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On II_1 factors arising from 2-cocycles of w-rigid groups

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Abstract

We consider II_1 factors $L_\mu(G)$ arising from 2-cocycles $\mu \in \mathrm{H}^2(G, \mathbb{T})$ on groups G containing infinite normal subgroups $H \subset G$ with the relative property (T) (i.e., G *w-rigid*). We prove that given any separable II_1 factor M, the set of 2-cocycles $\mu|_H \in \mathrm{H}^2(H, \mathbb{T})$ with the property that $L_\mu(G)$ is embeddable into M is at most countable. We use this result, the relative property (T) of $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$ for $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable and the fact that every cocycle $\mu_\alpha \in \mathrm{H}^2(\mathbb{Z}^2, \mathbb{T}) \simeq \mathbb{T}$ extends to a cocycle on $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$, to show that the one parameter family of II_1 factors $M_\alpha(\Gamma) = L_{\mu_\alpha}(\mathbb{Z}^2 \rtimes \Gamma)$, $\alpha \in \mathbb{T}$, are mutually non-isomorphic, modulo countable sets, and cannot all be embedded into the same separable II_1 factor. Other examples and applications are discussed.

Keywords: II_1 factors; Property (T) groups; 2-Cocycles

0. Introduction

Ever since Connes' celebrated "rigidity" paper [6], groups with the property (T) of Kazhdan have played an important rôle in operator algebra, being used to obtain a plethora of rigidity results and interesting examples (see, e.g., [3,4,20,22–25,29]), especially in the theory of II_1 factors. More recently, a weaker version of the property (T), merely requiring the existence of

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a "large" subgroup with the relative property (T) of Kazhdan and Margulis [16,18], proved to be equally important (cf. [23–25]). The prototype of such group is $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, with $\mathbb{Z}^2$ its relative property (T) subgroup (cf. [16,18]). Thus, it is shown in [23] that the II_1 factors associated with this arithmetic group, and more generally with the groups $\mathbb{Z}^2 \rtimes \Gamma$, for Γ non-amenable finitely generated subgroups of $SL(2,\mathbb{Z})$, have trivial fundamental group and are non-isomorphic if the groups Γ have different ℓ^2-Betti numbers, e.g., $\Gamma = \mathbb{F}_n$, $n = 2, 3, \ldots$. (For $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable the inclusion of groups $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$ was shown to have relative property (T) in [2].) This provided the first examples of factors with trivial fundamental group [19].

More generally, in [23], see also [25], one considers a one parameter family of II_1 factors $M_\alpha(\Gamma)$, $\alpha \in \mathbb{T}$, associated with $\mathbb{Z}^2 \rtimes \Gamma$, for each $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable, and one proves several rigidity properties and classification results for $M_\alpha(\Gamma)$. We continue in this paper the analysis of this interesting class of II_1 factors.

The factors $M_\alpha(\Gamma)$ are defined to be crossed product II_1 factors of the form $M_\alpha(\Gamma) = R_\alpha \rtimes_{\sigma_\alpha} \Gamma$, where $\alpha \in \mathbb{T}$, R_α is the finite von Neumann algebra generated by two unitaries $u, v \in R_\alpha$ satisfying the relation $uv = \alpha vu$ and trace $\tau(u^k v^l) = 0$, $\forall (k,l) \neq (0,0)$, Γ is an arbitrary non-amenable subgroup of $SL(2,\mathbb{Z})$ and the action σ_α is implemented by the restriction to Γ of the action of $SL(2,\mathbb{Z})$ on R_α given by $\sigma_\alpha(g)(u^k v^l) = \alpha^{\frac{1}{2}(kl-(ak+bl)(ck+dl))} u^{ak+bl} v^{ck+dl}$, where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

(see [1]).

If α is a primitive root of unity of order n, then R_α is isomorphic to $L((n\mathbb{Z})^2) \otimes M_{n\times n}(\mathbb{C})$ and $M_\alpha(\Gamma) \simeq L((n\mathbb{Z})^2 \rtimes \Gamma) \otimes M_{n\times n}(\mathbb{C})$ [23, Corollary 5.2.1]. If Γ is finitely generated and α' is another primitive root of 1 of order n' then by [23] $M_\alpha(\Gamma) \simeq M_{\alpha'}(\Gamma)$ if and only if $n = n'$. If, in turn, $\alpha = e^{2\pi i\theta} \in \mathbb{T}$ with $\theta \in [0, 1/2)$ irrational then R_α is isomorphic to the hyperfinite II_1 factor, represented as the *irrational rotation* von Neumann algebra R_α [26]. The factors $M_\alpha(\Gamma)$ are called *irrational* (respectively *rational*) *rotation* HT *factors* when $\alpha = e^{2\pi i\theta}$ with $\theta \in [0, 1/2) \setminus \mathbb{Q}$ (respectively $\theta \in \mathbb{Q}$). By [23], if Γ is non-amenable then an irrational rotation HT factor $M_\alpha(\Gamma)$ cannot be embedded into a rational rotation HT factor $M_{\alpha'}(\Gamma')$.

The problem of classifying the family of factors $M_\alpha(\Gamma)$, in terms of the embedding $\Gamma \subset SL(2,\mathbb{Z})$ and the parameter $\alpha \in \mathbb{T}$, is quite natural. In this respect, it has been conjectured in [23] that for each fixed $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable (notably for $\Gamma = SL(2,\mathbb{Z})$), the factors $M_\alpha(\Gamma)$, $\alpha \in \mathbb{T}$, irrational, are mutually non-isomorphic. In this paper we will give a partial, positive answer to this problem, by showing that for each fixed non-amenable group $\Gamma \subset SL(2,\mathbb{Z})$ the factors $M_\alpha(\Gamma)$, $\alpha \in \mathbb{T}$, are mutually non-stably isomorphic, modulo countable sets, i.e., there are at most countably many α's in $\mathbb{T}$ such that $M_\alpha(\Gamma) \simeq M_{\alpha_0}(\Gamma)$, for a fixed, arbitrary $\alpha_0 \in \mathbb{T}$.

