A FINITENESS RESULT FOR COMMUTING SQUARES WITH LARGE SECOND RELATIVE COMMUTANT

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Abstract. We prove that there exist only finitely many commuting squares of finite dimensional $*$-algebras of fixed dimension, satisfying a "large second relative commutant" condition. We show this by studying the local minima of $w \to \dim(A \cap wBw^*)$, where A, B are fixed sub-algebras of some $*$-algebra C and $w \in C$ is a unitary.

When applied to lattices arising from subfactors satisfying a certain extremality-like condition, our result yields Ocneanu's finiteness theorem for the standard invariants of such finite depth subfactors.

1. Introduction

Commuting squares were introduced by S. Popa in [Po1] (see also [Po2], [JS]). They arise naturally in subfactor theory, as invariants and construction data for subfactors. A *commuting square* is a square of inclusions of finite dimensional $*$-algebras:

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup, \tau \\ Q_{-1} & \subset & Q_0 \end{pmatrix}$$

with a faithful trace τ on P_0, such that

$$P_{-1} \ominus Q_{-1} \perp Q_0 \ominus Q_{-1}$$

i.e. the vector spaces $P_{-1} \ominus Q_{-1}$ and $Q_0 \ominus Q_{-1}$ are orthogonal with respect to the inner product defined by τ on P_0.

In this paper we consider commuting squares that satisfy a "large second relative commutant" (LRC) condition. These are commuting squares with a λ-Markov trace τ such that, after doing Jones' basic

construction ([Jon]):

$$\mathfrak{L} = \begin{pmatrix} P_{-1} & \subset & P_0 & \overset{e}{\subset} & P_1 & \\ \cup & & \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & Q_0 & \overset{e}{\subset} & Q_1 & \end{pmatrix}$$

the relative commutant $R_1 = P'_{-1} \cap Q_1$ satisfies

$$E_{P_0}(R'_1 \cap P_1) = P_{-1}$$

where E_{P_0} denotes the projection from the vector space P_1 onto P_0, with respect to the inner product defined by τ. Since in general $P_{-1} = E_{P_0}(P_{-1}) \subset E_{P_0}(R'_1 \cap P_1)$, we can interpret the equality as a restriction on R'_1 and thus a largeness condition for R_1.

We prove that there exist only finitely many such LRC commuting squares with $\dim(P_0)$ fixed (Theorem 2.4).

The proof is based on derivation techniques similar to those we introduced in [Ni]. We also need some properties of the local minima of maps $w \to \dim(A \cap wBw^*)$, where A, B are fixed sub-algebras of some $*$-algebra C. More precisely, we show that if $\dim(A \cap B) \leq \dim(A \cap w_n B w_n^*)$ for some unitaries $w_n \to I$ in C, then any direction of convergence of w_n (in the sense of Definition 2.8) belongs to $A + B + (A \cap B)' \cap C$ (Proposition 3.3).

The finiteness result of this paper, as well as the LRC condition, are motivated by the case of commuting squares arising in the standard invariant of a subfactor.

Let us recall the definition of the standard invariant. Let $N \subset M$ be an inclusion of II_1 factors of finite index with trace τ, and let $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset ...$ be the tower of factors obtained by iterating Jones' basic construction (see [Jon]), where $e_1, e_2, ...$ denote the Jones projections. The *standard invariant* $\mathcal{G}_{N,M}$ is then defined as the trace preserving isomorphism class of the following sequence of *commuting squares* of inclusions of finite dimensional $*$-algebras:

$$\begin{pmatrix} N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & N' \cap M_3 & \subset & ... & \\ & & \cup & & \cup & & \cup & & & , \tau \\ & & M' \cap M_1 & \subset & M' \cap M_2 & \subset & M' \cap M_3 & \subset & ... & \end{pmatrix}$$

together with the Jones projections $e_i \in N' \cap M_i$ and the trace τ.

If the subfactor is of finite depth n, by [Po1] the commuting square

$$\mathfrak{C}_n = \begin{pmatrix} N' \cap M_{n-1} & \subset & N' \cap M_n \\ \cup & & \cup \\ M' \cap M_{n-1} & \subset & M' \cap M_n \end{pmatrix}, \tau$$

uniquely determines the subfactor. In particular, the isomorphism class of a depth 2 subfactor is uniquely determined by $\mathfrak{C}_1$. Moreover, if $N \subset M$ is of depth n then $N \subset M_n$ has depth 2. This allows to work only with depth 2 subfactors for the purpose of finiteness results.

By a seminal result of S. Popa ([Po2],[Po4]), the standard invariant of a subfactor can be thought of as an abstract group-like object, described by a set of axioms. One of these axioms is based on the following equality:

$$(N' \cap M_1)' \cap (M' \cap M_3) = M_1' \cap M_3$$

Thus, for the commuting square $\mathfrak{C} = \mathfrak{C}_1$ of a depth 2 subfactor the relative commutant $P_{-1}' \cap Q_1$ is large, of dimension equal to $\dim(P_{-1})$. This is our inspiration for looking for some "large relative commutant" condition.

When applied to such commuting squares $\mathfrak{C}$, the LRC condition is equivalent to the following extremality condition:

$$E_{(N' \cap M_1)' \cap (N' \cap M_2)}(e_2) \in \mathbb{C}I$$

Thus our theorem yields a finiteness result for the standard invariant of such (depth 2) subfactors. This extremality condition is not automatically true for any depth 2 subfactor, as we will show in the last section. We show however that it is true if $N' \cap M$ is a (type $\mathrm{I_k}$) factor. In particular it is true when $N' \cap M = \mathbb{C}$, i.e. for those subfactors arising from Hopf algebras ([Szy]). It is also true for a larger class of depth 2 subfactors, those admiting an orthonormal basis for $N' \cap M \subset N' \cap M_1$ which is invariant under taking adjoints.

We mention that this finiteness theorem for standard invariants of finite depth subfactors is well known to specialists, as a theorem of A. Ocneanu, even without the extra assumption $E_{(N' \cap M_1)' \cap (N' \cap M_2)}(e_2) \in \mathbb{C}I$. See also [EtNiOs]. The proof we obtain here, as a consequence of our main theorem, is elementary in nature and does not use the languages of paragroups or tensor categories.

