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ON THE FINITENESS OF THE NUMBER OF N-DIMENSIONAL HOPF C^*-ALGEBRAS

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ABSTRACT. We give a new and elementary proof of the finiteness of the number of N-dimensional Hopf C^*-algebras.

1. INTRODUCTION

In this paper we give an elementary proof of the following theorem:

Theorem 1.1 (D. Stefan). *Let A be a finite dimensional C^*-algebra. Then there are only finitely many Hopf C^*-algebra structures on A.*

The original proof, as well as the proofs of the more general Ocneanu's Theorem (A. Wasserman, E. Blanchard in [2, 1], P. Etingof, D. Nikshych, V. Ostrik in [3]), use a cohomological framework and arguments. We give an explicit proof, in the language of the standard invariant of a subfactor. Also, our arguments can be refined to give an estimate of the number of Hopf structures on the finite dimensional C^*-algebra A.

Let $N \subset M$ be an inclusion of type II_1 factors with finite Jones index, $[M : N] < \infty$. Let $N \subset M \subset M_1 \subset M_2 \subset \cdots$ be its associated tower of factors obtained by iterating the Jones basic constructions [4].

The *standard invariant* $\mathcal{G}_{N,M}$ of the subfactor $N \subset M$ is then defined as the trace preserving isomorphism class of the following sequence of commuting squares of finite dimensional C^*-algebras, together with the trace and the Jones projections $e_n \in N' \cap M_n$

$$\begin{array}{ccccccccc} N'\cap M & \subset & N'\cap M_1 & \subset & N'\cap M_2 & \subset & N'\cap M_3 & \subset & \cdots \\ \cup & & \cup & & \cup & & \cup & & \\ M'\cap M & \subset & M'\cap M_1 & \subset & M'\cap M_2 & \subset & M'\cap M_3 & \subset & \cdots \end{array}$$

The principal parts of the Bratelli diagrams describing the rows of inclusions above are called the *principal graphs* of the inclusion $N \subset M$. The subfactor $N \subset M$ is said to have *depth n* if the principal graphs are finite of length n, i.e. $N' \cap M_{n-2} \subset N' \cap M_{n-1} \subset N' \cap M_n$ is a basic construction (with the projection e_n) and so is $M' \cap M_{n-1} \subset M' \cap M_n \subset M' \cap M_{n+1}$ (with projection e_{n+1}). For such finite depth inclusions of subfactors (and more generally for amenable inclusions), the standard invariant is a complete invariant (see [10]).

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If the depth is n, the n^{th} commuting square

$$\begin{matrix} N'\cap M_{n-1} & \subset & N'\cap M_n \\ \cup & & \cup \\ M'\cap M_{n-1} & \subset & M'\cap M_n \end{matrix}$$

is called a *standard commuting square*, and uniquely determines the standard invariant and thus the inclusion $N\subset M$.

By a result of [14], the Hopf C^*-structures on an n-dimensional C^*-algebra A correspond to standard commuting squares of the form:

$$\begin{matrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ \mathbb{C} & \subset & Q_0 \end{matrix}$$

with $P_{-1}=A$. Because $\mathbb{C}\subset P_{-1}\subset P_0$ is a basic construction, it follows $P_0=M_n(\mathbb{C})$. Such a commuting square is called *irreducible*, and its depth is always 2. Thus, to prove D. Stefan's theorem it is enough to show that there are only finitely many isomorphism classes of standard irreducible commuting squares of fixed upper-left corner A (or just of fixed dimension). Here isomorphism means unitary conjugation of the commuting square.

To prove the theorem, we show that any standard irreducible commuting square is isolated (modulo isomorphisms) among all standard commuting squares. The proof goes by contradiction: assume that

$$\begin{matrix} P^t_{-1} & \subset & P^t_0 \\ \cup & & \cup \\ \mathbb{C} & \subset & Q^t_0 \end{matrix}$$

is a converging sequence of standard commuting squares. It is immediate to see that one can assume $P^t_{-1}=P_{-1}$, $Q^t_0=U^*_tQ_0U_t$ for some unitaries $U_t\in P_0$, converging to the identity. Write the standardness condition for all t's, then take the "derivative" of this relations along some direction of convergence of U_t's, to reach a contradiction.

The main technical difficulty is finding intrinsic conditions that characterize the standardness of the commuting square, which we do by using S. Popa's abstract characterization of the standard invariant of a subfactor [12].

2. Characterizations of depth 2 commuting squares

In this section we find necessary and sufficient conditions for a commuting square to be a *standard depth* 2 commuting square. We also discuss other properties of standard commuting squares.

By a theorem of S. Popa [11, 12] the standard invariant of a subfactor can be regarded as an abstract group-like object, characterized by the Jones–Markov axioms and the commutation conditions given in [12]. In the following theorem we refine these axioms for the case when the subfactor has depth 2, so it is determined by a standard depth 2

Theorem 2.1. *Let*

$$\mathfrak{C}=\begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup\,,\tau \\ Q_{-1} & \subset & Q_0 \end{pmatrix}$$

be a commuting square of finite dimensional von Neumann algebras. Then $\mathfrak{C}$ is a standard commuting square of depth two if and only if the following are satisfied:

There exists $\lambda>0$, $\lambda^{-1}\in\{4\cos^2(\pi/n)\mid n\geqslant 3\}\cup[4,\infty)$ and projections $e_1\in P_{-1}$, $e_2\in Q_0$ satisfying the Jones relations $e_1e_2e_1=\lambda e_1$, $e_2e_1e_2=\lambda e_2$, such that if $P_{-2}=\{e_2\}'\cap P_{-1}$, $e_3=e_{P_{-1}}$, $P_1=\langle P_0,e_3\rangle$ and $Q_1=\langle Q_0,e_3\rangle\subset P_1$ we have:

(i) *$e_2xe_2=E_{P_{-2}}(x)e_2$ for all $x\in P_{-1}$, $P_0e_1=Q_0e_1$,* $\dim Q_{-1}e_1=\dim Q_{-1}$;

(ii) $[P_{-2},Q_0]=0$, $\{e_1\}'\cap Q_j=P'_{-1}\cap Q_j$, $j=-1,0,1$.

Proof. Let $P_{-1}\subset P_0\subset P_1\subset P_2\subset\cdots$ be the Jones tower obtained by reiterating the basic construction, with projections $e_i\in P_{i-1}$, and define $Q_i=\langle Q_{i-1},e_i\rangle$ inside P_i. We have to show that the system (P_i,Q_j,e_i) is a standard λ-sequence of commuting squares. The Jones-Markov axioms of [12] are clearly satisfied, so we only need to check the commutation conditions.