We will alternatively view a factor $M_\alpha(\Gamma)$ as a cocycle group von Neumann algebra $L_{\mu_\alpha}(\mathbb{Z}^2 \rtimes \Gamma)$ (see [7]) corresponding to a projective left regular representation λ_{μ_α} with the scalar 2-cocycle $\mu_\alpha \in \mathrm{H}^2(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{T})$ depending on $\alpha \in \mathbb{T}$. To explain this, let us first recall some definitions.

Let G be a discrete group and $\mu \in \mathrm{H}^2(G,\mathbb{T})$ a 2-cocycle on G, i.e., $\mu: G \times G \to \mathbb{T}$ satisfies $\mu_{g,h}\mu_{gh,k} = \mu_{h,k}\mu_{g,hk}$, $\forall g, h, k \in G$. One associates to μ the *projective left regular representation* $\lambda_\mu : G \to \mathcal{U}(l^2(G))$, defined by $\lambda_\mu(g)(\sum_{h\in G} c_h\xi_h) = \sum_{h\in G} c_h \mu_{g,h}\xi_{gh}$, where $\{\xi_h\}_{h\in G}$ is the canonical basis of $l^2(G)$. Denote by $L_\mu(G) = \lambda_\mu(G)''$ the cocycle group von Neumann algebra of (G,μ). It is well known that one has an isomorphism $\mathrm{H}^2(\mathbb{Z}^2,\mathbb{T}) \cong \mathbb{T}$, taking $\alpha \in \mathbb{T}$ to $\mu_\alpha \in \mathrm{H}^2(\mathbb{Z}^2,\mathbb{T})$, where

$$\mu_\alpha\big((k,l),(k',l')\big) = \alpha^{\frac{1}{2}(kl'-k'l)}.$$

If we define R_{μ_α} to be the cocycle group von Neumann algebra $L_{\mu_\alpha}(\mathbb{Z}^2)$, then R_{μ_α} is generated by the unitary elements $u=\lambda_{\mu_\alpha}(1,0)$, $v=\lambda_{\mu_\alpha}(0,1)$, which satisfy the relation $uv=\alpha vu$, thus being naturally isomorphic to R_α. Moreover, μ_α is invariant to the action σ of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$, thus σ implements an action σ_{μ_α} of $SL(2,\mathbb{Z})$ on $R_{\mu_\alpha}=R_\alpha$ which coincides with the action σ_α defined above.

Since any $\Gamma \subset SL(2,\mathbb{Z})$ has Haagerup's compact approximation property [12], by [23, 6.9.1] it follows that $M_\alpha(\Gamma)$ has Haagerup's property relative to R_α (as defined in [23, 2.1]). Also, by [2, Example 2, p. 62] the pair $(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)$ has the relative property (T) for any non-amenable subgroup $\Gamma \subset SL(2,\mathbb{Z})$ and thus, by [23, 6.9.1], the embedding $R_\alpha \subset M_\alpha(\Gamma)$ is rigid in the sense of [23, Definition 4.2].

Since the action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$ is outer, by [23, 3.3.2(ii)] σ_α are properly outer actions of $SL(2,\mathbb{Z})$ (thus of Γ as well) on R_α. Furthermore, since the stabilizer of any non-trivial element in $\mathbb{Z}^2$ is a cyclic group, it follows that if Γ leaves a finite subset $\neq \{(0,0)\}$ of $\mathbb{Z}^2$ invariant, then it is almost cyclic. Thus, by [23, 3.3.2(i)] any non-amenable $\Gamma \subset SL(2,\mathbb{Z})$ acts ergodically on R_α. Thus, $R_\alpha \subset M_\alpha(\Gamma)$ satisfies $R'_\alpha \cap M_\alpha(\Gamma) \subset R_\alpha$. In particular, when α is irrational, $R_\alpha \subset M_\alpha(\Gamma)$ are irreducible inclusions of II_1 factors, and they are HT inclusions in the sense of [23, 6.1].

By [23] the factors $M_\alpha(\Gamma)$ are non-Γ and by [21] they are prime, i.e., they cannot be decomposed into a tensor product of II_1 factors. It was shown in [25] that two factors $M_\alpha(\Gamma)$ with Γ torsion free are isomorphic iff $\sigma_\alpha(\Gamma)$ are cocycle conjugate in $\mathrm{Out}(R)$. In particular, isomorphism between irrational rotation HT factors $M_\alpha(\Gamma)$, with torsion free Γ, implies isomorphism of the corresponding groups Γ. Also, it follows from [23] that if Γ is torsion free then $M_\alpha(\Gamma)$ has countable fundamental group (see Appendix A for a more general result).

The factors $M_\alpha(\Gamma)$ are easily seen to be "approximately embeddable" into the hyperfinite II_1 factor R (in the sense of Connes [5]), i.e., $M_\alpha(\Gamma) \subset R^\omega$. Indeed, let m_k/n_k be a sequence of rational numbers such that $\alpha_k = \exp(2\pi i m_k/n_k) \to \alpha$ and $n_k \to \infty$. Let π_k be the projective representation with 2-cocycle μ_{α_k}, of the group $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ on $\mathcal{H}_n = \ell^2((\mathbb{Z}/n_k\mathbb{Z})^2 \rtimes SL(2,\mathbb{Z}/n_k\mathbb{Z}))$. Then $g \mapsto (\pi_n(g))_n$ is an embedding of $L_\alpha(\mathbb{Z}^2) \rtimes SL(2,\mathbb{Z})$ into $\Pi_n \mathcal{B}(\mathcal{H}_n) \subset R^\omega$. However, we have:

0.1. Theorem. *Let M be a separable II_1 factor. For each fixed $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable there exist at most countably many $\alpha \in \mathbb{T}$ such that $M_\alpha(\Gamma)$ is embeddable into M (not necessarily unitaly). In particular, the factors $\{M_\alpha(\Gamma)\}_{\alpha\in\mathbb{T}}$ are non-stably isomorphic modulo countable sets.*

Note that the above theorem gives an alternative proof to Ozawa's result on the non-existence of universal separable II_1 factors in [20], without using Gromov's property (T) groups. More precisely, in the same spirit as the results in [20], the above theorem shows that there exists no separable finite von Neumann algebra M that can contain an uncountable set of projective unitary representations $\{\pi_j\}_{j\in J}$ of $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ with distinct scalar 2-cocycles $\{\mu_{\pi_j}\}_j$. Theorem 0.1 will follow as a special case of the following theorem.

