

$$f_0(x, y) = xy$$

$$f^{p,q}(x, y) = \tau(xpyq) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

The following proposition is just Proposition 3.1 rewritten for sequences instead of analytic families. The proof is identical, as it just relies on the commuting square characterization of complex Hadamard matrices (Proposition 2.2).

Proposition 5.1. If $\mathcal{F} = \{f^{p,q} : p \in \mathcal{C}_n, q \in \mathcal{D}_n\} \cup \{f_0\}$, then H_k is a sequence of Hadamard matrices passing through F_n if and only if for $U_k = \frac{1}{\sqrt{k}} H_k F_n^*$ we have $f(U_k, U_k^*) = f(I, I)$ for all $f \in \mathcal{F}$.

We recall the definition of a direction of convergence for $U_k \to I$, from [10]:

Definition 5.1. Let $\{U_k\}_{k\geq 0}$ be a sequence of unitary matrices in $\mathrm{M_n}(\mathbb{C})$ such that $U_k \to I$ as $k \to \infty$ and $U_k \neq I$ for all k. We say that $a \in \mathrm{M_n}(\mathbb{C})$ is a *direction of convergence* of $\{U_k\}$ if a subsequence of $a_k = \dfrac{U_k - I}{\|U_k - I\|}$ converges to a.

Note that, since $\|a_k\| = 1$ for all k, by a compactness argument we are guaranteed that some subsequence of a_k converges. We will replace the original sequence of unitaries U_k by the corresponding subsequence, so we may assume $a_k = \frac{U_k - I}{\|U_k - I\|}$ converges to some matrix a.

Let $t_k = \|U_k - I\| \to 0$. Rearranging $a_k = \dfrac{U_k - I}{t_k}$, we have $U_k = t_k a_k + I$ and applying Proposition 5.1 we must have for all $f \in \mathcal{F}$ that

$$f(t_k a_k + I, t_k a_k^* + I) = f(I, I).$$

Expanding this relation becomes

$$t_k^2 f(a_k, a_k^*) + t_k(f(I, a_k^*) + f(a_k, I)) = 0.$$

Dividing both sides by t_k and taking a limit as $k \to \infty$ we see that a must satisfy

$$f(a, I) + f(I, a^*) = 0.$$

For $f = f_0$ the condition above becomes $a + a^* = 0$. For $f = f^{p,q}$ the condition above becomes $\tau([p, q]a) = 0$. Hence overall the condition $f(a, I) + f(I, a^*) = 0$ for all $f \in \mathcal{F}$ is equivalent to saying that $a \perp [\mathcal{C}_n, \mathcal{D}_n]$ and $a + a^* = 0$. In other words, we showed:

Proposition 5.2. Let $(H_k)_{k\geq 1}$ be a sequence of $n \times n$ complex Hadamard matrices approaching $\sqrt{n}F_n$, and such that $H_k \neq \sqrt{n}F_n$ for all k. Let $U_k = \frac{1}{\sqrt{k}}H_kF_n^*$. If a is a direction of convergence of $\{U_k\}$, then a satisfies the first order relations (i.e. $a \in \mathcal{A}_n$).

We now move on to showing the second order relations. Denote

$$b_k = \frac{\frac{U_k - I}{t_k} - a}{t_k} = \frac{a_k - a}{t_k}.$$

In contrast to the first order relations, we now have no control over $\|b_k\|$, and we cannot deduce that b_k has a convergent subsequence. However, we know that $t_kb_k \to 0$ since $a_k \to a$. Rearranging the equation defining b_k gives $t_k^2b_k + t_ka + I = U_k$, and by applying Proposition 5.1 we obtain:

$$f(t_k^2b_k + t_ka + I, t_k^2b_k^* + t_ka^* + I) = f(I, I) \text{ for all } f \in \mathcal{F}.$$

Expanding this relation, we have

$$t_k^4f(b_k, b_k^*) + t_k^3(f(b_k, a^*) + f(a, b_k^*)) + t_k^2(f(b_k, I) + f(I, b_k^*) + f(a, a^*)) + t_k(f(a, I) + f(I, a^*)) = 0.$$

From Proposition 5.2, we have that $f(a, I) + f(I, a^*) = 0$, which cancels out the last two terms of the equation above. Dividing the rest of the equation by t_k^2, letting $k \to \infty$, and using $t_kb_k \to 0$, we obtain:

$$\lim_{k\to\infty} (f(b_k, I) + f(I, b_k^*)) = -f(a, a^*) \text{ for all } f \in \mathcal{F}.$$