Let $j>1$. Since $[P_{-2},Q_1]=0$, $[P_{-2},e_i]=0$ for all $i\geqslant 2$ and $Q_j=\langle Q_1,e_2,\dots,e_j\rangle$ we have $[P_{-2},Q_j]=0$ which proves the first set of commutation conditions.

To check the second set, let $R_k=\{e_1\}'\cap Q_k$ for $k\geqslant 0$. Then $R_0\subset R_1\subset R_2\subset R_3\subset\cdots$ is a Jones tower with projections $e_4,e_5,\dots$. We have to show that $\{e_1,\dots,e_{i+2}\}'\cap Q_j=P'_i\cap Q_j,\forall i,j\geqslant -1$. In fact it is enough to show $\{e_1,\dots,e_{i+2}\}'\cap Q_j\subset P'_i\cap Q_j$, since the other inclusion always holds true. According to the hypothesis, this is true for $i=-1$, $j=-1,0,1$.

If $i=-1$, $j\geqslant 2$ then $R_j=\{e_1\}'\cap Q_j=\langle R_1,e_4,\dots,e_{j-2}\rangle$ commutes with P_{-1} so $R_j\subset P'_{-1}\cap Q_j$.

If $i\geqslant 0$, $j\geqslant 2$ then

$$\begin{aligned}\{e_1,\dots,e_{i+2}\}'\cap Q_j=\{e_1\}'\cap Q_j\cap\{e_2,\dots,e_{i+2}\}'&=P'_{-1}\cap Q_j\cap\{e_2,\dots,e_{i+2}\}'\\ &=P'_i\cap Q_j,\end{aligned}$$

since $P_i=\langle P_{-1},e_2,\dots,e_{i+2}\rangle$. □

Observation 2.2. If $\mathfrak{C}$ is a spin model the conditions in the theorem above hold always true, with the exception of the commutation condition $\{e_1\}'\cap Q_1=P'_{-1}\cap Q_1$. Thus, this is the main restriction characterizing standardness of the commuting square. In the following lemmas we find further properties of the standard commuting squares, that in some sense encode this commutation condition.

Lemma 2.3. *Let*

$$\mathfrak{C}=\begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup\,,\tau \\ Q_{-1} & \subset & Q_0 \end{pmatrix}$$

be a standard commuting square of depth 2, with Jones projections $e_1\in P_{-1}$, $e_2\in Q_0$. For any $x_i,y_i\in Q_0$ for $i=1,2,\dots,k$, if $\sum_{i=1}^k x_ie_1y_i\in P_{-1}$, then $\sum_{i=1}^k x_ipy_i\in P_{-1}$ for all $p\in P$

Proof. Since $\mathfrak{C}$ is a standard depth 2 commuting square, there exists a tower of II_1 factors $N \subset M \subset M_1 \subset M_2 \subset \cdots$ such that $e_1 = e_N, e_2 = e_M$ and

$$\mathfrak{C} = \begin{pmatrix} N' \cap M_1 & \subset & N' \cap M_2 & \\ \cup & & \cup & ,\tau \\ M' \cap M_1 & \subset & M' \cap M_2 & \end{pmatrix}$$

Let $x_i, y_i \in Q_0$ satisfying $\sum_{i=1}^k x_i e_1 y_i \in P_{-1}$ and let $p \in P_{-1}$. Because $P_{-1} = N' \cap M_1$ we have $p \in M_1 = \langle M, e_1 \rangle$ so $p = \sum_{j=1}^l a_j e_1 b_j, a_j, b_j \in M$. Using that a_j, b_j commute with $x_i, y_i \in Q_0 = M' \cap M_2$ we have

$$\sum_{i=1}^n x_i p y_i = \sum_{i=1}^k \sum_{j=1}^l x_i a_j e_1 b_j y_i = \sum_{j=1}^l a_j \Big(\sum_{i=1}^k x_i e_1 y_i \Big) b_j \in M_1$$

and since $\sum_{i=1}^k x_i p y_i \in P_0$ commutes with N it follows

$$\sum_{i=1}^k x_i p y_i \in N' \cap M_1 = P_{-1}. \qquad \square$$

In the rest of this section we will restrict to irreducible commuting squares, and obtain alternative characterizations of the depth 2 condition, that are more suitable for computations. Let

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & \mathbb{M}_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & Q_0 & \end{pmatrix}$$

be an irreducible commuting square with Jones projections $e_1 \in P_{-1}, e_2 \in Q_0$, such that $\mathbb{C} \subset P_{-1} \subset \mathbb{M}_n(\mathbb{C})$ is a basic construction with projection e_2 and $\mathbb{C} \subset Q_0 \subset \mathbb{M}_n(\mathbb{C})$ is a basic construction with projection e_1. Denote $I = I_n$ the identity matrix in $\mathbb{M}_n(\mathbb{C})$ and let $\lambda^{-1} = n$. We have $E_{P_{-1}}(e_2) = \lambda I$, $E_{Q_0}(e_1) = \lambda I$ so the Jones projections e_1, e_2 are minimal projections in P_0 of trace $1/n$.

Since $\mathbb{C} \subset Q_0 \subset P_0$ is a basic construction with projection e_1, for every $p \in P_{-1}$ there exists $q \in Q_0$ such that $pe_1 = qe_1$ (q is the "pull down" of $p \in P_0$). Applying $E_{P_{-1}}$ we have $pe_1 = \tau(q)e_1$, where we used the commuting square condition

(2.1) $$E_{P_{-1}}(q) = E_{P_{-1}}(E_{Q_0}(q)) = E_{\mathbb{C}}(q) = \tau(q)I.$$

It follows $e_1 p e_1 = \tau(q) e_1 = pe_1$, so p and e_1 commute. With a similar argument for e_2, we have

$$e_1 \in Z(P_{-1}), \quad e_2 \in Z(Q_0).$$

We can realize the subalgebra P_{-1} of $M_n(\mathbb{C})$ as $P_{-1} = \bigoplus_i M_{r_i}(\mathbb{C})$, where $\sum_i r_i^2 = n$. To simplify computations, we will abuse this notation and just write $P_{-1} = \bigoplus_r M_r(\mathbb{C})$. Consider $(p^r_{kl})_{1 \leqslant k,l \leqslant r}$ matrix units for $M_r(\mathbb{C})$, so $p^r_{kl} p^s_{ij} = \delta^s_r \delta^i_l p^r_{kj}$. For each $1 \leqslant k, l \leqslant r$ let $q^r_{kl} \in Q_0$ be the unique element satisfying

(2.2) $$p^r_{kl} e_2 e_1 = q^r_{kl} e_1.$$

Multiplying by e_2 to the right and using $e_2 e_1 e_2 = (1/n) e_1$, we also have

(2.3)

And after projecting (2.2) on Q_0

(2.4) $$q^r_{kl} = n E_{Q_0}(p^r_{kl} e_2 e_1).$$

From (2.3) it easily follows that $(q^r_{kl})_{k,l,r}$ is an orthogonal basis of Q_0. We will call $(q^r_{kl})_{k,l,r}$ the *dual basis* of $(p^r_{kl})_{k,l,r}$ with respect to the commuting square $\mathfrak{C}$.