0.2. Theorem. *Let $H \subset G$ be an inclusion of discrete groups with the relative property* (T). *Let M be a separable finite von Neumann algebra. Let J be the set of scalar 2-cocycles $\mu \in \mathrm{H}^2(G, \mathbb{T})$ such that $L_\mu(G)$ can be embedded into M* (*not necessarily unitaly*). *Then the set $\{\mu|_H \mid \mu \in J\} \subset \mathrm{H}^2(H, \mathbb{T})$ is countable.*

We prove this result in Section 1, by using a separability argument similar to [6,11,20,22,23] and a characterization of the relative property (T) in terms of projective representations. In Section 2 we give examples of pairs of groups $H \subset G$ with the relative property (T) with the torus $\mathbb{T}$ embedded as a subgroup of 2-cocycles $\mathbb{T} \subset \mathrm{H}^2(G, \mathbb{T})$, such that $\mathbb{T} \ni \mu \mapsto \mu|_H \in \mathrm{H}^2(H, \mathbb{T})$ is one-to-one. In Section 3 we give an explicit description of the disintegration of type II_1 von Neumann algebras from the property (T) groups Λ constructed by Serre, see [13, p. 40], as central extensions of property (T) groups Γ, in terms of factors $L_\alpha(\Gamma)$ associated with 2-cocycles of Γ. We also show that the factors in the disintegration of the algebra $L(\Lambda)$ of an arbitrary property (T) group Λ are mutually non-isomorphic, modulo countable sets, by using a separability argument similar to Ozawa's proofs in [20].

There are strong indications from [23] and results in this paper that for each $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable the factors $\{M_\alpha(\Gamma)\}_{\alpha \in I}$, where $I = \{e^{2\pi i t} \mid t \in [0, 1/2) \setminus \mathbb{Q}\}$, are all mutually non-stably isomorphic and have trivial fundamental group, and that if the normalizer of Γ in $GL(2, \mathbb{Z})$ is equal to Γ then $\mathrm{Out}(M_\alpha(\Gamma))$ is isomorphic to the character group of Γ. Furthermore, if $\Gamma_1, \Gamma_2 \subset SL(2, \mathbb{Z})$ non-amenable and $\alpha_1, \alpha_2 \in I$ then $M_{\alpha_1}(\Gamma_1)$ should be isomorphic to $M_{\alpha_2}(\Gamma_2)$ if and only if $\alpha_1 = \alpha_2$ and Γ_1, Γ_2 are conjugate in $SL(2, \mathbb{Z})$, by an automorphism of $SL(2, \mathbb{Z})$ (thus, by an element in $GL(2, \mathbb{Z})$). In this respect, note that by [25] the non-isomorphism of the factors $M_\alpha(\Gamma)$ for a fixed Γ amounts to showing that if $\alpha_1 \neq \alpha_2$ then $\sigma_{\alpha_1}, \sigma_{\alpha_2}$ are not cocycle conjugate. While we cannot prove this fact, we obtain in Section 4 the following result, whose proof is inspired from an argument in [3].

0.3. Theorem. *Let $\Gamma \subset SL(2, \mathbb{Z})$ be a subgroup of $SL(2, \mathbb{Z})$ containing a parabolic element a and an element b that does not commute with a. If for some $\alpha_1, \alpha_2 \in I$ the actions σ_{α_i} of Γ on R_{α_i}, $i = 1, 2$, are conjugate, then $\alpha_1 = \alpha_2$.*

1. A cocycle characterization of relative property (T)

Recall that an inclusion of discrete groups $H \subset G$ has the *relative property* (T) of Kazhdan–Margulis if there exist $\delta_0 > 0$ and a finite subset $F_0 \subset G$ such that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G on the Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g)\xi - \xi\| < \delta_0$ for all $g \in F_0$, then there exists a vector $\xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0$ for all $h \in H$. Note that in case $H = G$ this amounts to G itself having Kazhdan's property (T). It is easy to see that if H is normal in G then the above condition is equivalent to the following:

(1.0) $\forall \varepsilon > 0$ there exist a finite subset $F(\varepsilon) \subset G$ and $\delta(\varepsilon) > 0$ such that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G on the Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g)\xi - \xi\| < \delta(\varepsilon)$, $\forall g \in F(\varepsilon)$, then there exists a unit vector $\xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| < \varepsilon$ and $\pi(h)\xi_0 = \xi_0$, $\forall h \in H$.

The above condition is in fact equivalent to the relative property (T) even for inclusions that are not necessarily normal (cf. [15]), but we will only use here the equivalence for normal inclusions.

We first show that if $H \subset G$ has the relative property (T) then the projective representations satisfy a property similar to (1.0). To state the result, recall that a projective (unitary) representation of the group G on the Hilbert space $\mathcal{H}$ is a map $\pi : G \to \mathcal{U}(\mathcal{H})$ satisfying $\pi(g)\pi(h) = \mu_{g,h}\pi(gh)$, $\forall g, h \in G$, for some scalar 2-*cocycle* μ on G, i.e., $\mu : G \times G \to \mathbb{T}$ satisfies $\mu_{g,h}\mu_{gh,k} = \mu_{h,k}\mu_{g,hk}$, $\forall g, h, k \in G$. It is immediate to see that the equivalence class of such a π only depends on the class of μ in $\mathrm{H}^2(G, \mathbb{T}) \overset{\text{def}}{=} \mathrm{Z}^1(G, \mathbb{T})/\mathrm{B}^1(G, \mathbb{T})$, where $\mathrm{Z}^1(G, \mathbb{T})$ denotes the multiplicative group of all scalar valued 2-cocycles and $\mathrm{B}^1(G, \mathbb{T})$ is the subgroup of *coboundaries*, $\mu_{g,h} = \lambda_g\lambda_h\overline{\lambda_{gh}}$, for some $\lambda : G \to \mathbb{T}$.