If we let $f = f_0$ (recall that $f_0(x, y) = xy$), we get

$$\lim_{k\to\infty} (b_k + b_k^*) = -aa^*.$$

If we let $f = f^{p,q}$ (recall that $f^{p,q}(x, y) = \tau(xpyq)$) we have, for all $p \in \mathcal{C}_n$, $q \in \mathcal{D}_n$:

$$\lim_{k\to\infty} (\tau(b_kpq) + \tau(pb_k^*q)) = -f(a, a^*), \text{ or equivalently } \lim_{k\to\infty} (\tau(b_k(pq - qp)) + \tau(p(b_k + b_k^*)q)) = -\tau(apa^*q).$$

After using $\lim_{k\to\infty}(b_k + b_k^*) = -aa^*$ in the relation above we obtain, for all $x \in [\mathcal{C}_n, \mathcal{D}_n]$:

$$\lim_{k\to\infty} \tau(b_k[p, q]) = \tau([p, a]a^*q).$$

We can uniquely write $b_k = b'_k + b''_k$ with $b'_k \in [\mathcal{C}_n, \mathcal{D}_n]$ and $b''_k \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$. The relation above becomes:

$$\lim_{k\to\infty} \tau(b'_k[p,q]) = \tau([p,a]a^*q).$$

Since these limits exist for any $p \in \mathcal{C}_n, q \in \mathcal{D}_n$, it follows that $\tau(b'_k z)$ converges for all $z \in [\mathcal{C}_n, \mathcal{D}_n]$. Hence b'_k converges weakly to some b'. Since we are working in a finite dimensional Hilbert space, we also have that $b'_k \to b'$ in norm. Thus there exists $b' \in [\mathcal{C}_n, \mathcal{D}_n]$ such that for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$

$$\tau\left(b'[p,q]\right) = \tau([p,a]a^*q) = \lim_{k\to\infty} \tau(b'_k[p,q]).$$

Note this says that a satisfies the second order trace relations together with b'. If we set

$$b = b' - \frac{b' + b'^* + aa^*}{2}$$

then a satisfies the second order trace relations together with b (since $b'+b'^*+aa^* \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$), as well as the rest of the second order relations (since $b + b^* = -aa^*$). We thus proved:

Proposition 5.3. Let $(H_k)_{k\geq 1}$ be a sequence of $n \times n$ complex Hadamard matrices approaching $\sqrt{n}F_n$, and such that $H_k \neq \sqrt{n}F_n$ for all k. Let $U_k = \frac{1}{\sqrt{k}} H_k F_n^*$. If a is a direction of convergence of $\{U_k\}$, then a satisfies the second order relations.

We continue this line of reasoning to show that the direction of convergence a must satisfy the third order relations.

We already established that a satisfies the second order relations, i.e. $a \in \mathcal{A}_n$ and there exists $b \in \mathrm{M_n}(\mathbb{C})$ such that, for all $f \in \mathcal{F}$,

$$f(b,I) + f(I,b^*) = -f(a,a^*).$$

Moreover, the matrix b we constructed can be written as $b = b' + b''$, with $b' \in [\mathcal{C}_n, \mathcal{D}_n]$ as found in the previous argument, and $b'' = -\frac{b'+b'^*+aa^*}{2} \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$.

Recall that we showed $b'_k \to b'$, but it is not clear if b_k converges (or even that it has a convergent subsequence).

Denote

$$c_k = \frac{\frac{\frac{U_k - I}{t_k} - a}{t_k} - b}{t_k} = \frac{\frac{a_k - a}{t_k} - b}{t_k} = \frac{b_k - b}{t_k}$$

which after rearranging gives

$$U_k = c_k t_k^3 + t_k^2 b + t_k a + I.$$

We have $t_k c_k = b_k - b$. Thus $t_k^2 c_k = t_k(b_k - b) = t_k b_k - t_k b \to 0$, as $t_k b_k = a_k - a \to 0$ and $t_k \to 0$.

We use again that for all $f \in \mathcal{F}$ we have

$$f(U_k, U_k^*) = f(I,I)$$

We expand this relation by using $U_k = c_k t_k^3 + t_k^2 b + t_k a + I$ and grouping terms together based on the powers of t_k. Since $f(b,I) + f(I,b^*) = -f(a,a^*)$ and $f(a,I) + f(I,a^*) = 0$, the terms corresponding to t_k and t_k^2 vanish, as does $f(I,I)$ from both sides of the equality.