In the next theorem we give an intrinsic characterization of the irreducible standard commuting squares.

Theorem 2.4. *Let*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & \mathbb{M}_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & Q_0 & \end{pmatrix}$$

be an irreducible commuting square. Then $\mathfrak{C}$ is a standard commuting square if and only if exist Jones projections $e_1 \in P_{-1}, e_2 \in Q_0$, such that $\mathbb{C} \subset P_{-1} \subset \mathbb{M}_n(\mathbb{C})$ is a basic construction with projection e_2 and $\mathbb{C} \subset Q_0 \subset \mathbb{M}_n(\mathbb{C})$ is a basic construction with projection e_1, $P'_{-1} \cap Q_0 = \mathbb{C}$, and if $(p^r_{kl})_{k,l,r}$ is a system of matrix units in P_{-1}, $(q^r_{kl})_{k,l,r}$ its dual basis in Q_0, and $V = \sum_{k,l,r} \frac{n}{r} q^r_{kl} \otimes p^r_{kl} \in Q_0 \otimes P_{-1}$, then V is a unitary operator satisfying

(2.5) $$V(P_{-1} \otimes I) V^* \in P_{-1} \otimes P_{-1}.$$

Proof. We first prove the left to right implication. Assume $\mathfrak{C}$ is a standard commuting square. Since $Q_{-1} = \mathbb{C}$ it follows that $\mathfrak{C}$ has depth 2.

Let $(p^r_{kl})_{k,l,r}$ be a system of matrix units for P_{-1} and $(q^r_{kl})_{k,l,r}$ its dual basis. Since $\tau(p^r_{kl}) = \delta^l_k (r/n)$ so $(\sqrt{n/r}\, p^r_{kl})_{k,l,r}$ is an orthonormal basis of $\mathbb{C} \subset P_{-1}$, and $\mathbb{C} \subset P_{-1} \subset P_0$ is a basic construction with projection e_2, we have $\sum_{i,j,s} (n/s) p^s_{ij} e_2 (p^s_{ij})^* = 1 \Rightarrow \sum_{i,j,s} (n/s) p^s_{ij} e_2 p^s_{ji} = 1 \Rightarrow$

$$p^r_{kl} = p^r_{kl} \sum_{i,j,s} \frac{n}{s} p^s_{ij} e_1 p^s_{ji} = \sum_j \frac{n}{r} p^r_{kj} e_2 p^r_{jl},$$

so

(2.6) $$p^r_{kl} = \sum_j \frac{n}{r} p^r_{kj} e_2 p^r_{jl}.$$

From (2.2) and (2.6) it follows

$$\sum_j q^r_{kj} e_1 (q^r_{lj})^* = \sum_j p^r_{kj} e_2 e_1 e_2 (p^r_{lj})^* = \frac{1}{n} \sum_j p^r_{kj} e_2 p^r_{jl} = \frac{r}{n^2} p^r_{kl},$$

which implies

$$\sum_j q^r_{kj} e_1 (q^r_{lj})^* \in P_{-1}, \quad \forall\, k, l, r.$$

Thus, using Lemma 2.3 we have

(2.7) $$\sum_j q^r_{kj} p (q^r_{lj})^* \in P_{-1}, \quad \forall\, p \in P_{-1}$$

which shows that (2.5) is satisfied.

If we apply E_{Q_0} to the equality

$$\sum_j q^r_{kj} e_1 (q^r_{lj})^* = \frac{r}{n^2} p^r_{kl}$$

we obtain
$$\sum_j q^r_{kj}(q^r_{lj})^* = n\frac{r}{n^2}\tau(p^r_{kl})I,$$
so
$$\sum_j q^r_{kj}(q^r_{lj})^* = \delta^l_k\frac{r^2}{n^2}I$$
which shows that V is unitary.

Let's now prove the right to left implication. We need to check if conditions (i) and (ii) from Theorem 2.1 are satisfied. (i) is clearly satisfied under the given hypothesis. Let $P_{-1}\subset P_0\subset P_1$ be the basic construction with Jones projection e_3, and $Q_1=\langle Q_0,e_3\rangle\subset P_1$. All we need to show is that $P'_{-1}\cap Q_1=\{e_1\}'\cap Q_1$.

Let $p\in P_{-1}$. (2.5) implies that for every fixed k,l,r, $\sum_j q^r_{kj}p(q^r_{lj})^*\in P_{-1}$. Since $[e_3,P_{-1}]=0$ it follows
$$\sum_j q^r_{kj}p(q^r_{lj})^*e_3 = e_3\sum_j q^r_{kj}p(q^r_{lj})^*.$$
If we fix $1\leq a,b\leq r$ and multiply the previous equality by q^{r*}_{kb} to the left and q^r_{la} to the right then sum up after k,l, using the fact that V is unitary, we obtain
$$p\Big(\sum_l q^{r*}_{lb}e_3q^r_{la}\Big)=\Big(\sum_k q^{r*}_{kb}e_3q^r_{ka}\Big)p.$$
Thus $\left(\sum_j q^{r*}_{jb}e_3q^r_{ja}\right)_{a,b,r}$ are orthogonal elements of Q_1 that commute with P_{-1}, so the dimension of $P'_{-1}\cap Q_1$ is at least $n=\dim(\{e_1\}'\cap Q_1)$. Since $P'_{-1}\cap Q_1\subset\{e_1\}'\cap Q_1$ it follows $P'_{-1}\cap Q_1=\{e_1\}'\cap Q_1$ which ends the proof. □

We end this section with a reformulation of a technical lemma from [14], describing the canonical conjugations in a standard irreducible commuting square:

Lemma 2.5. *Let*
$$\begin{matrix} P_{-1} & \subset & P_0\\ \cup & & \cup\\ \mathbb{C} & \subset & Q_0\end{matrix}$$
be a standard commuting square with Jones projections e_1,e_2 and define an anti-linear multiplicative isomorphism $\Phi_1:P_0\to P_0$ by $\Phi_1(p_1e_2p_2)=p_1^*e_2p_2^*$ for $p_1,p_2\in P_{-1}$. Similarly, define $\Phi_2:P_0\to P_0$, $\Phi_2(q_1e_1q_2)=q_1^*e_1q_2^*$ for all $q_1,q_2\in Q_0$. Then
$$\Phi_1(P_{-1})=P'_{-1}\cap P_0,\Phi_1(Q_0)=Q_0,\quad \Phi_2(Q_0)=Q'_0\cap P_0,\Phi_2(P_{-1})=P_{-1}.$$