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2. Preliminaries

In this section we recall the definition of a commuting square and we introduce several notations and normalizations. All the algebras considered will be matrix algebras, i.e. *-closed unital subalgebras of $\mathbb{M}_n(\mathbb{C})$ for some $n \geq 1$. Such an algebra is always of the form $\oplus_i \mathbb{M}_{n_i}(\mathbb{C})$, with n_i positive integers.

For a unitary inclusion of matrix algebras $B \subset A$ we will use the notations:

$$B' \cap A = \{\text{all } b \in B \text{ such that } ba = ab \text{ for all } a \in A\}$$

$$\mathcal{N}_A(B) = \{u \in A \text{ unitary such that } uBu^* \subset B\}$$

If τ is a fixed trace on A, we denote by $E_B = E_B^A$ the τ-invariant conditional expectation of A onto B.

If B_1, B_2 are subspaces of A, we denote by $E_{B_1}(B_2)$ the projection of B_2 onto B_1, with respect to the trace τ. While this may be considered an abuse of notation, it is consistent with the conditional expectation notation.

If V, W are vector subspaces of the matrix algebra A we denote:

$$V + W = \{v + w : v \in V, w \in W\}$$

$$V \cdot W = \text{span}\{vw : v \in V, w \in W\}$$

$$[V, W] = \text{span}\{vw - wv : v \in V, w \in W\}$$

We recall the definition of a commuting square (see [Po1],[Po2]):

Definition 2.1. *A commuting square of matrix algebras is a square of unital inclusions:*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ Q_{-1} & \subset & Q_0 \end{pmatrix}, \tau$$

where P_0, P_{-1}, Q_0, Q_{-1} are matrix algebras and τ is a trace on P_0, $\tau(1) = 1$, satisfying the condition:

$$E_{P_{-1}} E_{Q_0} = E_{Q_0} E_{P_{-1}} = E_{Q_{-1}}$$

We say that the commuting square $\mathfrak{C}$ is *non-degenerate* if $P_0 = P_{-1}Q_0$. We will assume all our commuting squares to be non-degenerate.

We now introduce the large second relative commutant (LRC) condition that we will consider.

Definition 2.2. *Let $\mathfrak{C}$ be a commuting square with a λ - Markov trace τ. With the previous notations, let $\mathfrak{L}$ denote the lattice obtained by doing Jones' basic construction (see* [JS]*) from $\mathfrak{C}$:*

$$\mathfrak{L} = \begin{pmatrix} P_{-1} & \subset & P_0 & \overset{e}{\subset} & P_1 & \\ \cup & & \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & Q_0 & \overset{e}{\subset} & Q_1 & \end{pmatrix}$$

where the extension of the trace τ to P_1 is still denoted by τ and e is the Jones projection of the basic construction. Let $R_1 = P'_{-1} \cap Q_1$ denote the second relative commutant associated to the commuting square $\mathfrak{C}$. We say that $\mathfrak{C}$ satisfies the LRC condition if the following two dual equalities hold:

$$E_{Q_1}(P'_{-1} \cap P_1) = R_1 \tag{1}$$

$$E_{P_0}(R'_1 \cap P_1) = P_{-1} \tag{2}$$

Remark 2.3. *For any $\mathfrak{C}$ we have: $R_1 = E_{Q_1}(P'_{-1} \cap Q_1) \subset E_{Q_1}(P'_{-1} \cap P_1)$. Thus condition (1) can be interpreted as a largeness condition on R_1, hence the name LRC (large relative commutant).*

Similarly, $P_{-1} = E_{P_0}(P_{-1}) \subset E_{P_0}(R'_1 \cap P_1)$, as $[P_{-1}, R_1] = 0$. Thus, condition (2) also requires that R_1 be large, since it is a restriction on the size of its commutant.

Asking for a "largeness" condition on the relative commutant is inspired by the case of the standard invariant of a subfactor, as it will be discussed in Section 5. We will see that in this context (1) and (2) are equivalent and dual to each other.

Our main result, proved in Section 4, states:

Theorem 2.4. *There exist only finitely many isomorphism classes of commuting squares $\mathfrak{C}$, with $\dim(P_0)$ fixed, satisfying the LRC condition.*

We will prove the theorem by contradiction: assuming that there exist infinitely many such commuting squares we find a convergent subsequence, to which we apply derivation techniques similar to those that we introduced in [Ni].

To make the notion of convergence of commuting squares precise, let us first recall the following definition and result from [Chr]:

Definition 2.5. *Let A be a matrix algebra with normalized trace τ. Denote $\mathcal{S}(A)$ = the set of all *-subalgebras of A containing the identity. For $B_1, B_2 \in \mathcal{S}(A)$ and $\delta > 0$ we say that B_1 is δ-contained in B_2 if for every element $x \in B_1$ of $\|x\| = 1$ there exists $y \in B_2$ such that*

$\|x-y\|_2 < \delta$. *Here* $\|\cdot\|_2$ *denotes the norm given by the trace* τ *on* A, *i.e.* $\|x\|_2 = \tau(x^*x)^{1/2}$.

If B_1 *is* δ*-contained in* B_2 *and* B_2 *is* δ*-contained in* B_1 *we write* $\|B_1 - B_2\|_{2,A} < \delta$.

Theorem 2.6. *With the previous notations, there exists a continuous increasing function* $f : [0,\infty) \to [0,\infty), f(0) = 0$, *such that if* δ *is small and* $\|B_1 - B_2\|_{2,A} < \delta$, *then* $B_2 = Ad(U)(B_1)$ *for some unitary element* $U \in A$, $\|U - I\|_2 < f(\delta)$.