1.1. Lemma. *Let $H \subset G$ be an inclusion of groups satisfying the relative property* (T). *Fix $1 \geqslant \varepsilon > 0$ and let $F(\varepsilon)$, $\delta(\varepsilon)$ be the constants given by* (1.0). *Denote $\tilde{F}(\varepsilon) = F(\varepsilon^2/28)$, $\tilde{\delta}(\varepsilon) = \delta(\varepsilon^2/28)/2$. Then the following holds true*:

If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a projective representation with scalar 2-cocycle $\mu \in \mathrm{H}^2(G, \mathbb{T})$, and $\xi \in \mathcal{H}$ is a unit vector satisfying $d(\pi(g)\xi, \mathbb{C}\xi) \leqslant \tilde{\delta}(\varepsilon)$, $\forall g \in \tilde{F}(\varepsilon)$, then $\exists \xi_0 \in \mathcal{H}$ and $\lambda : H \to \mathbb{T}$, such that

$$\|\xi - \xi_0\| < \varepsilon, \qquad \pi(h)\xi_0 = \lambda_h\xi_0 \quad \textit{and} \quad \mu_{h,h'} = \lambda_h\lambda_{h'}\bar{\lambda}_{hh'}, \quad \forall h, h' \in H.$$

In particular, if $\delta_1 = \frac{1}{2}\delta(\frac{1}{28})$, $F_1 = F(\frac{1}{28})$, then whenever $\pi : G \to \mathcal{U}(\mathcal{H})$ is a projective representation with scalar 2-cocycle μ such that $\|\pi(g)\xi - \xi\| < \delta_1$, $\forall g \in F_1$, for some unit vector $\xi \in \mathcal{H}$, $\mu|_H$ follows coboundary.

Proof. Note first that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a projective representation of the group G on the Hilbert space $\mathcal{H}$, then $\pi \otimes \bar{\pi} : G \to \mathcal{U}(\mathcal{H} \otimes \overline{\mathcal{H}})$ is a genuine representation of G on the Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$. Identify $\mathcal{H} \otimes \overline{\mathcal{H}}$ with $\mathcal{HS}$, the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}$, and then note that $\pi \otimes \bar{\pi}$ can be extended to all $\mathcal{B}(\mathcal{H})$ by the formula $(\pi \otimes \bar{\pi})(g)(T) = \pi(g)T\pi(g)^*$ for all $g \in G$ and for all T in $\mathcal{B}(\mathcal{H})$.

Fix $\varepsilon > 0$ and let $\tilde{F}(\varepsilon)$ and $\tilde{\delta}(\varepsilon)$ be defined as in the second part of the statement. Let $\xi \in \mathcal{H}$ be a unit vector such that $d(\pi(g)\xi, \mathbb{C}\xi) \leqslant \tilde{\delta}(\varepsilon)$, $\forall g \in \tilde{F}(\varepsilon)$. Then for all $g \in F(\varepsilon^2/28)$

$$\big\|(\pi \otimes \bar{\pi})(g)(\xi \otimes \bar{\xi}) - \xi \otimes \bar{\xi}\big\| = \big\|\pi(g)\xi \otimes \bar{\pi}(g)\bar{\xi} - \xi \otimes \bar{\xi}\big\|_{\mathcal{HS}} < \delta\big(\varepsilon^2/28\big).$$

By the relative property (T) applied to the representation $\pi \otimes \bar{\pi}$ on $\mathcal{HS}$, there exists a Hilbert–Schmidt operator T of $\mathcal{HS}$-norm equal to 1 such that $(\pi \otimes \bar{\pi})(h)(T) = T$, $\forall h \in H$ and $\|T - \xi \otimes \bar{\xi}\| \leqslant \|T - \xi \otimes \bar{\xi}\|_{\mathcal{HS}} < \varepsilon^2/28$.

Thus, $\pi(h)T\pi(h)^* = T$, $\forall h \in H$, implying that the operators $\pi(h)$ and T on the Hilbert space $\mathcal{H}$ commute, $\forall h \in H$. But then T^* and TT^* also commute with $\pi(h)$, $\forall h \in H$. Thus, all the spectral projections of TT^* are in the commutant of $\pi(H)$ in $\mathcal{B}(\mathcal{H})$. Since TT^* is a trace class operator, its spectral projections have finite trace, i.e., they are finite-dimensional.

Since $\|TT^* - \xi \otimes \bar{\xi}\| < 2\varepsilon^2/28$, it follows that $\|(TT^*)^2 - \xi \otimes \bar{\xi}\| < 4\varepsilon^2/28$ and $\|(TT^*)^2 - TT^*\| < 6\varepsilon^2/28$. Thus, if $\varepsilon < 1$, then there exist a non-zero spectral projection P of TT^* with finite rank such that $\|P - TT^*\| < 12\varepsilon^2/28$. This implies that $\|P - \xi \otimes \bar{\xi}\| < \varepsilon^2/2$.

In particular P has to be a rank one projection, i.e., P is of the form $\xi' \otimes \bar{\xi}'$ and $|\langle \xi', \xi\rangle| > 1 - \varepsilon^2/2$. Taking $\alpha \in \mathbb{T}$ such that $\alpha\langle \xi', \xi\rangle > 0$ we get that $\|\alpha\xi' - \xi\| < \varepsilon$.