Proof. Let $N\subset M\subset M_1\subset M_2$ be an irreducible tower of II_1 factors such that $P_{-1}=N'\cap M_1, P_0=N'\cap M_2, Q_0=M'\cap M_2$ and $e_1=e_N, e_2=e_M$. Let M_1 act on $L^2(M_1,\tau)$ by left multiplication. We can realize M_2 as the basic construction of $M\subset M_1$ inside $B(L^2(M_1,\tau))$
$$N\subset M\subset M_1\subset M_2\subset B(L^2(M_1,\tau))$$
and define J
$$L^2(M_1,\tau)\to L^2(M_1,\tau),\quad J(\hat{x})=\hat{x}*$$
the modular involution, where $\hat{x}$ is the element $x\in M_1$ as regarded in the vector space $L^2(M_1,\tau)$. It is easy to check that $J^2=\mathrm{id}$, J is anti-linear isomorphism,

Denote by J_1 the restriction of J to the invariant subspace $L^2(N'\cap M_1,\tau)$. Note that J_1 can be defined just in terms of the basic construction $\mathbb{C}\subset P_{-1}\subset P_0$, by realizing P_0 as $B(L^2(P_{-1},\tau))$ and J_1 the corresponding modular involution (identifying x in P_{-1} with $\hat{x}=x\xi\in L^2(P_{-1},\tau)$, where ξ is the vector corresponding to the 1-dimensional projection e_1). We denote by $L(x),R(x)$ the operators of left/right multiplication by elements x on $L^2(P_{-1},\tau)$.

Let then $\Phi_1(x)=J_1xJ_1=R(x^*)$ for $x\in P_{-1}$, and extend it to $P_0=\operatorname{span}P_{-1}e_2P_{-1}$ by $\Phi_1(xe_2y)=x^*e_2y^*$, which will have the mentioned properties, as $xe_2=L(x)e_2=R(x)e_2=J_1x^*J_1e_2$.

Because of the symmetry of the standard invariant, the second part follows in a similar manner. □

Corollary 2.6. *Under the hypothesis from Lemma* 2.5 *the following is a standard commuting square*
$$\begin{matrix} P'_{-1}\cap P_0 & \subset & P_0\\ \cup & & \cup\\ \mathbb{C} & \subset & Q_0.\end{matrix}$$

Observation 2.7. We can write an explicit formula for $\Phi_1|Q_0$, in terms of the dual bases of P_{-1},Q_0: $p^r_{kl}e_2e_1=q^r_{kl}e_1\Rightarrow\Phi_1(p^r_{kl}e_2e_1)=\Phi_1(q^r_{kl}e_1)\Rightarrow p^r_{lk}e_2e_1=\Phi_1(q^r_{kl})e_1$, so
$$\Phi_1(q^r_{kl})=q^r_{lk}. \tag{2.8}$$

3. A technical lemma

In this section we give a technical lemma that makes clear the concept of "direction of convergence".

Lemma 3.1. *Let $P_0=\mathbb{M}_n(\mathbb{C})$ and let $U_t, t=1,2,3,\dots$ be unitaries in P_0, converging to I as $t\to\infty$. Assume that*
$$\mathfrak{C}_t=\begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau\\ \mathbb{C} & \subset & U_t^*Q_0U_t & \end{pmatrix}$$
are non-isomorphic commuting squares with Jones projections $e^t_1\in P_{-1}, e^t_2\in Q_0$. Then there exists a subsequence $0<t_1<t_2<t_3<\cdots$ of $\mathbb{N}$ and unitaries $\widetilde{U}_{t_1},\widetilde{U}_{t_2},\widetilde{U}_{t_3},\ldots\in P_0$ such that
$$\widetilde{\mathfrak{C}}_{t_k}=\begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau\\ \mathbb{C} & \subset & \widetilde{U}^*_{t_k}Q_0\widetilde{U}_{t_k} & \end{pmatrix}$$
are commuting squares with Jones projections $\tilde{e}^t_1,\tilde{e}^t_2$ such that $\tilde{e}^t_1=e_1$ is constant, $\mathfrak{C}_k$ isomorphic to $\mathfrak{C}_{t_k}$ for all k and
$$\lim_{k\to\infty}\frac{\widetilde{U}_{t_k}-I}{\mathrm{i}\,\|\widetilde{U}_{t_k}-I\|}=\tilde{h}\in Q'_{-1}\cap P_0,$$
$$E_{P'_{-1}\cap P_0}(\tilde{h})=E_{Q'_0\cap P_0}(\tilde{h})=E_{Q'_{-1}\cap P_{-1}}(\tilde{h})=E_{Q'_{-1}\cap Q_0}(\tilde{h})=0.$$

Proof. Without changing the isomorphism class of the commuting square we may assume $e_2^t = e_2$ are constant, since e_2^t are projections orthogonal on P_{-1} so $e_2^t = \mathrm{Ad}(p_t)(e_2)$ for some unitaries $p_t \in P_{-1}$ and we may conjugate the commuting square by p_t^*.

The projections e_1^t can also be assumed constant for all t's, since the commuting square condition implies $e_1^t \in Z(P_{-1})$ which has finitely many projections; so by passing to a subsequence we may assume $e_1^t = e_1$.

We will first show that we may modify U_t without changing the isomorphism class of $\mathfrak{C}_t$, such that they commute with e_1, e_2. Since

$$e_2 \in Z(U_t^* Q_0 U_t) = U_t^* Z(Q_0) U_t \Rightarrow U_t e_2 U_t^* \in Z(Q_0)$$

and $Z(Q_0)$ has finitely many projections, we may assume by passing to a subsequence that $U_t e_2 U_t^*$ is constant and since it converges to e_2 we must have $U_t e_2 U_t^* = e_2$, so $[U_t, e_2] = 0$. We show we can change U_t to $q_t U_t = \widetilde{U_t}$ ($q_t \in Q_0$ unitaries converging to I) such that $[e_1, q_t U_t] = 0$. Note that $q_t U_t$ will still commute with e_2 as $[Q_0, e_2] = 0$.