If Theorem 2.4 is false then there exist infinitely many non-isomorphic LRC commuting squares

$$\mathfrak{C}_n = \begin{pmatrix} P_{-1}^n & \subset & P_0 \\ \cup & & \cup \\ Q_{-1}^n & \subset & Q_0^n \end{pmatrix}, \tau$$

By using Theorem 2.6 together with the compactness of the unit ball of P_1 it follows that the inclusions $Q_{-1}^n \subset P_{-1}^n \subset P_0$ are unitarily conjugate for infinitely many n. Thus, after conjugating each $\mathfrak{C}_n$ by a unitary and eventually passing to a subsequence, we may assume that:

$$\mathfrak{C}_n = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ Q_{-1} & \subset & Q_0^n \end{pmatrix}, \tau$$

By a similar compactness argument, after passing to a subsequence we may assume that $Q_0^n = u_n Q_1 u_n^*$, with $u_n \to I$ unitaries in P_0. We have $u_n^* Q_{-1} u_n \subset Q_1$ and Lemma 3.2 shows that $u_n = q_n' q_n$ for n large, with $q_n \to I$ unitaries in Q_0 and $q_n' \to I$ unitaries in $Q_{-1}' \cap P_0$. Since $u_n Q_1 u_n^* = q_n' q_n Q_1 q_n^* (q_n')^* = q_n' Q_1 (q_n')^*$, by substituting $u_n \to q_n'$ we may assume that $u_n \in Q_{-1}' \cap P_0$.

In the following proposition we show that the notion of LRC commuting square behaves well to limits. This is somewhat surprising, since it is not true in general that $P_{-1}' \cap Q_1^n = R_1^n$ $(n \geq 1)$ implies $P_{-1}' \cap Q_1 = R_1$. However, $P_{-1}' \cap Q_1 = R_1$ will follow from conditions (1), (2), which have nice continuity properties.

Proposition 2.7. *With the previous notations, if* $\mathfrak{C}_n$ *are LRC commuting squares for all* $n \geq 1$ *then so is* $\mathfrak{C}$.

Proof. If $B \subset A$ are matrix algebras and $u_n \to I$ are unitaries in A then the conditional expectation satisfies the following continuity property:

$$\lim_{n\to\infty} E_{u_n B u_n^*}(a) = E_B(a), \text{ for all } a \in A$$

This is easy to see since $E_{u_n B u_n^*}(a) = u_n E_B(u_n^* a u_n) u_n^*$.

We may assume the Jones projection e to be the same for all the commuting squares $\mathfrak{C}_n$, i.e. the projection $e \in B(L^2(P_0, \tau))$ implementing the conditional expectation $E_{P_{-1}}^{P_0}$. By the continuity of the conditional expectation, taking the limit of the relations $E_{P_{-1}}E_{Q_0^n} = E_{Q_0^n}E_{P_{-1}} = E_{Q_{-1}}$ implies that $\mathfrak{C}$ is a commuting square.

Let $P_1 =< P_0, e >$, $Q_1 =< Q_0, e >= \text{span}Q_0eQ_0$ and $Q_1^n =< Q_0^n, e >= \text{span}Q_0^neQ_0^n \subset P_1$. Let $x_1, ..., x_N$ be a basis of Q_0. Then $u_nx_ku_n^*eu_nx_lu_n^*$ $(1 \leq k, l \leq N)$ span Q_1^n, x_kex_l $(1 \leq k, l \leq N)$ span Q_1 and $||u_nx_ku_n^*eu_nx_lu_n^* - x_kex_l||_2 \to 0$ as $n \to \infty$. The finiteness of the spanning set implies that $||Q_1^n - Q_1||_{2,P_1} \to 0$, so there exist unitaries $w_n \to I$ in P_1 such that $Q_1^n = w_nQ_1w_n^*$.

Let $R_1^n = P_{-1}' \cap Q_1^n$. By eventually passing to a subsequence, we may assume that all R_1^n are unitary conjugate inside P_1. By the compactness of the unit ball of P_1, after passing again to a subsequence we may assume that these unitaries converge. It follows that $R_1^n = v_nR_1v_n^*$ with $v_n \to I$ unitaries in P_1, where R_1 is some subalgebra of P_1. Since $R_1^n \subset Q_1^n$ we have $R_1 \subset Q_1$. However, it is not obvious that $P_{-1}' \cap Q_1 = R_1$ and in fact this will only follow because of the LRC conditions.

We know that $E_{Q_1^n}(P_{-1}' \cap P_1) = R_1^n$ for all $n \geq 1$. Thus $v_n^*E_{w_nQ_1w_n^*}(x)v_n \in R_1$ for all $x \in P_{-1}' \cap P_1$. From the continuity of the conditional expectation we obtain $E_{Q_1}(P_{-1}' \cap P_1) \subset R_1$. On the other hand, $R_1^n = P_{-1}' \cap Q_1^n \subset P_{-1}' \cap P_1$ implies $[R_1, P_{-1}] = 0$ so $R_1 \subset P_{-1}' \cap P_1$. Using this and the previous inclusion it follows $E_{Q_1}(P_{-1}' \cap P_1) = R_1$.

Notice that $R_1 \subset P_{-1}' \cap Q_1 = E_{Q_1}(P_{-1}' \cap Q_1) \subset E_{Q_1}(P_{-1}' \cap P_1) = R_1$ also shows that $R_1 = P_{-1}' \cap Q_1$.

Since $E_{P_0}((R_1^n)' \cap P_1) = P_{-1}$ and $(R_1^n)' \cap P_1 = (v_nR_1v_n^*)' \cap P_1 = (v_nR_1v_n^*)' \cap (v_nP_1v_n^*) = v_n(R_1' \cap P_1)v_n^*$, by taking the limit we obtain $E_{P_0}(R_1' \cap P_1) \subset P_{-1}$. On the other hand, $[R_1, P_{-1}] = 0$ so $P_{-1} \subset R_1' \cap P_1$, which together with the previous inclusion yields $E_{P_0}(R_1' \cap P_1) = P_{-1}$. This also implies that $P_{-1} = R_1' \cap P_0$.