Let $\xi_0 = \alpha\xi'$. Then $\pi(h)\xi_0 \otimes \overline{\pi(h)(\xi_0)} = (\pi \otimes \bar{\pi})(h)(\xi_0 \otimes \bar{\xi}_0) = \xi_0 \otimes \bar{\xi}_0$ for all $h \in H$ which implies that $\pi(h)(\xi_0) = \lambda_h \xi_0$ with $\lambda_h \in \mathbb{T}$. Since $\pi(h)\pi(h')\xi_0 = \mu_{h,h'}\pi(hh')\xi_0$ we get $\lambda_h\lambda_{h'} = \mu_{h,h'}\lambda_{hh'}$ for all $h, h' \in H$. □

1.2. Theorem. *Let M be a separable finite von Neumann algebra. Let G be a discrete group with a subgroup H such that (G, H) has the relative property* (T). *Let $\{\pi_j\}_{j\in J}$ be projective representations of G into the unitary group of $p_j M p_j$, with scalar 2-cocycles $\{\mu_j\}_{j\in J}$, where $p_j \in \mathcal{P}(M)$ are non-zero projections in M. Then the image of the set $\{\mu_j|_H\}_{j\in J}$ in $\mathrm{H}^2(H, \mathbb{T})$ is at most countable.*

Proof. Let $J_0 \subset J$ be such that the cocycles $\mu_j|_H$, $j \in J_0$, are distinct in $\mathrm{H}^2(H, \mathbb{T})$ and $\{\mu_j|_H\}_{j\in J_0} = \{\mu_j|_H\}_{j\in J}$. We have to prove that J_0 is countable.

Assume it is not. Then there exists $c > 0$ such that the set $J_1 = \{j \in J_0 \mid \tau(p_j) \geqslant c\}$ is uncountable. Let F_1 and δ_1 be as in Lemma 1.1. Let also τ be a normal faithful trace state on M and denote as usual by $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x \in M$ the corresponding Hilbert norm on M. Since M is separable and J_1 is uncountable, there exist $j_1, j_2 \in J_1$ such that

$$\|p_{j_1} - p_{j_2}\|_2 < \delta_1 c/4,$$

$$\left\|\pi_{j_1}(g) - \pi_{j_2}(g)\right\|_2 < \delta_1 c/4, \quad \forall g \in F_1.$$

In particular, the first inequality shows that

$$\|p_{j_1}\|_2 - \|p_{j_1}p_{j_2}\|_2 \leqslant \left\|p_{j_1}(p_{j_1} - p_{j_2})\right\|_2 \leqslant \delta_1 c/4 \leqslant c/2,$$

implying that $\|p_{j_1}p_{j_2}\|_2 \geqslant c - c/2 = c/2$.

For $x \in M$, denote by $L(x)$, $R(x)$ the operators of left, respectively right multiplication by x on $L^2(M, \tau)$. Define $\pi : G \to \mathcal{B}(p_{j_1}L^2(M, \tau)p_{j_2})$ by $\pi(g)\eta = L(\pi_{j_1}(g))R(\pi_{j_2}(g)^*)\eta$. Then π is a projective representation of cocycle $\mu_{j_1}\bar{\mu}_{j_2}$ and if we denote $\xi = \|p_{j_1}p_{j_2}\|_2^{-1}(p_{j_1}p_{j_2})\hat{}$ then ξ has norm one and we have for all $g \in F_1$ the estimates:

$$\begin{aligned}\left\|\pi(g)\xi - \xi\right\| &= \|p_{j_1}p_{j_2}\|_2^{-1}\left\|\pi_{j_1}(g)p_{j_2} - p_{j_1}\pi_{j_2}(g)\right\|_2 \\ &= \|p_{j_1}p_{j_2}\|_2^{-1}\left\|\big(\pi_{j_1}(g) - \pi_{j_2}(g)\big)p_{j_2} + (p_{j_2} - p_{j_1})\pi_{j_2}(g)\right\|_2 \\ &\leqslant (c\delta_1/4 + c\delta_1/4)\|p_{j_1}p_{j_2}\|_2^{-1} \leqslant (c\delta_1/2)\|p_{j_1}p_{j_2}\|_2^{-1} \leqslant \delta_1.\end{aligned}$$

From Lemma 1.1 it follows that the cocycle $\mu_{j_1}\bar{\mu}_{j_2}$ is a coboundary in H, which contradicts $\mu_{j_1}|_H \neq \mu_{j_2}|_H$. □

For the next corollary, recall that given any scalar 2-cocycle μ on a discrete group G, one associates to it the *projective left regular representation* $\lambda_\mu : G \to \mathcal{U}(L^2(G))$, with scalar 2-cocycle μ, defined by $\lambda_\mu(g)(\sum_h c_h\xi_h) = \sum_h c_h\mu_{g,h}\xi_{gh}$. We denote $L_\mu(G) = \lambda_\mu(G)''$ the corresponding von Neumann algebra, as considered by Connes and Jones in [7].

1.3. Corollary. *Let $H \subset G$ be an inclusion of discrete groups with the relative property* (T). *Let M be a separable finite von Neumann algebra. Let J be the set of scalar 2-cocycles $\mu \in$*

$\mathrm{H}^2(G,\mathbb{T})$ *such that* $L_\mu(G)$ *can be embedded into* M (*not necessarily unitaly*)*. Then the set* $\{\mu|_H \mid \mu \in J\} \subset \mathrm{H}^2(H,\mathbb{T})$ *is countable.*

Proof. It is enough to show that for every n the set: $\{\mu|_H \mid \mu \in \mathrm{H}^2(G,\mathbb{T})$ with $L_\mu(G)$ embeddable (not necessarily unitaly) in $M_n(M)\}$ is at most countable. Since for any scalar 2-cocycle μ for G the μ-twisted left regular representation λ_μ is a projective representation with scalar 2-cocycle μ and whose von Neumann algebra is $L_\mu(G)$, the statement follows from the previous theorem. □

Note that when applied to the case $M = M_{n\times n}(\mathbb{C})$ the proof of Lemma 1.1 gives an estimate of the number of certain sets of scalar 2-cocycles of H in terms of the constants of rigidity of $H \subset G$. We emphasize this in the next proposition, where we also include an estimate of the number of projective representations of dimension n of a group with the property (T), generalizing a result in [14].