To do this, let $q_t = nE_{Q_0}(e_1 U_t^*)$ be the unique element of Q_0 satisfying $e_1 q_t = e_1 U_t^*$. If we project $q_t^* e_1 q_t = U_t e_1 U_t^*$ on Q_0 we get

$$q_t^* q_t = nE_{Q_0}(U_t e_1 U_t^*) = nU_t E_{U_t^* Q_0 U_t}(e_1) U_t^* = n\tau(e_1) I_n = I_n$$

which shows that q_t are unitaries. We have

$$e_1 q_t = e_1 U_t^* \Rightarrow e_1 q_t U_t = e_1 \Rightarrow U_t^* q_t^* e_1 = e_1 \Rightarrow e_1 = q_t U_t e_1 \Rightarrow [q_t U_t, e_1] = 0.$$

$((U_t - I)/\|U_t - I\|)_{t=1,2,3,\ldots}$ are norm 1 vectors, so by the compactness of the unit ball in the finite dimensional Banach space P_0 there exists a convergent subsequence of U_t (which for convenience we will still denote as $U_1, U_2, U_3, \ldots$) such that $\lim_{t\to\infty}\big((U_t - I)/(\mathrm{i}\|U_t - I\|)\big) = h$. h follows hermitian since

$$h^* = -\lim_{t\to\infty} \frac{U_t^* - I}{\mathrm{i}\|U_t^* - I\|} = -\lim_{t\to\infty} \frac{U_t^*(I - U_t)}{\mathrm{i}\|U_t^* - I\|} = h.$$

Let $U_t = \exp(\mathrm{i}h_t)$, h_t hermitians converging to 0. Since

$$\lim_{t\to\infty} \frac{\|\exp(\mathrm{i}h_t) - I\|}{\mathrm{i}\|h_t\|} = 1$$

we have

$$h = \lim_{t\to\infty} \frac{h_t}{\|h_t\|}.$$

We will refer to h as the *direction of convergence* of $(U_t)_t$. According to the above we may assume that $[h, e_1] = [h, e_2] = 0$. By changing U_t to $\det(U_t^*)U_t$, we may assume that $\det(U_t) = 1$ for all t, so $\tau(h_t) = 0$ and thus $\tau(h) = 0$.

We will show how to change $(U_t)_t$ so that h satisfies the conditions in the lemma. First we prove that we can modify $(U_t)_t$ such that

$$h \notin P_{-1} + P'_{-1} + Q_0 + Q'_0 \tag{3.1}$$

and h still commutes with the Jones projections e_1, e_2.

With the notations introduced in the previous section, let

$$\mathfrak{A} = \{p\Phi_1(p) = L(p)R(p^*) : p \in \mathfrak{U}(P_{-1})\}$$

$\mathfrak{A}$ is closed under multiplication because $\Phi_1(p) \in P'_{-1}$). $\mathfrak{A}$ is compact, being closed and included in the unit ball of P_0. Also, for $a \in \mathfrak{A}$ we have $[a, e_1] = [a, e_2] = 0$ since $L(p)R(p^*)e_2 = pp^* e_2 = e_2$. Similarly, let $\mathfrak{B}$ be the set of unitaries of the form $q\Phi_2(q)$ with $q \in \mathfrak{U}(Q_0)$. Let a_t, b_t be such that

$$\|b_t U_t a_t - I\|_2 = \inf_{(b,a)\in\mathfrak{U}(\mathfrak{B})\times\mathfrak{U}(\mathfrak{A})} \|bU_t a - I\|_2.$$

We show that if we change U_t to $b_t U_t a_t$, which still commute with e_1, e_2, the new direction of convergence (that we will still denote by h) satisfies (3.1). Note that $b_t U_t a_t$ are unitaries converging to I, since $\|b_t U_t a_t - I\|_2 \leqslant \|U_t - I\|_2$. If we denote $\mathrm{Re}\,\tau$ the real part of τ we have

$$\|b_t U_t a_t - I\|_2 = 2 - 2\,\mathrm{Re}\,\tau(b_t U_t a_t)$$

and using the definition of a_t, b_t it follows $\mathrm{Re}\,\tau(bU_t a - b_t U_t a_t) \leqslant 0$, which we can rewrite

$$\mathrm{Re}\,\tau((b - b_t)U_t a + b_t U_t (a - a_t)) \leqslant 0.$$

Let now $p \in P_{-1}$ be hermitian of trace 0 and $p' = -\Phi_1(p)$. Similarly let $q \in Q_0$ be hermitian of trace 0 and $q' = -\Phi_2(q)$. For λ real close to zero

$$a_t \exp(\mathrm{i}\lambda(p + p')) = a_t \exp(\mathrm{i}\lambda p)\Phi_1(\exp(\mathrm{i}\lambda p)) \in \mathfrak{A},$$

$$\exp(\mathrm{i}\lambda(q + q'))b_t = \exp(\mathrm{i}\lambda q)\Phi_2(\exp(\mathrm{i}\lambda q))b_t \in \mathfrak{B}$$

$$\Rightarrow \mathrm{Re}\,\tau((\exp(\mathrm{i}\lambda(q + q')) - I)b_t U_t a + b_t U_t a_t(\exp(\mathrm{i}\lambda(p + p')) - I)) \leqslant 0.$$

Dividing by $\lambda > 0$ and taking the limit as $\lambda \to 0$ we have $\mathrm{Re}\,\tau(\mathrm{i}((q+q')b_t U_t a_t + b_t U_t a_t(p + p'))) \leqslant 0$.

Doing the same for $\lambda < 0$ it follows $\mathrm{Re}\,\tau(\mathrm{i}((q + q')b_t U_t a_t + b_t U_t a_t(p + p'))) = 0$, so $\mathrm{Re}\,\tau(\mathrm{i}((q + q')(b_t U_t a_t - I) + (b_t U_t a_t - I)(p + p'))) = 0$ and after dividing by $\|b_t U_t a_t - I\|$ and using

$$\lim_{t\to\infty} \frac{b_t U_t a_t - I}{\|b_t U_t a_t - I\|} = \mathrm{i}h$$

we obtain $\mathrm{Re}\,\tau(\mathrm{i}((q + q')(\mathrm{i}h) + (\mathrm{i}h)(p + p'))) = 0$, so

$$\mathrm{Re}\,\tau(h(p - \Phi_1(p) + q - \Phi_2(q)) = 0 \tag{3.2}$$

for all $p \in P_{-1}, q \in Q_0$ hermitian of trace 0.