□

Definition 2.8. *We say that the sequence of unitaries $u_n \to I$, $u_n \neq I$, has direction of convergence h if*

$$h = lim_{n\to\infty}\frac{u_n - I}{i\|u_n - I\|}$$

Let $v_n \to I$, $v_n \neq I$ $(n \geq 1)$ be a sequence of unitaries in P_0. Write $v_n = \exp(ik_n)$, with $k_n \in P_0$ hermitian non-zero elements, $k_n \to 0$. Consider any subsequence $(h_n)_{n\geq 1}$ of $(k_n)_{n\geq 1}$ such that $\frac{h_n}{\|h_n\|}$ converges to some norm one hermitian element h of P_0. Such sequences $(h_n)_{n\geq 1}$

exist by a standard compactness argument and any such h will be referred to as *a direction of convergence* for $(v_n)_{n\geq 1}$. This terminology is compatible with the previous definition, as h is the direction of convergence of the subsequence $u_n = \exp(ih_n)$ of $(v_n)_{n\geq 1}$. Indeed:

$$h = \lim_{n\to\infty} \frac{u_n - I}{i\|h_n\|} = \lim_{n\to\infty} \frac{u_n - I}{i\|u_n - I\|}$$

We end this section by recalling a result from [Ni], which gives normalizations on the direction of convergence of a sequence of commuting squares. These will be essential for the proof of Theorem 2.4.

Proposition 2.9. *Let P_0 be a matrix algebra with trace τ and let*

$$\mathfrak{C}_n = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ Q_{-1} & \subset & u_n Q_0 u_n^* \end{pmatrix}, \ \tau$$

be a sequence of non-isomorphic commuting squares, where $u_n \to I$ are unitaries in $Q'_{-1} \cap P_0$. After eventually replacing u_n by one of its subsequences, we have:

There exist unitaries $q_n \in Q'_{-1} \cap Q_0, q'_n \in Q_0' \cap P_{-1}, p_n \in Q'_{-1} \cap P_{-1}, p'_n \in P_{-1}' \cap P_0$ such that:

$$\tilde{u_n} = p_n {p_n}' u_n q'_n q_n \to I$$

$$\lim_{n\to\infty} \frac{\tilde{u_n} - I}{i\|\tilde{u_n} - I\|} = \tilde{h} \in Q'_{-1} \cap P_0$$

$$E_{P'_{-1}\cap P_0}(\tilde{h}) = E_{Q'_0\cap P_0}(\tilde{h}) = E_{Q'_{-1}\cap P_{-1}}(\tilde{h}) = E_{Q'_{-1}\cap Q_0}(\tilde{h}) = 0$$

Remark 2.10. *Note that the change $u_n \to \tilde{u_n} = p_n {p_n}' u_n q'_n q_n$ preserves the isomorphism class of the commuting square $\mathfrak{C}_n$.*

3. Local minima of matrix algebras intersections

In this section we deal with the main technical ingredient of the paper. Let C be a matrix algebra and A, B two $*$-closed subalgebras of C. Consider the algebras $A \cap w_n B w_n^*$, where w_n are unitaries in C approaching the identity. We find restrictions on the directions of convergence of w_n such that $A \cap B$ can be unitarily embedded in $A \cap w_n B w_n^*$ for all n large. In other words, we study when $A \cap B$ is what one might call a local minimum along the curve $A \cap wBw^*$, with $w \to I$ unitaries in the direction h.

In the next results we will often use the following relation that holds true for every a, b, c in a matrix algebra (C, τ):

$$\tau([a, b]c) = \tau(a[b, c]) = \tau([c, a]b) \tag{3}$$

as it can easily be checked: $\tau([a,b]c) = \tau(abc-bac) = \tau(abc)-\tau(bac) = \tau(abc)-\tau(acb) = \tau(a[b,c]) = \tau(cab)-\tau(acb)$

Since a lot of the derivatives of the relations we will consider are commutator relations, the following lemma will be very useful.

Lemma 3.1. *Let (C,τ) be a matrix algebra with a trace τ. Let $A \subset B$ be $*$-closed sub-algebras of C. For $c \in C$ we have:*

$$[c,A] \subset B \text{ if and only if } c \in B + A' \cap C$$

Proof. The right to left implication is clearly true, as

$$[B + A' \cap C, A] \subset [B,A] + [A' \cap C, A] = [B,A] \subset [B,B] \subset B$$

Assume that c is such that $[c,A] \subset B$. Let $c_0 = E_B(c) \in B$. Since $[c_0, A] \subset [B,B] \subset B$, we have $[c-c_0, A] \subset B$. On the other hand, $[c-c_0, A]$ is perpendicular to B. Indeed, for $a \in A$ and $b \in B$ equation (3) implies:

$$\tau([c-c_0, a]b^*) = \tau((c-c_0)[a,b^*]) = 0$$

as $[a,b^*] \in B$ and $c-c_0$ is orthogonal to B.

It follows that $[c-c_0, A] = 0$ so $c-c_0 \in A' \cap C$. Thus $c \in B + A' \cap C$, which ends the proof.

□

A consequence of Lemma 3.1 is the following:

Lemma 3.2. *Let (C,τ) be a matrix algebra with a trace τ. Let $A \subset B$ be $*$-closed sub-algebras of C. Let $c_n \to I$ be unitaries in C such that $c_n A c_n^* \subset B$ for all n. Then there exist unitaries $b_n \to I$ in B and $a_n' \to I$ in $A' \cap C$ such that $c_n = b_n a_n$ for all n large.*

Proof. Let $\mathfrak{X} = \mathcal{U}(B) \times \mathcal{U}(A' \cap C)$. Since $\mathfrak{X}$ is compact in $\| \ \|_2$, for every n there exist elements $b_n \in B, a_n' \in A' \cap C$ that realize the minimum:

$$\|b_n^* c_n a_n'^* - I\|_2 = \inf_{(b,a') \in \mathfrak{X}} \|b^* c_n a'^* - I\|_2$$

Let $u_n = b_n^* c_n (a_n')^*$. Clearly $u_n \to I$, since for $b = a' = I$ we have: $\|u_n - I\|_2 \leq \|c_n - I\|_2$. If $u_n = I$ for all n large we are done. Assume, by eventually passing to a subsequence, that $u_n \neq I$ for all n. By passing again to a subsequence, we may assume that $h = \lim_{n\to\infty} \frac{u_n - I}{i\|u_n - I\|}$ exists. Clearly $\|h\| = 1$.