After dividing the remaining equality by t_k^3 we obtain:

$$\begin{aligned} t_k^3 f(c_k, c_k^*) + t_k^2(f(b,c_k^*) + f(c_k,b^*)) + t_k(f(c_k,a^*) + f(a,c_k^*) + f(b,b^*)) + f(c_k,I) + f(I,c_k^*) \\ + f(a,b^*) + f(b,a^*) = 0. \end{aligned} \tag{1}$$

If we let $f = f_0$ in this relation (recall $f_0(x,y) = xy$), and isolate $c_k^* = f_0(I, c_k^*)$, we have

$$c_k^* = -c_k - ab^* - ba^* - t_k(c_k a^* + a c_k^* + bb^*) - t_k^2(b c_k^* + c_k b_k^*) - t_k^3 c_k c_k^*.$$

By applying $f(I,\cdot)$ to this equality, for some $f \in \mathcal{F}$, we have:

$$\begin{aligned} f(I,c_k^*) = &- f(I,c_k) - f(I,ab^*) - f(I,ba^*) - t_k(f(I,c_k a^*) + f(I,ac_k^*) + f(I,bb^*)) - t_k^2(f(I,bc_k^*) \\ &+ f(I,c_k b_k^*)) - t_k^3 f(c_k,c_k^*). \end{aligned}$$

We now substitute this expression for $f(I,c_k^*)$ in (1), and group terms together by powers of t_k:

$$\begin{aligned} &f(I,ab^*) - f(a,b^*) + f(I,ba^*) - f(b,a^*) = \\ &\ t_k^3(f(c_k,c_k^*) - f(I,c_k c_k^*)) \\ &+ t_k^2(f(b,c_k^*) - f(I,bc_k^*) + f(c_k,b^*) - f(I,c_k b^*)) \\ &+ t_k(f(c_k,a^*) - f(I,c_k a^*) + f(a,c_k^*) - f(I,ac_k^*) + f(b,b^*) - f(I,bb^*)) \\ &+ f(c_k,I) - f(I,c_k), \end{aligned}$$

Taking the limit as $k \to \infty$, after using $t_k^2 c_k \to 0$ and $t_k \to 0$, we obtain:

$$\begin{aligned} &f(I,ab^*) - f(a,b^*) + f(I,ba^*) - f(b,a^*) = \lim_{k\to\infty} (t_k^3(f(c_k,c_k^*) - f(I,c_k c_k^*)) + \\ &t_k(f(c_k,a^*) - f(I,c_k a^*) + f(a,c_k^*) - f(I,ac_k^*)) + f(c_k,I) - f(I,c_k)). \end{aligned}$$

We can uniquely decompose each $c_k = c_k' + c_k''$ with $c_k' \in [\mathcal{C}_n, \mathcal{D}_n]$ and $c_k'' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$. Since $t_k c_k' + t_k c_k'' = t_k c_k = b_k - b = (b_k' - b') + (b_k'' - b'')$ with $(b_k' - b'), t_k c_k' \in [\mathcal{C}_n, \mathcal{D}_n]$ and $t_k c_k'', (b_k'' - b'') \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, it follows $t_k c_k' = b_k' - b'$. In particular $t_k c_k' \to 0$. By writing $c_k = c_k' + c_k''$ and using $t_k c_k' \to 0$, $t_k^2 c_k \to 0$, the previous limit simplifies to

$$\begin{aligned} &f(I,ab^*) - f(a,b^*) + f(I,ba^*) - f(b,a^*) = \lim_{k\to\infty} t_k^3(f(c_k'', c_k''^*) - f(I, c_k'' c_k''^*)) + \\ &t_k(f(c_k'',a^*) - f(I,c_k'' a^*) + f(a,c_k''^*) - f(I,ac_k''^*)) + f(c_k,I) - f(I,c_k). \end{aligned} \tag{2}$$

We now let $f = f^{p,q}$ in (2). Note that $f^{p,q}(x,y) - f^{p,q}(I,xy) = \tau(xpyq) - \tau(pxyq) = -\tau([p,x]yq)$, which allows us to rewrite each of the differences $f(x,y) - f(I,xy)$ in an easier form for $f = f^{p,q}$.

The left side of the equality from (2) becomes (for $f = f^{p,q}$):

$$f(I, ab^*) - f(a, b^*) + f(I, ba^*) - f(b, a^*) = \tau(pab^*q) - \tau(apb^*q) + \tau(pba^*q) - \tau(bpa^*q)$$
$$= \tau([p, a]b^*q) + \tau([p, b]a^*q). \tag{3}$$

and the right side of the equality from (2) becomes for ($f = f^{p,q}$):

$$\lim_{k\to\infty} t_k^3(f(c_k'', c_k''^*) - f(I, c_k''c_k''^*)) + t_k(f(c_k'', a^*) - f(I, c_k''a^*) + f(a, c_k''^*) - f(I, ac_k''^*)) + f(c_k, I) - f(I, c_k)$$
$$= \lim_{k\to\infty} - t_k^3\tau([p, c_k'']c_k''^*q) - t_k(\tau([p, c_k'']a^*q) + \tau([p, a]c_k''^*q)) + \tau(c_k[p, q]). \tag{4}$$