If $h = p_0 + p'_0 + q_0 + q'_0 \in P_{-1} + P'_{-1} + Q_0 + Q'_0$ we may assume $\tau(p_0) = \tau(q_0) = \tau(p'_0) = \tau(q'_0) = 0$, so

$$E_{P_{-1}}(h) = E_{P_{-1}}(p_0 + p'_0) = E_{P_{-1}}(p'_0) + p_0 = E_{Z(P_{-1})}(p'_0) + p_0$$

and by substituting p_0 with $p_0 + E_{Z(P_{-1})}(p'_0)$ and p'_0 with $p'_0 - E_{Z(P_{-1})}(p'_0)$ we may take

$$p_0 = E_{P_{-1}}(h), \quad p'_0 = E_{P'_{-1}}(h) - E_{Z(P_{-1})}(h)$$

and make similar assumptions for q_0, q'_0. In particular all these elements are self-adjoint.

$$[h, e_2] = [q_0, e_2] = [q'_0, e_2] = 0 \Rightarrow [p_0 + p'_0, e_2] = 0 \Rightarrow [p_0 + \Phi_1(p'_0), e_2] = 0$$

and using $p_0 + \Phi_1(p'_0) \in P_{-1}$ it follows $p_0 + \Phi_1(p'_0) = 0$. Reasoning similarly for q_0 we get $h = p_0 - \Phi_1(p_0) + q_0 - \Phi_2(q_0)$ which together with relation (3.2) leads to $\mathrm{Re}\,\tau(h^2) = 0$, so

To finish the proof of the lemma consider

$$p_t = \exp(-\mathrm{i}\|U_t - I\| E_{P_{-1}}(h)), \quad p'_t = \exp(-\mathrm{i}\|U_t - I\|(E_{P'_{-1}}(h) - E_{Z(P_{-1})}(h))),$$
$$q_t = \exp(-\mathrm{i}\|U_t - I\| E_{Q_0}(h)), \quad q'_t = \exp(-\mathrm{i}\|U_t - I\|(E_{Q'_0}(h) - E_{Z(Q_0)}(h))),$$

which are unitaries since h is hermitian. Let $\widetilde{U_t} = q_t q'_t U_t p_t p'_t$. Then

$$\begin{matrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ \mathbb{C} & \subset & \widetilde{U_t}^* Q_0 \widetilde{U_t} \end{matrix}$$

are commuting squares with Jones projections $\widetilde{e}_1{}^t = e_1, \widetilde{e}_2{}^t = p'^*_t p^*_t e_2 p_t p'_t$ and

$$h_0 = \lim_{t\to\infty} \frac{\widetilde{U_t} - I}{i\|U_t - I\|}$$
$$= h - E_{P_{-1}}(h) - E_{P'_{-1}}(h) + E_{Z(P_{-1})}(h) - E_{Q_0}(h) - E_{Q'_0}(h) + E_{Z(Q_0)}(h)$$

satisfies $E_{P_{-1}}(h_0) = E_{P'_{-1}}(h_0) = E_{Q_0}(h_0) = E_{Q'_0}(h_0) = 0$ and h_0 is non-zero, since h is not in the span of the four algebras according to (3.1). Thus:

$$\widetilde{h} = \lim_{t\to\infty} \frac{\widetilde{U_t} - I}{\mathrm{i}\|\widetilde{U_t} - I\|} = \frac{h_0}{\|h_0\|}$$

satisfies the conditions in the lemma. □

4. On the finiteness of the number of n-dimensional Hopf C^*-algebras

In this section we give the proof of D. Stefan's theorem, stating that for every integer $n > 1$ there exist only finitely many n-dimensional Hopf C^*-algebras. Also, our proof can be refined to obtained an estimate on the number of such Hopf structures.

As shown in [14], every Hopf algebra can be encoded in a standard irreducible commuting square, so one can restate D. Stefan's theorem as:

Theorem 4.1. *Let n be a positive integer. Then there exist only finitely many isomorphism classes of standard irreducible commuting squares*

$$\begin{pmatrix} P_{-1} & \subset & \mathbb{M}_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & Q_0 & \end{pmatrix}.$$

Proof. It is enough to show that each such irreducible standard commuting square $\mathfrak{C}$ is isolated among the standard commuting squares (modulo isomorphisms). Assume not. This implies the existence of unitaries $U_t \in P_0 = \mathbb{M}_n(\mathbb{C}), t = 1,2,3,\ldots$ converging to the identity I_n, such that

$$\mathfrak{C}_t = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & U_t^* Q_0 U_t & \end{pmatrix}$$

are non-isomorphic standard commuting squares, that can be assumed to satisfy the conditions with Jones projections $e_1^t = e_1, e_2^t = U_t^* e_2 U_t$.

Theorem 2.4 implies the existence of unitary elements $V, V_1, V_2, V_3, \ldots$ in $Q_0 \otimes P_{-1}$ such that $V_t \to V$ as $t \to \infty$ and

$$V_t(p \otimes I)V_t^* \in P_{-1} \otimes P_{-1}, \quad \forall p \in P_{-1}. \tag{4.1}$$

We will denote the unitary V, associated to the initial commuting square, by V_0. We will identify the elements $x \in P_0$ with $x \otimes I \in P_0 \otimes P_{-1}$. Also, by abuse of notation we will sometimes denote the trace $\tau \otimes \tau$ on $P_0 \otimes P_{-1}$ by τ.

$V_t = \sum_{k,l,r}(n/r) q_{kl}^{r,t} \otimes p_{kl}^r$ has the components $q_{kl}^{r,t} \in U_t^* Q_0 U_t = Q_0^t$ uniquely determined by relation (2.2)

$$p_{kl}^r e_2^t e_1 = q_{kl}^{r,t} e_1.$$

If we denote by

$$e = \sum_{k,l,r} \frac{n}{r} p_{kl}^r \otimes p_{kl}^r \in P_{-1} \otimes P_{-1}$$

then we can rewrite the previous equality

$$V_t e_1 = e e_2^t e_1 \Rightarrow V_t = n E_{Q_0^t \otimes P_{-1}}(e e_2^t e_1). \tag{4.2}$$

$(1/n)e$ is a projection since $e^2 = ne$, being in fact the basic construction projection for the inclusion $P'_{-1} \cap P_0 \subset P_0$.

Letting $t \to \infty$ in the previous equality we obtain

$$V_0 e_1 V_0^* = e e_2 e_1 e_2 e = \frac{1}{n} e e_2 e = \frac{1}{n} e. \tag{4.3}$$

Let

$$H_0 = \lim_{t\to\infty} \frac{V_t - V_0}{\mathrm{i}\|U_t - I\|} \in P_0 \otimes P_{-1}.$$

We will show that this limit exists and find an explicit formula for H_0; note that in general H_0 is not hermitian.