Let $\Re\tau$ denote the real part of the trace τ. For every u unitary we have: $\|u - I\|_2^2 = 2 - 2\Re\tau(u)$. It follows:

$$\Re\tau(u_n) \geq \Re\tau(b^* c_n a'^*), \text{ for all } (b,a') \in \mathfrak{X}$$

Let λ be a real number, let $b_0 \in B$ be a hermitian element, and let $b = (\exp(i\lambda b_0)b_n^*)^*$, $a' = a'_n$. The previous inequality implies:

$$\Re\tau(u_n) \geq \Re\tau(exp(i\lambda b_0)u_n) \Longrightarrow \Re\tau((exp(i\lambda b_0) - I)u_n) \leq 0$$

After dividing by $\lambda > 0$ and taking limit as λ approaches 0, we obtain $\Re\tau(ib_0u_n) \leq 0$. Similarly, after dividing by $\lambda < 0$ we have $\Re\tau(ib_0u_n) \geq 0$. It follows:

$$\Re\tau(ib_0u_n) = 0$$

Since for hermitians b_0 we have $\Re\tau(ib_0) = 0$, we can rewrite the previous equality as: $\Re\tau(ib_0(u_n - I)) = 0$ and after dividing by $\|u_n - I\|$ and taking the limit we obtain $\Re\tau(b_0h) = 0$. Since $\tau(b_0h) = \tau(hb_0) = \tau((b_0h)^*)$, it follows that $\tau(b_0h)$ is a real number and thus $\tau(b_0h) = 0$. Consequently:

$$E_B(h) = 0$$

Similar arguments show that $E_{A'\cap C}(h) = 0$. Also, note that $u_nAu_n^* \subset B$ for all $n \geq 1$.

For every $a \in A$ we have: $(u_n - I)au_n^* + a(u_n - I) = u_nau_n^* - a \in B$ for all $n \geq 1$. After dividing by $i\|u_n - I\|$ and taking the limit, we obtain:

$$[h, a] \in B \text{ for all } a \in A$$

and Lemma 3.1 implies

$$h \in B + (A' \cap C)$$

Since the vector space $B + (A' \cap C)$ can be written as the sum of two orthogonal subspaces $(B \ominus (A' \cap B)) \oplus (A' \cap C)$ and h is orthogonal on both these subspaces, we obtain $h = 0$ which is a contradiction. Thus, $u_n = I$ for all n large, or equivalently $c_n = b_na_n$.

□

We now present the main result of this section, dealing with local minima of intersections of algebras.

Proposition 3.3. *Let (C, τ) be a matrix algebra with a trace τ. Let A, B be $*$-closed sub-algebras of C. Let $w_n \to I$ be a sequence of unitaries in C such that $w_n \neq I$ and $h = \lim_{n\to\infty} \frac{w_n - I}{i\|w_n - I\|}$ exists. Assume that $A \cap B$ unitarily embeds into $A \cap w_nBw_n^*$ for all $n \geq 1$. Then*

$$h \in A + B + (A \cap B)' \cap C$$

Proof. Let $v_n \to I$ be unitaries in C such that:

$$v_n(A \cap B)v_n^* \subset A \cap w_nBw_n^*$$

for all $n \geq 1$.After eventually passing to a subsequence we may assume that $v_n \to v \in C$. By taking the limit it follows that $v(A\cap B)v^* \subset A\cap B$

and because both sides have the same dimension we must have equality. Then $\tilde{v}_n = v_n v^* \to I$ and $\tilde{v}_n(A\cap B)\tilde{v}_n^* = v_n(A\cap B)v_n^* \subset A \cap w_n B w_n^*$. This shows that we can assume, by substituting $v_n \to \tilde{v}_n$, that $v_n \to I$.

Assume that infinitely many of v_n are different from I. By eventually passing to a subsequence, we may assume that the limit: $\tilde{h} = \lim_{n\to\infty} \frac{v_n - I}{i\|v_n - I\|}$ exists. Notice that by modifying $v_n \to v_n s_n s_n'$ with unitaries $s_n \in A\cap B, s_n' \in (A\cap B)'\cap C$, $s_n, s_n' \to I$, we do not change the algebra $v_n(A\cap B)v_n^*$. Thus, an argument similar to Proposition 2.9 shows that we may assume $\tilde{h}$ orthogonal to $A\cap B$ and $(A\cap B)'\cap C$.

For $n \geq 1$ let $r_n = \sup(\|w_n - I\|, \|v_n - I\|)$. Clearly $r_n \neq 0$. By eventually passing to a subsequence, we may assume that the following limits exist:

$$h_w = \lim_{n\to\infty} \frac{w_n - I}{ir_n}, h_v = \lim_{n\to\infty} \frac{v_n - I}{ir_n}$$

Notice that from the definition of r_n it follows that at least one of h_w, h_v must be non-zero.

Since $\frac{\|w_n - I\|}{r_n} \cdot \frac{w_n - I}{i\|w_n - I\|} = \frac{w_n - I}{ir_n}$, we have $h_w = ch$ for some positive scalar c (which may be 0). In particular h_w is hermitian. A similar argument shows that $h_v = d\tilde{h}$ for some $d \geq 0$, thus h_v is hermitian and orthogonal to $A\cap B$, $(A\cap B)'\cap C$.

Let $s \in A\cap B$. We have: $(v_n - I)sv_n^* + s(v_n - I)^* = v_n s v_n^* - s \in A$. Consequently:

$$\frac{v_n - I}{ir_n} s v_n^* - s(\frac{v_n - I}{ir_n})^* \in A$$

After taking the limit of this relation as $n \to \infty$ and using $h_v = h_v^*$ we obtain:

$$[h_v, s] \in A \text{ for all } s \in A\cap B$$

and Lemma 3.1 implies

$$h_v \in A + (A\cap B)'\cap C \tag{4}$$

For $s \in A\cap B$ we also have: $v_n s v_n^* \in w_n B w_n^*$. Equivalently:

$$w_n^* v_n s (w_n^* v_n)^* \in B \text{ for all } n \geq 1 \tag{5}$$

Observe that

$$\frac{w_n^* v_n - I}{ir_n} = -(\frac{w_n - I}{ir_n})^* v_n + \frac{v_n - I}{ir_n} \to h_v - h_w$$

Thus, after dividing by ir_n and taking the limit, equation (5) yields

$$[h_v - h_w, s] \in B \text{ for all } s \in A\cap B$$

and after applying again Lemma 3.1 we obtain:

$$h_v - h_w \in B + (A\cap B)'\cap C \tag{6}$$

Combining (4) and (6) yields:

$$h_w \in A + B + (A \cap B)' \cap C$$

Since $h_w = ch$, we only need to argue that $c \neq 0$ to finish the proof. If $c = 0$ then $h_w = 0$ and equation (6) becomes

$$h_v \in B + (A \cap B)' \cap C$$

Thus $h_v = b + s'$ with $b \in B$ and $s' \in (A \cap B)' \cap C$. We have:

$$0 = E_{(A\cap B)'\cap C}(h_v) = E_{(A\cap B)'\cap C}(b) + s'$$

which implies $s' = -E_{(A\cap B)'\cap C}(b) \in (A \cap B)' \cap B$ so $h_v = b + s' \in B$.