We need the following notations. For G a discrete group and n a positive integer, we denote by $\mathcal{PR}(G,n)$ the set of equivalence classes of projective representations of G of dimension n. Also, we denote by $\mathrm{H}^2(G,n)$ the set of scalar 2-cocycles $\mu \in \mathrm{H}^2(G,\mathbb{T})$ for which there exists a projective representation $\pi \in \mathcal{PR}(G,n)$ with cocycle μ.

1.4. Proposition.

(1) *Let* G *be a discrete group with property* (T)*. There exists a constant* $c > 1$*, which depends only on the constants of rigidity of* G*, such that* $\mathcal{PR}(G,n)$ *has at most* c^{n^2} *elements,* $\forall n \geqslant 1$.
(2) *Let* $H \subset G$ *be an inclusion of discrete groups such that* (G,H) *has the relative property* (T)*. There exists a constant* d *such that the subset* $\{\mu|_H \mid \mu \in \mathrm{H}^2(G,n)\}$ *of* $\mathrm{H}^2(H,\mathbb{T})$ *has at most* d^{n^2} *elements,* $\forall n \geqslant 1$.

Proof. (1) Let (F_1,δ_1) be as in Lemma 1.1, for $H = G$. By [30] there exists $c_0 > 1$ such that one can cover the unit sphere in $\mathbb{R}^{2n^2}$ with $c_0^{n^2}$ balls of radius $\delta_1/2$, $\forall n \geqslant 1$. Thus one can cover the sphere of radius $\sqrt{n}$ with $c_0^{n^2}$ balls of radius $\sqrt{n}\delta_1/2$.

We first show that there are at most $c_0^{|F_1|n^2}$ irreducible projective representations of dimension n. Assume not. Regarding unitaries in $M_n(\mathbb{C})$ as vectors of norm $\sqrt{n}$ in $\mathbb{R}^{2n^2}$, the pigeonhole principle implies that there exist π_1, π_2 irreducible such that $\|\pi_1(g) - \pi_2(g)\|_2 < \delta_1$, $\forall g \in F_1$.

As in the proof of Theorem 1.2, define $\pi : G \to \mathcal{B}(L^2(\mathbb{M}_n(\mathbb{C}),\tau))$ by

$$\pi(g)\eta = L\big(\pi_1(g)\big)R\big(\pi_2(g)^*\big)\eta, \quad \forall \eta \in L^2\big(M_{n\times n}(\mathbb{C}),\tau\big),$$

where τ is the normalized trace. Then π is a projective representation of G with cocycle $\mu_\pi = \mu_{\pi_1}\bar{\mu}_{\pi_2}$ and it satisfies $\|\pi(g)\hat{1} - \hat{1}\|_2 < \delta_1$, $\forall g \in F_1$. Lemma 1.1 implies that there exist $\lambda : G \to \mathbb{T}$ and a unit vector $\xi_0 \in \mathbb{M}_n(\mathbb{C})$ such that if we define $\pi_1'(g) = \bar{\lambda}_g\pi_1(g)$ then $\pi_1'(g)\xi_0 = \xi_0\pi_2(g)$, $\forall g$, and π_1', π_2 have the same cocycle μ'. Taking the adjoint and noticing that $\pi_1'(g)\pi_1'(g^{-1}) = \pi_2(g)\pi_2(g^{-1}) = \mu'_{g,g^{-1}}1$, $\forall g \in G$, it follows $\xi_0^*\pi_1'(g) = \pi_2(g)\xi_0^*$, $\forall g \in G$. The two relations imply that $\xi_0^*\xi_0$ commutes with $\pi_2(g)$, $\forall g \in G$, and using the irreducibility of π_2 it follows that π_1', π_2 are conjugate.

Thus, for every n there are at most $c_0^{|F_1|n^2}$ irreducible projective representations of order n. This implies that the number of projective representations of dimension n is at most $2^n c_0^{|F_1|n^2}$, so $c = 2c_0^{|F_1|}$ will do.

(2) Let (F_1, δ_1) be the constants from Lemma 1.1 for (G, H), and let c_0 be as before. Let $d = c_0^{|F_1|}$. Assume that $\{\mu|_H \mid \mu \in \mathrm{H}^2(G, n)\}$ has more than d^{n^2} elements. By the pigeonhole principle if follows that there exist $\pi_{j_1}, \pi_{j_2} \in \mathcal{PR}(G, n)$ with cocycles μ_{j_1}, μ_{j_2} such that $\mu_{j_1}|_H \neq \mu_{j_2}|_H$ and $\|\pi_{j_1}(g) - \pi_{j_2}(g)\|_2 < \delta_1$, $\forall g \in F_1$. This leads to a contradiction, as in the proof of Lemma 1.1. □

1.5. Remark. The same proof as for part (1) of the above proposition shows that the similar result for groups G with the property (τ) of Lubotzky holds true (see [17, 1.3] for the definition of property (τ) and [17, 1.4.3] for related statements).

2. Examples

Recall that the groups $SL(n, \mathbb{Z})$, $n > 2$, and $Sp(2n, \mathbb{Z})$, $n > 1$, have the property (T) of Kazhdan [16]. Obvious examples of inclusions with relative property (T) are $H \subset H \times \Gamma$ with H a property (T) group and Γ an arbitrary discrete group. It is shown in [16,18,27] that $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ has the relative property (T). More generally, by a result of Burger [2], any inclusion of the form $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$, with $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable, has the relative property (T). Recently Valette [28] showed that if Γ is an arithmetic lattice in an absolutely simple Lie group, then there exists an embedding of Γ in $SL(m, \mathbb{Z})$ for some m, such that $\mathbb{Z}^m \subset \mathbb{Z}^m \rtimes \Gamma$ has the relative property (T). Fernos [10] constructed other examples of inclusions of groups $\mathbb{Z}^m \subset \mathbb{Z}^m \rtimes \Gamma$ with the relative property (T), for $\Gamma \subset GL(m, \mathbb{Z})$.

More examples of pairs of groups having the relative property (T) come out from the following easy observation.