Using (4.2) we have

$$\begin{aligned} V_t - V_0 &= n E_{Q_0^t \otimes P_{-1}}(e e_2^t e_1) - n E_{Q_0 \otimes P_{-1}}(e e_2 e_1) \\ &= n U_t^*(E_{Q_0 \otimes P_{-1}}(U_t e e_2^t e_1 U_t^*)U_t - n E_{Q_0 \otimes P_{-1}}(e e_2 e_1) \\ &= n((U_t - I)^*(E_{Q_0 \otimes P_{-1}}(U_t e e_2^t e_1 U_t^*)U_t \\ &\quad + E_{Q_0 \otimes P_{-1}}((U_t - I) e e_2^t e_1 U_t^*)U_t + E_{Q_0 \otimes P_{-1}}(e(e_2^t - e_2)e_1 U_t^*)U_t \\ &\quad + E_{Q_0 \otimes P_{-1}}(e e_2 e_1 (U_t - I)^*)U_t + E_{Q_0 \otimes P_{-1}}(e e_2 e_1)(U_t - I)). \end{aligned}$$

And since

$$e_2^t - e_2 = U_t^* e_2 U_t - e_2 = (U_t - I)^* e_2 U_t + e_2(U_t - I)$$

so

$$\lim_t \frac{e_2^t - e_2}{\mathrm{i}\|U_t - I\|} = [e_2, h]$$

we get

$$H_0 = n([E_{Q_0 \otimes P_{-1}}(e e_2 e_1), h] - E_{Q_0 \otimes P_{-1}}([e e_2 e_1, h]) + E_{Q_0 \otimes P_{-1}}(e[e_2, h]e_1), \tag{4.4}$$

which can be rewritten

$$H_0 = [V_0, h] - n E_{Q_0 \otimes P_{-1}}([V_0 e_1, h]) + n E_{Q_0 \otimes P_{-1}}(e[e_2, h]e_1) \tag{4.5}$$

since $nE_{Q_0\otimes P_{-1}}(ee_2e_1) = V_0$ and $ee_2e_1 = V_0e_1$.

We will now do the main trick of the proof, which is taking the "derivative" of relation (4.1) along the direction H_0. For $p \in P_{-1}$ we have

$$V_t p V_t^* \in P_{-1}\otimes P_{-1} \Rightarrow (V_t - V_0)pV_t^* + V_0 p(V_t - V_0)^* \in P_{-1}\otimes P_{-1}$$

and after dividing by $\mathrm{i}\,\|U_t - I\|$ and taking the limit as $t \to \infty$

$$H_0 p V_0^* - V_0 p H_0^* \in P_{-1}\otimes P_{-1}. \tag{4.6}$$

Let $H = H_0V_0^*$. H is hermitian since $H = \lim_{t\to\infty}(V_tV_0^* - I)/(\mathrm{i}\,\|U_t - I\|)$, so $H = V_0H_0^*$. (4.6) can be rewritten as

$$[H, V_0pV_0^*] \in P_{-1}\otimes P_{-1}, \quad \forall p \in P_{-1}. \tag{4.7}$$

We now use the following lemma:

Lemma 4.2. *Let $X \subset Y \subset Z$ be finite dimensional C^*-algebras with trace τ, and let $H \in Z$ be an element such taht $[H, x] \in Y$ for all $x \in X$. Then there exist elements $y_0 \in Y$ and $x_0' \in X' \cap Z$ such that $H = y_0 + x_0'$.*

Proof. Let $y_0 = E_Y(H)$ and $x_0' = H - y_0$. We need to show $x_0' \in X' \cap Z$. Because $[H, x] \in Y$ and $[y_0, x] \in Y$ we must have $[x_0', x] \in Y$. But, for y in Y, $\tau([x_0', x]y) = \tau(x_0'xy - xx_0'y) = \tau(x_0'(xy - yx)) = 0$ as $xy - yx \in Y$. So by taking $y = [x_0', x]^*$ we get $[x_0', x] = 0$ which shows that $x' \in X' \cap Z$. □

If we apply the lemma to $X = V_0P_{-1}V_0^*$, $Y = P_{-1}\otimes P_{-1}$, $Z = P_0\otimes P_{-1}$, since $X' \cap Z = V_0(P_{-1}'\otimes P_{-1})V_0^*$ we get:

$$H = p_0 + V_0p_0'V_0^* \tag{4.8}$$

with $p_0 = E_{P_{-1}\otimes P_{-1}}(H) \in P_{-1}\otimes P_{-1}$, $p_0' \in P_{-1}'\otimes P_{-1}$.

However, we will show that H is orthogonal on both $P_{-1}\otimes P_{-1}$ and $V_0(P_{-1}'\otimes P_{-1})V_0^*$, which will lead to a contradiction and end the proof.

We have

$$\begin{aligned} H &= H_0V_0^* \\ &= [V_0, h]V_0^* - nE_{Q_0\otimes P_{-1}}([V_0e_1, h]V_0^*) + nE_{Q_0\otimes P_{-1}}(e[e_2, h]e_1V_0^*) \\ &= (V_0hV_0^* - h) - nE_{Q_0\otimes P_{-1}}([V_0e_1, h]V_0^*) + nE_{Q_0\otimes P_{-1}}(e[e_2, h]e_1V_0^*). \end{aligned}$$

We first show that

$$E_{P_{-1}\otimes P_{-1}}(H) \in \mathbb{C}\otimes Z(P_{-1}). \tag{4.9}$$

Using $E_{P_{-1}}E_{Q_0} = E_{\mathbb{C}}$ we have

$$E_{P_{-1}\otimes P_{-1}}(H) = E_{P_{-1}\otimes P_{-1}}(V_0hV_0^* - h) - nE_{\mathbb{C}\otimes P_{-1}}([V_0e_1, h]V_0^*) + nE_{\mathbb{C}\otimes P_{-1}}(e[e_2, h]e_1V_0^*),$$

but $E_{P_{-1}\otimes P_{-1}}(V_0hV_0^* - h) = E_{P_{-1}\otimes P_{-1}}(V_0hV_0^*) = 0$, since $E_{P_{-1}}(h) = 0$ and for any $p\otimes p_{kl}^r \in P_{-1}\otimes P_{-1}$ we have

$$\begin{aligned} \tau(V_0hV_0^*(p\otimes p_{kl}^r)) &= \tau(hV_0^*(p\otimes p_{kl}^r)V_0) = \left(\frac{n}{r}\right)^2\sum_{i,j}\tau(hq_{ki}^{r*}pq_{lj}^r)\tau(p_{ik}^rp_{kl}^rp_{lj}^r) \\ &= \left(\frac{n}{r}\right)^2\sum_{i,j}\tau(hq_{ki}^{r*}pq_{lj}^r)\tau(p_{ij}^r) = \frac{1}{r}\tau\Big(h\sum_i q_{ki}^{r*}pq_{li}^r\Big) = 0, \end{aligned}$$

were we used that $\sum_i q_{ki}^{r*}pq_{li}^r \in P_{-1}$, which follows from

$$\begin{aligned} &\frac{r}{n^2}p_{kl}^r = \sum_i q_{ki}^re_1q_{li}^{r*} \in P_{-1} \\ &\Rightarrow \Phi_2\Big(\frac{r}{n^2}p_{kl}^r\Big) = \sum_i q_{ki}^{r*}e_1q_{li}^r \in \Phi_2(P_{-1}) = P_{-1}. \end{aligned}$$