Similarly, equation (4) implies $h_v \in A$. Thus $h_v \in A \cap B$, which implies $h_v = 0$ since h_v is orthogonal to $A \cap B$. We thus obtained that both h_v, h_w are 0, which is impossible.

We still have to deal with the case when infinitely many v_n are equal to I. This yields:

$$A \cap B \subset A \cap w_n B w_n^*$$

Thus for every $s \in A \cap B$ we have

$$(w_n - I)sw_n^* + s(w_n - I)^* = w_n^* s w_n - s \in B$$

Dividing by $i\|w_n - I\|$ and taking the limit yields $[h, s] \in B$ for all $s \in A \cap B$. Thus in this case we easily obtain $h \in B + (A \cap B)' \cap C$. $\square$

Remark 3.4. *Similar arguments as in Lemma 3.2 can be used to show that $w_n = a_n s_n b_n$, with $a_n \in A$, $b_n \in B$, $s_n \in (A \cap B)' \cap C$. However, to deduce the conclusion of the proposition one still needs a proof in the same lines, to control the speed of convergence of a_n, b_n, s_n.*

We end this section with a lemma that will be useful towards proving the main result.

Lemma 3.5. *Let (C, τ) be a matrix algebra with a trace τ. Let A, B be $*$-closed sub-algebras of C and $D = A \cap B$. Assume that the commuting square condition holds: $E_A E_B = E_B E_A = E_D$. Then we have the following equality of vector spaces:*

$$D' \cap (A + B) = D' \cap A + D' \cap B$$

Proof. We just have to show that "$\subset$" holds. Let $a \in A, b \in B$ be such that $a + b \in D' \cap C$. Since $D \subset A \subset C$, we have $E_A(D' \cap C) = D' \cap A$. On the other hand $E_A(a + b) = a + E_A(b) = a + E_D(b)$. It follows that $a + E_D(b) \in D' \cap A$. If we rewrite $a + b = a_1 + b_1$, with $a_1 = a + E_D(b) \in A$ and $b_1 = b - E_D(b) \in B$, we have $a_1 \in D' \cap C$ which also implies $b_1 \in D' \cap C$. This ends the proof.

$\square$

4. The Main Result

We are now ready to prove Theorem 2.4, stating that there exist only finitely many isomorphism classes of commuting squares $\mathfrak{C}$, with $\dim(P_0)$ fixed, satisfying the LRC condition. Since we fix $\dim(P_0)$, it is clear that we may in fact assume that both the algebra P_0 and the λ-Markov trace τ are fixed, without changing the finiteness result.

We assume, by contradiction, that the theorem is false. The discussion from Section 2 shows then that there exist non-isomorphic commuting squares

$$\mathfrak{C}_n = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & u_n Q_0 u_n^* & \end{pmatrix}, \mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

all satisfying the LRC condition, where $u_n \to I$ are unitaries in $Q'_{-1} \cap P_0$, $u_n \neq I$. We may also assume, by eventually passing to a subsequence, that the unitaries u_n converge in direction h_0, i.e.

$$h_0 = \lim_{n\to\infty} \frac{u_n - I}{i\|u_n - I\|}$$

Proposition 2.9 shows that we may take h_0 orthogonal to $Q'_{-1} \cap Q_0, Q'_{-1} \cap P_{-1}, P'_{-1} \cap P_0, Q'_0 \cap P_0$.

Moreover, we may assume that the lattices $\mathfrak{L}_n, \mathfrak{L}$ obtained by doing the basic construction from $\mathfrak{C}_n, \mathfrak{C}$ are of the form:

$$\mathfrak{L}_n = \begin{pmatrix} P_{-1} & \subset & P_0 & \overset{e}{\subset} & P_1 & \\ \cup & & \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & u_n Q_0 u_n^* & \overset{e}{\subset} & w_n Q_1 w_n^* & \end{pmatrix}$$

$$\mathfrak{L} = \begin{pmatrix} P_{-1} & \subset & P_0 & \overset{e}{\subset} & P_1 & \\ \cup & & \cup & & \cup & , \ \tau \\ Q_{-1} & \subset & Q_0 & \overset{e}{\subset} & Q_1 & \end{pmatrix}$$

where e is the Jones projection of the basic construction $P_{-1} \subset P_0 \subset P_1$, $u_n \to I$ are unitaries in P_0 and $w_n \to I$ are unitaries in P_1. Also, $R_1^n = P'_{-1} \cap w_n Q_1 w_n^*$ is unitarily conjugate to $R_1 = P_{-1} \cap Q_1$ for all n.

Since $P_0 \cap w_n Q_1 w_n^* = u_n Q_0 u_n^*$, it is clear that $w_n \neq I$ for n large. By eventually passing to a subsequence, we may assume the existence of the limit

$$h_1 = \lim_{n\to\infty} \frac{w_n - I}{i\|w_n - I\|}$$

Notice that the algebra $w_n Q_1 w_n^*$ does not change if we modify w_n by multiplying it to the right with unitaries of Q_1 or $Q'_1 \cap P_0$. Thus,

similar arguments to Proposition 2.9 and Lemma 3.2 show that we may assume h_1 orthogonal to $Q_1, Q_1' \cap P_1$.