2.1. Lemma. *Let $\sigma : \Gamma \to \mathrm{Aut}(H)$ and $\sigma' : \Gamma \to \mathrm{Aut}(H')$ be actions of a Γ on H, H' and denote by $\tilde{\sigma} : \Gamma \to \mathrm{Aut}(H \times H')$ the diagonal action, $\tilde{\sigma}(g)(x, y) = (\sigma(g)x, \sigma'(g)y)$, $x \in H$, $y \in H'$, $g \in \Gamma$.*

(1) *If $H \subset H \rtimes_\sigma \Gamma$ and $H' \subset H' \rtimes_{\sigma'} \Gamma$ have the relative property* (T) *then $(H \times H') \subset (H \times H') \rtimes_{\tilde{\sigma}} \Gamma$ has the relative property* (T).
(2) *Assume $H \subset H \rtimes \Gamma$ has the relative property* (T). *Let $\beta \in \mathrm{Aut}(\Gamma)$. Denote $\sigma' = \sigma \circ \beta$ and $\tilde{\sigma}$ the diagonal action $\tilde{\sigma}(g) = \sigma(g) \times \sigma'(g)$ of G on $H \times H$. Then $(H \times H) \subset (H \times H) \rtimes_{\tilde{\sigma}} \Gamma$ has the relative property* (T).
(3) *If Γ is a subgroup of $GL(n, \mathbb{Z})$ such that $\mathbb{Z}^n \subset \mathbb{Z}^n \rtimes \Gamma$ has the relative property* (T) *then $\mathbb{Z}^{2n} \subset \mathbb{Z}^{2n} \rtimes_\theta \Gamma$ also has the relative property* (T), *where for each $g \in \Gamma$ $\theta(g) \in SL(2n, \mathbb{Z})$ is the matrix $\begin{pmatrix} g & 0 \\ 0 & (g^{-1})^t \end{pmatrix}$.*

Proof. (1) If π is a unitary representation of $(H \times H') \rtimes \Gamma$ on $\mathcal{H}$ with an almost invariant unit vector $\xi \in \mathcal{H}$ then by (1.0) ξ follows uniformly almost invariant to $H \times \{e'\}$ and to $\{e\} \times H'$. Since

$$\begin{aligned}\|\pi(h, h')\xi - \xi\| &\leqslant \|\pi(h, e')\pi(e, h')\xi - \pi(h, e')\xi\| + \|\pi(h, e')\xi - \xi\| \\ &\leqslant \|\pi(e, h')\xi - \xi\| + \|\pi(h, e')\xi - \xi\|\end{aligned}$$

for all $h \in H$, $h' \in H'$, it follows that ξ is uniformly almost invariant to $H \times H'$. Thus, if ξ_0 is the element of minimal norm in $\overline{\mathrm{co}}^w\{\pi(h,h')\xi\colon h \in H,\ h' \in H'\} \subset \mathcal{H}$, then ξ_0 is invariant to $\pi(H \times H')$ and $\xi_0 \neq 0$.

(2) Since the inclusions $H \subset H \rtimes_\sigma \Gamma$ and $H \subset H \rtimes_{\sigma'} \Gamma$ are isomorphic, and the first inclusion has the relative property (T), the second one has this property as well. Thus, part (1) applies to get the conclusion.

(3) Apply (2) to $(H \subset H \rtimes \Gamma)$=$(\mathbb{Z}^n \subset \mathbb{Z}^n \rtimes \Gamma)$ and $\beta(g) = (g^{-1})^t$. □

Of all these examples of inclusions of groups $H \subset G$ with the relative property (T) we are interested in those for which the set of restrictions of 2-cocycles $\{\mu|_H\colon \mu \in \mathrm{H}^2(G, \mathbb{T})\}$ is "large" (uncountable), so we can take advantage of Corollary 1.3. There are difficulties in obtaining such examples. First it is difficult to calculate second cohomology groups. Secondly it is hard to control the size of this group when restricted to H. We overcome these difficulties by looking at inclusions of the form $(H \subset G) = (\mathbb{Z}^{2n} \subset \mathbb{Z}^{2n} \rtimes \Gamma)$, and the 2-cocycles on G arise as extensions to G of Γ-invariant 2-cocycles in H. A similar construction has been considered in [4].

Denote with J the matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) \in GL(2n, \mathbb{Z})$. It defines a 2-cocycle $\nu : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$ by the formula $\nu(x, y) = x^t J y$. For each $\alpha \in \mathbb{T}$ we denote with ν_α the $\mathbb{T}$-valued 2-cocycle defined by $\nu_\alpha \stackrel{\mathrm{def}}{=} \alpha^{\nu/2}$. Since ν_α is a coboundary iff $\alpha = 1$, $\alpha \mapsto \nu_\alpha$ is an embedding of $\mathbb{T}$ into $\mathrm{H}^2(\mathbb{Z}^{2n}, \mathbb{T})$.

The set of invertible matrices that leave ν (and also ν_α) invariant is the symplectic group $Sp(2n, \mathbb{Z})$. Thus, given any subgroup Γ of $Sp(2n, \mathbb{Z})$, ν_α can be extended to a 2-cocycle on $\mathbb{Z}^{2n} \rtimes \Gamma$, which we still denote ν_α, by the formula $\nu_\alpha((x_1, \gamma_1), (x_2, \gamma_2)) \stackrel{\mathrm{def}}{=} \nu_\alpha(x_1, \gamma_1 x_2)$.