To further simplify (4), recall that $c_k'', c_k''^*, a, a^* \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, and Proposition 4.1 yields:

$$\tau(\varphi_2(c_k'', c_k'')[p, q]) = \tau([p, c_k'']c_k''^*q)$$
$$\tau(\varphi_2(c_k'', a^*)[p, q]) = \tau([p, c_k'']a^*q)$$
$$\tau(\varphi_2(a, c_k''^*)[p, q]) = \tau([p, a]c_k''^*q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. If we set $\tilde{c_k} = c_k - t_k^3\varphi_2(c_k'', c_k'') - t_k\varphi_2(c_k'', a^*) - t_k\varphi_2(a, c_k''^*)$, (4) becomes:

$$\lim_{k\to\infty} t_k^3(f(c_k'', c_k''^*) - f(I, c_k''c_k''^*)) + t_k(f(c_k'', a^*) - f(I, c_k''a^*) + f(a, c_k''^*) - f(I, ac_k''^*)) + f(c_k, I) - f(I, c_k)$$
$$= \lim_{k\to\infty} - t_k^3\tau([p, c_k'']c_k''^*q) - t_k(\tau([p, c_k'']a^*q) + \tau([p, a]c_k''^*q)) + \tau(c_k[p, q])$$
$$= \lim_{k\to\infty} - t_k^3\tau(\varphi_2(c_k'', c_k'')[p, q]) - t_k(\tau(\varphi_2(c_k'', a^*)[p, q]) + \tau(\varphi_2(a, c_k''^*)[p, q]) + \tau(c_k[p, q])$$
$$= \lim_{k\to\infty} \tau((c_k - t_k^3\varphi_2(c_k'', c_k'') - t_k\varphi_2(c_k'', a^*) - t_k\varphi_2(a, c_k''^*))[p, q])$$
$$= \lim_{k\to\infty} \tau(\tilde{c_k}[p, q]).$$

Combining (3) and (4) via the equality from (2), we obtain:

$$\lim_{k\to\infty} \tau(\tilde{c_k}[p, q]) = \tau([p, a]b^*q) + \tau([p, b]a^*q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

We can uniquely write $\tilde{c_k} = \tilde{c_k}' + \tilde{c_k}''$ with $\tilde{c_k}' \in [\mathcal{C}_n, \mathcal{D}_n]$ and $\tilde{c_k}'' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$. The limit above becomes:

$$\lim_{k\to\infty} \tau(\tilde{c_k}'[p, q]) = \tau([p, a]a^*q).$$

Since $\lim_{k\to\infty} \tau(\tilde{c_k}'[p, q])$ exists for any $p \in \mathcal{C}_n, q \in \mathcal{D}_n$, it follows that $\tau(\tilde{c_k}'z)$ converges for all $z \in [\mathcal{C}_n, \mathcal{D}_n]$. Hence $\tilde{c_k}'$ converges weakly to some c. Since we are working in a finite dimensional Hilbert space, we also have that $\tilde{c_k}' \to c$ in norm. Thus there exists $c \in [\mathcal{C}_n, \mathcal{D}_n]$ such that for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$

$$\tau(c[p, q]) = \tau([p, a]b^*q) + \tau([p, b]a^*q).$$

In other words, we showed that a (together with b, c) satisfies the third order trace relations. By Proposition 4.3, it follows that a satisfies the third order relations. This gives us the main result of this section:

Theorem 5.1. Let $(H_k)_{k\geq 1}$ be a sequence of $n \times n$ complex Hadamard matrices approaching $\sqrt{n}F_n$, and such that $H_k \neq \sqrt{n}F_n$ for all k. Let $U_k = \frac{1}{\sqrt{k}}H_kF_n^*$. If a is a direction of convergence of $\{U_k\}$, then a satisfies the third order relations.

6 On the structure of $\mathcal{H}(n)$ for $n = 30$

In this section we show that, for $n = 30$, not every norm one hermitian a in the enveloping tangent space $\tilde{\mathrm{T}}_{\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ is a direction of convergence of a sequential family of complex Hadamard matrices approaching $\sqrt{n}F_n$. From here we deduce that the dimension of any differentiable family of complex Hadamard matrices containing $\sqrt{n}F_n$ (with $n = 30$) is *strictly less* than the dimension of the tangent space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ (i.e., the defect of F_n). This is quite surprising, considering that for every n the space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ admits a basis of directions of convergence for one-parameter analytic families of complex Hadamard matrices (see [14]).

In other words, for $n = 30$ there exist $d(F_{30}) = 135$ independent one-parameter families of Hadamard deformations of $\sqrt{30}F_{30}$ that cannot be "joined" into a 135-dimensional family of Hadamard deformations of $\sqrt{30}F_{30}$.