Applying Proposition 3.3 for $A = P_{-1}' \cap P_1$, $B = Q_1$, $R_1 = A \cap B$ yields:

$$h_1 \in P_{-1}' \cap P_1 + Q_1 + R_1' \cap P_1$$

For $n \geq 1$ let $r_n = \sup(\|w_n - I\|, \|u_n - I\|)$. Clearly $r_n \neq 0$. By eventually passing to a subsequence, we may assume that the following limits exist:

$$h_w = \lim_{n\to\infty} \frac{w_n - I}{ir_n}, h_u = \lim_{n\to\infty} \frac{u_n - I}{ir_n}$$

Notice that from the definition of r_n it follows that at least one of h_w, h_u must be non-zero. Also, arguments similar to those from Proposition 3.3 show that $h_u = ch_0$, $h_w = dh_1$ for some positive (but possibly equal to zero!) scalars c, d. It follows that h_u, h_v also satisfy:

$$h_u \in Q_{-1}' \cap P_0 \tag{7}$$

$$h_w \in P_{-1}' \cap P_1 + Q_1 + R_1' \cap P_1 \tag{8}$$

$$h_u \perp Q_{-1}' \cap Q_0, Q_{-1}' \cap P_{-1}, P_{-1}' \cap P_0, Q_0' \cap P_0 \tag{9}$$

$$h_w \perp Q_1, Q_1' \cap P_1 \tag{10}$$

We have :

$$\lim_{n\to\infty} \frac{w_n^* u_n - I}{ir_n} = h_u - h_w \tag{11}$$

Since $u_n Q_0 u_n^* \subset w_n Q_1 w_n^*$, it follows that $(w_n^* u_n) Q_0 (w_n^* u_n)^* \subset Q_1$ for all n. Thus

$$(w_n^* u_n - I) q_0 (w_n^* u_n)^* + q_0 (w_n^* u_n - I)^* \in Q_1 \text{ for all } q_0 \in Q_0$$

After dividing by ir_n, using (11) and taking the limit we obtain:

$$[h_u - h_w, q_0] \in Q_1 \text{ for all } q_0 \in Q_0$$

Using Lemma 3.1 yields:

$$h_u - h_w \in Q_1 + Q_0' \cap P_1 \tag{12}$$

Thus, we may write $h_w = h_u + h$, where h is a hermitian in $Q_1 + Q_0' \cap P_1$.

After projecting on P_0 and using the LRC condition $E_{P_0}(R_1' \cap P_1) = P_{-1}$, equation (8) yields:

$$E_{P_0}(h_w) \in P_{-1}' \cap P_0 + Q_0 + P_{-1}$$

On the other hand, $E_{P_0}(h_w) = h_u + E_{P_0}(h)$ and $E_{P_0}(h) \in E_{P_0}(Q_1) + E_{P_0}(Q_0' \cap P_1) \subset Q_0 + Q_0' \cap P_0$, which implies:

$$h_u \in P_{-1}' \cap P_0 + Q_0 + P_{-1} + Q_0' \cap P_0$$

We also know that $h_u \in Q'_{-1} \cap P_0$. Using the previous relation and Lemma 3.5 for $A = Q_0$, $B = P_{-1}$, $D = Q_{-1}$, we obtain:

$$h_u \in P'_{-1} \cap P_0 + Q'_{-1} \cap Q_0 + Q'_{-1} \cap P_{-1} + Q'_0 \cap P_0$$

This together with equation (9) implies $h_u = 0$. Thus

$$h_w = h \in Q_1 + Q'_0 \cap P_1$$

We can write:

$$h_w \in Q_1 + Q'_0 \cap P_1 = Q_1 \ominus (Q'_0 \cap Q_1) \oplus Q'_0 \cap P_1$$

and the two vector spaces are orthogonal: $Q_1 \ominus (Q'_0 \cap Q_1) \perp Q'_0 \cap P_1$. On the other hand, we know that $h_w \perp Q_1$. We obtain

$$h_w \in Q'_0 \cap P_1 \tag{13}$$

We now use the existence of the Jones projection $e \in w_n Q_1 w_n^*$. Since $(w_n - I)^* e w_n + e(w_n - I) = w_n^* e w_n - e \in Q_1$ for all n, after dividing by ir_n and taking the limit as $n \to \infty$ we obtain:

$$[h_w, e] \in Q_1$$

However, $E_{Q_1}([h_w, e] = [E_{Q_1}(h_w), e] = 0$, which shows that $[h_w, e] = 0$. Thus $h_w \in e' \cap P_1$. Together with equation (13) this yields:

$$h_w \in Q'_0 \cap e' \cap P_1 =< Q_0, e >' \cap P_1 = Q'_1 \cap P_1$$

which together with (10) implies $h_w = 0$.

We thus obtained that both h_v, h_w are 0, which is a contradiction.

5. Examples

Let $N \subset M$ be a subfactor of depth 2, i.e. $N' \cap M \subset N' \cap M_1 \subset N' \cap M_2$ is a basic construction. By a result of [Po1], [Oc], the commuting square

$$\mathfrak{C} = \begin{pmatrix} N' \cap M_1 & \subset & N' \cap M_2 \\ \cup & & \cup \\ M' \cap M_1 & \subset & M' \cap M_2 \end{pmatrix}, \tau$$

uniquely determines the isomorphism class of $N \subset M$. $\mathfrak{C}$ is called a *standard* commuting square. If we let $P_0 = N' \cap M_2, P_{-1} = N' \cap M_1, Q_0 = M' \cap M_2, Q_{-1} = M' \cap M_1$ we have $P_1 =< P_0, e_3 >= N' \cap M_3$ and $Q_1 =< Q_0, e_3 >= M' \cap M_3$.

We have: $P'_{-1} \cap Q_1 = (N' \cap M_1)' \cap (N' \cap M_3) = M'_1 \cap M_3$. The last equality is easy to check, since $(N' \cap M_1)' \cap (N' \cap M_3) \subset e'_1 \cap (M' \cap M_3) =< M, e_1 >' \cap M_3 = M_1 \cap M_3$ and clearly $M'_1 \cap M_3 \subset (N' \cap M_1)' \cap (N' \cap M_3)$. Thus:

$$R_1 = P'_{-1} \cap Q_1 = M'_1 \cap M_3$$

which is anti-isomorphic (and therefore isomorphic) to $N' \cap M_1$. This justifies asking for a largeness condition on R_1.

We now investigate when does $\mathfrak{C}$ satisfy the LRC condition.