Using (4.3) and $\tau(hp_{kl}^r) = 0)$ we have

$$\begin{aligned} E_{\mathbb{C}\otimes P_{-1}}([V_0e_1, h]V_0^*) &= E_{\mathbb{C}\otimes P_{-1}}(V_0e_1hV_0^* - hV_0e_1V_0^*) \\ &= E_{\mathbb{C}\otimes P_{-1}}\Big(V_0e_1hV_0^* - \frac{1}{n}he\Big) = E_{\mathbb{C}\otimes P_{-1}}(V_0e_1hV_0^*) - \sum_{k,l,r}\frac{1}{n}\tau\Big(\frac{n}{r}hp_{kl}^r\Big)I\otimes p_{kl}^r \\ &= E_{\mathbb{C}\otimes P_{-1}}(V_0e_1hV_0^*) = \sum_{k,l,r,i}\frac{n^2}{r^2}\tau(q_{ki}^re_1hq_{li}^{r*})I\otimes p_{kl}^r \\ &= \sum_{k,l,r,i}\frac{n^2}{r^2}\tau(e_1hq_{li}^{r*}q_{ki}^r)I\otimes p_{kl}^r = \sum_{k,r}\tau(e_1h)I\otimes p_{kk}^r = 0 \end{aligned}$$

since $e_1 \in P_{-1}$ and h is orthogonal on P_{-1}. Finally, we have

$$E_{\mathbb{C}\otimes P_{-1}}(e[e_2, h]e_1V_0^*) = \sum_{k,l,r,i}\frac{n^2}{r^2}\tau(p_{ki}^r[e_2, h]e_1q_{li}^{r*})I\otimes p_{kl}^r$$

and using $e_1q_{li}^{r*} = e_1e_2p_{il}^r$

$$\begin{aligned} E_{\mathbb{C}\otimes P_{-1}}(e[e_2, h]e_1V_0^*) &= \sum_{k,l,r,i}\frac{n^2}{r^2}\tau(p_{ki}^r[e_2, h]e_1e_2p_{il}^r)I\otimes p_{kl}^r \\ &= \sum_{k,l,r,i}\frac{n^2}{r^2}\tau([e_2, h]e_1e_2p_{il}^rp_{ki}^r)I\otimes p_{kl}^r = \sum_{k,r,i}\frac{n^2}{r^2}\tau([e_2, h]e_1e_2p_{ii}^r)I\otimes p_{kk}^r \\ &= \sum_{k,r}\frac{n^2}{r}\tau\Big([e_2, h]e_1e_2\sum_i p_{ii}^r\Big)I\otimes p_{kk}^r = \sum_r\frac{n^2}{r}\tau([e_2, h]e_1e_2z_r)I\otimes z_r \in \mathbb{C}\otimes Z(P_{-1}), \end{aligned}$$

where $z_r = \sum_k p_{kk}^r$ are central projections of P_{-1}.

This shows (4.9). We now show that

$$E_{P_{-1}'\otimes P_{-1}}(V_0^*HV_0) \in \mathbb{C}\otimes P_{-1} \tag{4.10}$$

Indeed, from the definition of H and using $E_{P_{-1}'}E_{Q_0} = E_{\mathbb{C}}$ we have

$$E_{P_{-1}'\otimes P_{-1}}(H) = E_{P_{-1}'\otimes P_{-1}}(-V_0^*hV_0 + h) - nE_{\mathbb{C}\otimes P_{-1}}(V_0^*[V_0e_1, h]) + nE_{\mathbb{C}\otimes P_{-1}}(V_0^*e[e_2, h]e_1)$$

belongs to $\mathbb{C}\otimes P_{-1}$, since

$$E_{P_{-1}'\otimes P_{-1}}(-V_0^*hV_0 + h) = -E_{P_{-1}'\otimes P_{-1}}(V_0^*hV_0) = 0.$$

This is true because for any $p'\otimes p_{kl}^r \in P_{-1}'\otimes P_{-1}$ we have

$$\begin{aligned} \tau(V_0^*hV_0(p'\otimes p_{kl}^r)) &= \tau(hV_0(p'\otimes p_{kl}^r)V_0^*) = \left(\frac{n}{r}\right)^2\sum_{i,j}\tau(hq_{ik}^rp'q_{jl}^{r*})\tau(p_{ik}^rp_{kl}^rp_{lj}^r) \\ &= \left(\frac{n}{r}\right)^2\sum\tau(hq_{ik}^rp'q_{jl}^{r*})\tau(p_{ij}^r) = \frac{1}{r}\tau\Big(h\sum q_{ik}^rp'q_{il}^{r*}\Big) = 0, \end{aligned}$$

were we used $\sum_i q^r_{ik} p' q^{r*}_{il} \in P'_{-1}$ which follows from (2.8) together with

$$\Phi_1\Big(\sum_i q^r_{ik} p' q^{r*}_{il}\Big) = \sum_i q^r_{ki} \Phi_1(p') q^{r*}_{li} \in P_{-1}.$$

Thus we proved (4.10) and (4.9).

Let $p_0 = E_{P_{-1}\otimes P_{-1}}(H) \in \mathbb{C} \otimes Z(P_{-1})$. In particular $p_0 = V_0 p_0 V_0^*$, so $H = V_0(p_0 + p_0')V_0^* \in V_0(P'_{-1} \otimes P_{-1})V_0^*$. Since we also have $E_{P'_{-1}\otimes P_{-1}}(V_0^* H V_0) \in \mathbb{C} \otimes P_{-1}$ it follows $V_0^* H V_0 \in \mathbb{C} \otimes P_{-1}$, so using the formula for H

$$h - V_0^* h V_0 \in Q_0 \otimes P_{-1},$$

which implies $[V_0, h] \in Q_0 \otimes P_{-1}$. Since q^r_{kl} span Q_0 and $[q^r_{kl}, h] \in Q_0$ it follows $[q, h] \in Q_0, \forall q \in Q_0$, so using argumnents similar to the previous lemma $h \in Q_0 + Q_0'$, which contradicts the assumptions $E_{Q_0}(h) = E_{Q_0'}(h) = 0$, $h \neq 0$. □

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