We note that 30 is the smallest integer with three distinct prime divisors. Based on numerical evidence for $n < 100$ from [3], it seems likely that the same might be true more generally for any n with three distinct prime divisors. Note that for $n = 6$, which only has two distinct prime divisors, the dephased defect of F_6 is equal to 4 and there exists a 4-dimensional smooth family of Hadamard matrices through $\sqrt{6}F_6$ (see [20]).

The next Theorem is the main result of this section: we construct an element $a \in \mathcal{A}_{30}$ (i.e., a satisfies the first order relations, hence also the second order relations) which does not satisfy the third order relations.

Theorem 6.1. Let $n = 30$. Using the notations from Lemma 4.1, let

$$a = q_3p_{10} + q_{10}p_3 + q_{15}p_2 - (q_{-3}p_{-10} + q_{-10}p_{-3} + q_{-15}p_{-2}).$$

Then $a \in \mathcal{A}_n$, $a \neq 0$, and a does not satisfy the third order relations. Thus $\frac{a}{\|a\|} \in \tilde{\mathrm{T}}_{\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ is a norm one element which is not a direction of convergence of a sequential family of complex Hadamard matrices approaching $\sqrt{n}F_n$.

Proof. From Lemma 4.1 it follows that $p_iq_j = q_jp_i$ whenever $ij = 0 \bmod n$. In particular, $(q_jp_i)^* = p_i^*q_j^* = p_{-i}q_{-j} = q_{-j}p_{-i}$. Moreover, from Lemma 4.2 we have $q_jp_i \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ whenever $ij = 0 \bmod n$. Since

$$3 \cdot 10 = 10 \cdot 3 = 15 \cdot 2 = (-3) \cdot (-10) = (-10) \cdot (-3) = (-15) \cdot (-2) = 0 \bmod 30$$

it follows that $a = -a^*$, and $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$.

Since $a \in \mathcal{A}_n$, a satisfies the first order relations, hence it also satisfies the second order relations. That is to say, there exists $b \in M_n(\mathbb{C})$ such that

$$b + b^* = a^2 \ \text{ and } \ \tau(b[p, q]) = \tau([p, a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Assume, by contradiction, that a satisfies the third order relations. Thus there exists $c \in M_n(\mathbb{C})$ such that, for all $p \in \mathcal{C}_n$ and $d \in \mathcal{D}_n$,

$$\tau(c[p, q]) = \tau([p, a]b^*q) + \tau([p, b]a^*q)$$

or equivalently (using $a^* = -a$):

$$\tau(c[p,q]) = \tau([p,a]b^*q) + \tau([b,p]aq)$$

Let $i, j \in \mathbb{Z}_n$ with $ij = 0 \mod n$. Since $[p_i, q_j] = 0$, the previous equation becomes

$$\tau([p_i,a]b^*q_j) + \tau([b,p_i]aq_j) = 0.$$

However, we will show that for $i = 15$ and $j = 2$ we have $\tau([p_i,a]b^*q_j) + \tau([p_i,b]a^*q_j) \neq 0$, thus obtaining a contradiction.

In order to make this computation easier to follow, we write a as

$$a = \sum_{xx'=0} \alpha_{x,x'} q_x p_{x'}$$

where

$$\alpha_{x,x'} = \begin{cases} 1 & \text{if } (x,x') \in \{(3,10),(10,3),(15,2)\} \\ -1 & \text{if } (x,x') \in \{(-3,-10),(-10,-3),(-15,-2)\} \\ 0 & \text{otherwise} \end{cases}$$

We use the second order relations, which a satisfies together with b, to rewrite $\tau([p_i,a]b^*q_j)$ and $\tau([p_i,b]a^*q_j)$ in terms of a only. We compute these two terms separately. In the following computations we use $xx' = 0 \mod 30$ and $ij = 15 \cdot 2 = 0 \mod 30$, hence $q_x p_{x'} = p_{x'} q_x$ and $q_i p_j = p_j q_i$.

We first rewrite $\tau([p_i,a]b^*q_j)$:

$$\begin{aligned}
\tau([p_i,a]b^*q_j) &= \tau(b^*q_jp_ia - b^*q_jap_i) \\
&= \sum_{xx'=0} \alpha_{x,x'} \tau(b^*(q_jp_iq_xp_{x'} - q_jq_xp_{x'}p_i)) \\
&= \sum_{xx'=0} \alpha_{x,x'} (\varepsilon^{jx'} - \varepsilon^{(j+x)(i+x')}) \tau(b^*p_{i+x'}q_{j+x}) \\
&= \sum_{xx'=0} \alpha_{x,x'} \varepsilon^{jx'} (1 - \varepsilon^{ix}) \tau(b^*p_{i+x'}q_{j+x}).
\end{aligned}$$