Proposition 5.1. *Let $\mathfrak{C}$ be the standard commuting square associated to the depth 2 subfactor $N \subset M$. Then $\mathfrak{C}$ is a LRC commuting square if and only if*

$$E_{(N'\cap M_1)'\cap(N'\cap M_2)}(e_2) \in \mathbb{C}I \tag{14}$$

Proof. Because of the duality in the lattice of relative commutants of a subfactor, conditions (1) and (2) from the definition of LRC commuting squares are equivalent for $\mathfrak{C}$. Indeed, if J is the conjugation map on $L^2(M_1,\tau)$ and we embed N, M, M_1, M_2, M_3 in $B(L^2(M_1,\tau))$, we have: $JP_{-1}J = R_1$, $JP_0J = Q_1$, $JP_1J = P_1$. This shows that

$$E_{P_0}(R_1' \cap P_1) = P_{-1} \text{ if and only if } E_{Q_1}(P_{-1}' \cap P_0) = R_1$$

Is is thus sufficient to work with $E_{Q_1}(P_{-1}' \cap P_0) = R_1$. Equivalently, this can be written

$$Q_1 \ominus R_1 \perp P_{-1}' \cap P_1 \tag{15}$$

Assume (15) holds. Let $\lambda^{-1} = [M : N]$. Since $e_2 \in Q_1$ and $E_{R_1}(e_2) = \lambda \cdot I$, we have $e_2 - \lambda I \in Q_1 \ominus R_1$ and thus we must have $e_2 - \lambda I \perp P_{-1}' \cap P_1$. Equivalently: $E_{P_{-1}'\cap P_1}(e_2) = \lambda I$. Since $e_2 \in P_0$, we have:

$$E_{P_{-1}'\cap P_0}(e_2) = E_{P_{-1}'\cap P_1}(e_2) = \lambda I \in \mathbb{C}I$$

which shows that the left to right implication holds true.

We now prove that $E_{P_{-1}'\cap P_0}(e_2) = \lambda I$ implies (15). Let $x \in P_{-1}' \cap P_1$. It is sufficient to show that $y = x - E_{R_1}(x) \in P_{-1}' \cap P_1$ is orthogonal to Q_1. Since $Q_1 =< R_1, e_2 >$, it is enough to show that y is orthogonal on elements of the form re_2r', with $r, r' \in R_1$. We have $\tau(yre_2r') = \tau(r'yre_2)$. Since $r'yr \in P_{-1}' \cap P_1$ and $E_{P_{-1}'\cap P_1}(e_2) = E_{P_{-1}'\cap P_0}(e_2) = \lambda I$, we obtain:

$$\tau(yre_2r') = \tau(r'yre_2) = \lambda\tau(r'yr) = \lambda\tau(yrr') = 0$$

as y is orthogonal to R_1. This ends the proof.

□

By combining the previous result with Theorem 2.4, we obtain the following finiteness result for the standard invariants of finite depth subfactors. We mention that this result is well known to specialists as a theorem of A. Ocneanu, even without the extra assumption $E_{(N'\cap M_1)'\cap(N'\cap M_2)}(e_2) \in \mathbb{C}I$.

Corollary 5.2. *There exist only finitely many isomorphism classes of standard commuting squares $\mathfrak{C}$, of fixed dimension* $\dim P_0$*, arising from depth* 2 *subfactors $N \subset M$ with $E_{(N' \cap M_1)' \cap (N' \cap M_2)}(e_2) \in \mathbb{C}I$.*

Remark 5.3. *In a depth* 2 *subfactor, the following is a basic construction: $N' \cap M \subset N' \cap M_1 \subset N' \cap M_2$. For any $A \subset B \overset{e}{\subset} C$ a basic construction of finite dimensional $*$-algebras, we can state the previous condition: $E_{B' \cap C}(e) \in \mathbb{C}I$. This does NOT always hold true. For instance, it fails for $A = M_k \oplus M_l$ with $k \neq l$ and $B = M_n$, $n = k + l$. We will see however that it holds true if $A \subset B$ has an ONB closed under taking adjoints, and in particular if $A = N' \cap M$ is a factor.*

Proposition 5.4. *With the previous notations, if $N' \cap M \subset N' \cap M_1$ admits an orthonormal basis closed under taking adjoints, then $\mathfrak{C}$ is LRC.*

Proof. Let $\{a_i\}_{1 \leq i \leq n}$ be such an orthonormal basis. Then a_i satisfy $\sum a_i e_2 a_i^* = \sum a_i^* e_2 a_i = 1$, $\sum a_i a_i^* = \lambda^{-1}$. For $a' \in (N' \cap M_1)' \cap (N' \cap M_2)$ we have:

$$\tau(e_2 a') = \lambda \tau(e_2 a' \sum a_i a_i^*) = \lambda \tau((\sum a_i^* e_2 a_i) a') = \lambda \tau(a')$$

This shows that $E_{(N' \cap M_1)' \cap (N' \cap M_2)}(e_2) \in \lambda I$, which ends the proof. □

Corollary 5.5. *If $N \subset M$ is a depth* 2 *subfactor with $N' \cap M$ a factor (of type* $\mathrm{I_n}$*), then the associated depth* 2 *commuting square is LRC.*

Proof. $B = N' \cap M_1$ must be a tensor product $B = A \otimes S$, where A is the factor $N' \cap M$ and S is some $*$-subalgebra of B. Any ONB of S which is closed under taking adjoints is also an ONB for $A \subset B$, closed under taking adjoints. □

In particular, the LRC condition holds if the first relative commutant is trivial, i.e. $N' \cap M = \mathbb{C}I$. By a result of [Szy], such commuting squares correspond precisely to the finite dimensional Hopf C^*-algebras. We thus obtain a new proof of the following theorem of D. Stefan:

Corollary 5.6. *For every $N \geq 1$ there exist only finitely many N-dimensional Hopf $*$-algebras.*

A somewhat more general class of LRC commuting squares arises from depth 2 subfactors $N = I_k \otimes N \subset M_k(\mathbb{C}) \otimes M$, where $N \subset M$ is a Hopf algebra cross product subfactor and I_k denotes the identity matrix of $M_k(\mathbb{C})$, $k \geq 2$.

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