Whenever $(j+x)(i+x') = 0 \mod n$, from Lemma 4.3 we have that $ix = jx' = 0 \mod n$, so we may remove the corresponding terms from our sum. The formula for $\tau([p_i,a]b^*q_j)$

becomes

$$\begin{aligned}
\tau([p_i,a]b^*q_j) &= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0}} \frac{\alpha_{x,x'}\varepsilon^{jx'}(1-\varepsilon^{ix})}{1-\varepsilon^{(j+x)(i+x')}}\tau(b^*[p_{i+x'},q_{j+x}])\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0}} \frac{\alpha_{x,x'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\overline{\tau(b[p_{-(i+x')},q_{-(j+x)}])}\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0}} \frac{\alpha_{x,x'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\overline{\tau([a,p_{-(i+x')}]aq_{-(j+x)})}\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0}} \frac{\alpha_{x,x'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\tau(q_{j+x}ap_{i+x'}a-q_{j+x}aap_{i+x'})\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0}} \frac{\alpha_{x,x'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\tau(aq_{j+x}ap_{i+x'}-a^2p_{i+x'}q_{j+x}))\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\tau(aq_{j+x}q_yp_{y'}p_{i+x'}-aq_yp_{y'}p_{i+x'}q_{j+x})\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\Big(\tau(aq_{j+x}q_yp_{y'}p_{i+x'})-\varepsilon^{y(i+x')}\tau(ap_{y'}p_{i+x'}q_yq_{j+x})\Big)\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\Big(\tau(aq_{j+x+y}p_{i+x'+y'})-\varepsilon^{y(i+x')}\tau(ap_{i+x'+y'}q_{j+x+y})\Big)\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)}{1-\varepsilon^{(j+x)(i+x')}}\Big(\tau(ap_{i+x'+y'}q_{j+x+y})-\varepsilon^{y(i+x')}\tau(ap_{i+x'+y'}q_{j+x+y})\Big)\\
&= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)(1-\varepsilon^{y(i+x')})}{1-\varepsilon^{(j+x)(i+x')}}\tau(ap_{i+x'+y'}q_{j+x+y}).
\end{aligned}$$

In the chain of equalities above we used that $\tau(aq_{j+x+y}p_{i+x'+y'})=\tau(ap_{i+x'+y'}q_{j+x+y})$. Let us explain why this is true. Consider first the case $(i+x'+y')(j+x+y)\neq 0 \mod n$. Then $p_{i+x'+y'}q_{j+x+y}, q_{j+x+y}p_{i+x'+y'}\in[\mathcal{C}_n,\mathcal{D}_n]$, as shown in Lemma 4.2. Since $a\in[\mathcal{C}_n,\mathcal{D}_n]^\perp$, it follows that $\tau(ap_{i+x'+y'}q_{j+x+y})=0=\tau(aq_{j+x+y}p_{i+x'+y'})$. Now consider the case $(i+x'+y')(j+x+y)=0 \mod n$. In this case, $p_{i+x'+y'}$ and q_{j+x+y} commute, so again we have

$\tau(ap_{i+x'+y'}q_{j+x+y}) = 0 = \tau(aq_{j+x+y}p_{i+x'+y'})$.

This argument also shows that in the sum above the only possibly nonzero terms are the terms with $(i+x'+y')(j+x+y) = 0 \mod n$. In this case we have

$$\begin{aligned}\tau(ap_{i+x'+y'}q_{j+x+y}) &= \sum_{zz'=0} \alpha_{z,z'}\tau(q_z p_{z'} p_{i+x'+y'} q_{j+x+y}) \\ &= \sum_{zz'=0} \alpha_{z,z'}\tau(p_{i+x'+y'+z'})\tau(q_{j+x+y+z}) \\ &= \alpha_{-(j+x+y),-(i+x'+y')}.\end{aligned}$$

Using this fact, the formula for $\tau([p_i,a]b^*q_j)$ becomes

$$\begin{aligned}\tau([p_i,a]b^*q_j) &= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\varepsilon^{jx'}(\varepsilon^{ix}-1)(1-\varepsilon^{y(i+x')})}{1-\varepsilon^{(j+x)(i+x')}}\tau(ap_{i+x'+y'}q_{j+x+y}) \\ &= \sum_{\substack{xx'=0\\(j+x)(i+x')\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\alpha_{-(j+x+y),-(i+x'+y')}\varepsilon^{jx'}(\varepsilon^{ix}-1)(1-\varepsilon^{y(i+x')})}{1-\varepsilon^{(j+x)(i+x')}}.\end{aligned}$$

We now compute $\tau([b, p_i]aq_j)$ in a similar way:

$$\begin{aligned}
\tau([b, p_i]aq_j) &= \tau(bp_iaq_j - baq_jp_i) \\
&= \sum_{xx'=0} \alpha_{x,x'}\tau(b(p_iq_xp_{x'}q_j - q_xp_{x'}q_jp_i)) \\
&= \sum_{xx'=0} \alpha_{x,x'}(1-\varepsilon^{ix})\tau(bp_{i+x'}q_{x+j}) \\
&= \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0}} \frac{\alpha_{x,x'}(1-\varepsilon^{ix})}{1-\varepsilon^{(j+x)(i+x')}}\tau(b[p_{i+x'}, q_{j+x}]) \\
&= \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0}} \frac{\alpha_{x,x'}(1-\varepsilon^{ix})}{1-\varepsilon^{(j+x)(i+x')}}\tau([a, p_{i+x'}]aq_{j+x}) \\
&= \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}(1-\varepsilon^{ix})}{1-\varepsilon^{(j+x)(i+x')}}\tau(aq_{j+x}q_yp_{y'}p_{i+x'} - aq_{j+x}p_{i+x'}q_yp_{y'}) \\
&= \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}(1-\varepsilon^{ix})}{1-\varepsilon^{(j+x)(i+x')}}(1-\varepsilon^{-y(i+x')})\tau(ap_{j+x+y}q_{i+x'+y'}) \\
&= \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0\\yy'=0}} \frac{\alpha_{x,x'}\alpha_{y,y'}\alpha_{-(j+x+y),-(i+x'+y')}(1-\varepsilon^{ix})(1-\varepsilon^{-y(i+x')})}{1-\varepsilon^{(j+x)(i+x')}}.
\end{aligned}$$

For ease of notation, for the values $i = 15$ and $j = 2$ that we will work with, denote

$$\xi(x, x', y, y') = \alpha_{x,x'}\alpha_{y,y'}\alpha_{-(j+x+y),-(i+x'+y')}.$$

Now combine the two terms of the 3rd order trace relation to obtain

$$\tau([p_i, a]b^*q_j) + \tau([b, p_i]aq_j) = \sum_{\substack{xx'=0\\(i+x')(j+x)\neq 0\\yy'=0}} \frac{\xi(x, x', y, y')(\varepsilon^{ix}-1)\left(\varepsilon^{jx'}(1-\varepsilon^{y(i+x')}) - (1-\varepsilon^{-y(i+x')})\right)}{1-\varepsilon^{(j+x)(i+x')}}$$

To simplify this expression, we note that, for a term in the sum to be non-zero, we must have:

$$(i + x' + y')(j + x + y) = ix + jx' + iy + jy' + xy' + yx' = 0 \mod n.$$

Using this fact, we can simplify:

$$\begin{aligned}
\varepsilon^{jx'}(1-\varepsilon^{y(i+x')}) - (1-\varepsilon^{-y(i+x')}) &= \varepsilon^{jx'}\left(1-\varepsilon^{y(i+x')} - \varepsilon^{-jx'} + \varepsilon^{-y(i+x')-jx'}\right) \\
&= \varepsilon^{jx'}\left(1-\varepsilon^{yi+yx'} - \varepsilon^{yi+yx'+ix+jy'+xy'} + \varepsilon^{ix+jy'+xy'}\right) \\
&= \varepsilon^{jx'}\left(\left(1-\varepsilon^{yi+yx'}\right) + \varepsilon^{ix+jy'+xy'}\left(1-\varepsilon^{yi+yx'}\right)\right) \\
&= \varepsilon^{jx'}\left(1-\varepsilon^{yi+yx'}\right)\left(1+\varepsilon^{ix+jy'+xy'}\right) \\
&= \varepsilon^{jx'}\left(1-\varepsilon^{y(i+x)'}\right)\left(1+\varepsilon^{(i+y')(j+x)}\right)
\end{aligned}$$

This shows that

$$\tau([p_i,a]b^*q_j) + \tau([b,p_i]aq_j) = \sum_{\substack{xx'=0 \\ (i+x')(j+x)\neq 0 \\ yy'=0}} \frac{\xi(x,x',y,y')\varepsilon^{jx'}(\varepsilon^{ix}-1)(1-\varepsilon^{y(i+x')})(1+\varepsilon^{(i+y')(j+x)})}{1-\varepsilon^{(j+x)(i+x')}}.$$

Note that, for any $xx'=0 \mod n$ and $yy'=0 \mod n$, the product

$$\xi(x,x',y,y') = \alpha_{x,x'}\alpha_{y,y'}\alpha_{-(2+x+y),-(15+x'+y')}$$

is non-zero only if

$(x,x'),(y,y'),(-(2+x+y),-(15+x'+y'))$ are a permutation of$(3,10),(10,3),(15,2)$.

This reduces the sum to 6 terms:

$$\begin{aligned}
\tau([p_{15},a]b^*q_2) + \tau([p_{15},b]a^*q_2) &= \frac{\varepsilon^{20}(1-\varepsilon^{15})(\varepsilon^{0}+1)(\varepsilon^{10}-1)}{1-\varepsilon^{5}} \\
&+ \frac{\varepsilon^{20}(1-\varepsilon^{15})(\varepsilon^{25}+1)(\varepsilon^{15}-1)}{1-\varepsilon^{5}} \\
&+ \frac{\varepsilon^{4}(1-\varepsilon^{15})(\varepsilon^{6}+1)(\varepsilon^{20}-1)}{1-\varepsilon^{19}} \\
&+ \frac{\varepsilon^{4}(1-\varepsilon^{15})(\varepsilon^{5}+1)(\varepsilon^{21}-1)}{1-\varepsilon^{19}} \\
&+ 0 \\
&+ 0 \\
&= -6i\sqrt{35+13\sqrt{5}-\sqrt{30(65+29\sqrt{5})}}.
\end{aligned}$$

Since this is nonzero, it follows that the a we constructed does not satisfy the third order relations. ■

We conclude with a corollary showing that, for $n = 30$, the dimension of any differentiable family of complex Hadamard matrices containing $\sqrt{n}F_n$ is *strictly less* than the dimension of the enveloping tangent space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ (i.e., the defect of F_n). This is particularly surprising since the space $\tilde{\mathrm{T}}_{\sqrt{\mathrm{n}}\mathrm{F_n}}\mathcal{H}(\mathrm{n})$ admits a basis of directions of convergence for one-parameter analytic families of complex Hadamard matrices (see [14]).

In other words, the $d(F_{30}) = 135$ independent one-parameter families of Hadamard deformations of $\sqrt{30}F_{30}$ found in [14] cannot be "joined" into a 135-dimensional family of Hadamard deformations of $\sqrt{30}F_{30}$. Recall that $\mathcal{H}(30)$ denotes the set of complex Hadamard matrices of size 30.

Corollary 6.1. Let $J \subset \mathbb{R}$ be an open interval containing 0. There does not exist a differentiable map $f : J^{135} \to \mathcal{H}(30)$, such that $f(0, 0, ...0) = \sqrt{30}F_{30}$ and such that the partial derivatives $\partial_i f(0, 0, ..., 0)$ are linearly independent (over $\mathbb{R}$) for $1 \leq i \leq 135$.

Proof. Assume, by contradiction, that such a map f exists. Denote $g(t) = \frac{1}{\sqrt{30}} f(t)F_{30}^*$ for all $t \in J$. Then g is differentiable, unitary-valued, and $g(0, 0, ..., 0) = I_{30}$. Moreover, since $g(t, 0, 0, 0, ...)$, $g(0, t, 0, 0, ..)$, $g(0, 0, t, 0, ...)$, etc are families of unitaries satisfying the hypothesis of Proposition 5.2. (for any sequence of t's approaching 0), it follows that their directions of convergence (which are nonzero scalar multiples of $\partial_1 g(0, 0, ..., 0)$, $\partial_2 g(0, 0, ..., 0)$, $\partial_3 g(0, 0, ..., 0)$, etc) belong to $\mathcal{A}_{30}$.

Since $\partial_i f(0, 0, ..., 0)$ are linearly independent (over $\mathbb{R}$), it follows that $\partial_i g(0, 0, ..., 0)$ are linearly independent elements of $\mathcal{A}_{30}$ (over $\mathbb{R}$), for $1 \leq i \leq 135$. The dimension of the real vector space $\mathcal{A}_{30}$ is the defect of F_{30}, which is known to be 135. Hence $\partial_i g(0, 0, ..., 0)$, $1 \leq i \leq 135$, form a basis for $\mathcal{A}_{30}$.

Consider the element $a = q_3 p_{10} + q_{10} p_3 + q_{15} p_2 - (q_{-3} p_{-10} + q_{-10} p_{-3} + q_{-15} p_{-2})$ that we constructed in Theorem 6.1. Since $a \in \mathcal{A}_{30}$, there exist real numbers r_i, $1 \leq i \leq 135$, with $a = \sum_{i=1}^{135} r_i \partial_i g(0, 0, ..., 0)$.

The map $t \to g(r_1 t, r_2 t, r_3 t, ...)$ is well defined on a neighborhood of 0 included in J, and its derivative at 0 is $\sum_{i=1}^{135} r_i \partial_i g(0, 0, ..., 0) = a$. Since the unitaries $g(r_1 t, r_2 t, r_3 t, ...)$ satisfy the hypothesis of Theorem 5.1 for any sequence of t's approaching 0, and the corresponding direction of convergence is a nonzero scalar multiple of $\partial_t g(r_1 t, r_2 t, r_3 t, ...)_{|t=0} = a$, it follows that the direction of convergence must be $\frac{a}{\|a\|}$. By Theorem 5.1 we have that $\frac{a}{\|a\|}$ satisfies the third order relations, contradicting Theorem 6.1.

■

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