2.2. Notations. For each subgroup $\Gamma \subset Sp(2n, \mathbb{Z})$ and each $\alpha \in \mathbb{T}$ let N_α and $M_\alpha(\Gamma, n)$ be the cocycle von Neumann algebras $N_\alpha \stackrel{\mathrm{def}}{=} L_{\nu_\alpha}(\mathbb{Z}^{2n}) \subset L_{\nu_\alpha}(\mathbb{Z}^{2n} \rtimes \Gamma) \stackrel{\mathrm{def}}{=} M_\alpha(\Gamma, n)$. Alternatively, $M_\alpha(\Gamma, n)$ can be regarded as the cross product von Neumann algebra $N_\alpha \rtimes_{\sigma_\alpha} \Gamma$, where the action σ_α is defined by $\sigma_\alpha(g)(\lambda_{\nu_\alpha}(x)) = \lambda_{\nu_\alpha}(gx)$ for all $x \in \mathbb{Z}^{2n}$ and all $g \in \Gamma$. Note that the isomorphism class of $M_\alpha(\Gamma, n)$ may in fact depend on the embedding $\Gamma \subset Sp(2n, \mathbb{Z})$. In other words it may depend on the way Γ acts on $\mathbb{Z}^{2n}$, a fact that is not well emphasized by the notation $M_\alpha(\Gamma, n)$. For instance, the group $\mathbb{F}_2$ can be embedded in $SL(2, \mathbb{Z})$ in many ways, giving different actions of $\mathbb{F}_2$ on $\mathbb{Z}^2$ and thus on $L_{\nu_\alpha}(\mathbb{Z}^2)$.

If α is a root of unity of order m then N_α is homogeneous of type I_{nm}, while if α is not a root of unity, then N_α is isomorphic to the hyperfinite II_1 factor R. $N_\alpha' \cap M_\alpha(\Gamma, n) = \mathcal{Z}(N_\alpha)$. Also, if either $\mathbb{Z}^{2n}$ has no Γ-invariant finite subsets other than $\{0\}$ or if α is not a root of unity then $M_\alpha(\Gamma, n)$ is a II_1 factor.

In the case when $n = 1$, $Sp(2, \mathbb{Z})$ is in fact equal to $SL(2, \mathbb{Z})$ and if we denote by u and v the canonical generators of $\mathbb{Z}^2$ we have that $\lambda_u \lambda_v = \nu_\alpha(u, v)\lambda_{uv} = \alpha^{1/2}\lambda_{uv}$ and $\lambda_v \lambda_u = \nu_\alpha(v, u)\lambda_{vu} = \alpha^{-1/2}\lambda_{uv}$ showing that $\lambda_u \lambda_v = \alpha \lambda_v \lambda_u$. So when $n = 1$ and $\alpha = \exp(2\pi \imath \theta)$ with θ irrational, $L_{\nu_\alpha}(\mathbb{Z}^2)$ is the hyperfinite II_1 factor represented as the irrational rotation algebra R_α of angle θ. Thus if Γ is an arbitrary non-amenable subgroup of $SL(2, \mathbb{Z})$, then $M_\alpha(\Gamma) \stackrel{\mathrm{def}}{=} M_\alpha(\Gamma, 1)$ is an irrational rotation HT factor, as considered in [23,25].

Recall that two finite von Neumann algebras M and N are *stably isomorphic* if M is isomorphic to an amplification N^t of N, i.e., if there exist $n \in \mathbb{N}$ and a projection $p \in M_n(N)$ such that M is isomorphic to $pM_n(N)p$ $(= N^t$, where $t = n\tau(p))$.

2.3. Corollary. *Let M be a separable II_1 factor. If Γ is a subgroup of $Sp(2n,\mathbb{Z})$ such that $\mathbb{Z}^{2n} \subset \mathbb{Z}^{2n} \rtimes \Gamma$ has the relative property* (T)*, then the set of $\alpha \in \mathbb{T}$ for which some amplification of $M_\alpha(\Gamma, n)$ can be embedded into M is at most countable. Thus, the factors $\{M_\alpha(\Gamma, n)\}_{\alpha\in\mathbb{T}}$ are non-stably isomorphic modulo countable sets.*

2.4. Corollary.

(1) *The irrational rotation HT factors $M_\alpha(\Gamma)$ cannot be all embedded into a separable II_1 factor and are non-stably isomorphic modulo countable sets.*
(2) *If Γ is a subgroup of $GL(n,\mathbb{Z})$ such that $\mathbb{Z}^n \subset \mathbb{Z}^n \rtimes \Gamma$ has the relative property* (T) *then the factors $M_\alpha(\Gamma, n) = L_{\nu_\alpha}(\mathbb{Z}^{2n} \rtimes_\theta \Gamma)$ where θ is as in* 2.1(3) *cannot be all embedded into a separable II_1 factor and are non-stably isomorphic modulo countable sets.*

3. Disintegration of rigid von Neumann algebras

We now use results from the previous section to derive some properties of the disintegration of the type II_1 von Neumann algebras coming from property (T) groups Λ with large (infinite) radical, i.e., for which $L(\Lambda)$ has diffuse center. Thus, we give an explicit description of the disintegration of the type II_1 von Neumann algebras from the property (T) groups Λ with large center constructed by Serre (see [13, p. 40]), which arise as central extensions of property (T) groups. We also use an argument from [20] to show that the factors in the disintegration of the algebra $L(\Lambda)$ of an arbitrary property (T) group Λ are mutually non-isomorphic, modulo countable sets.

We first recall some facts about the disintegration theory of a von Neumann algebra (see [8, Chapter 2] for a detailed treatment). Thus, let $(\mathcal{Z}, \mu)$ be a Borel space with a positive measure and $\xi \to \mathcal{H}_\xi$ be a measurable field of Hilbert spaces on $\mathcal{Z}$. We denote by $\mathcal{H} = \int_{\xi\in\mathcal{Z}}^{\oplus} \mathcal{H}_\xi \, d\mu$ the corresponding direct integral Hilbert space. An operator field $\xi \to T_\xi$, $T_\xi \in \mathcal{B}(\mathcal{H}_\xi)$ is called diagonalizable if it is of the form $\xi \to c(\xi) I_{\mathcal{H}_\xi}$ where $c : \mathcal{Z} \to \mathbb{C}$ is measurable. An operator T acting on $\mathcal{H}$ is called decomposable if it comes from a measurable operator field $\xi \to T_\xi$, in which case we write $T = \int_{\xi\in\mathcal{Z}}^{\oplus} T_\xi \, d\mu$. An operator T is decomposable if and only if it commutes with the set of diagonalizable operators.

Now assume that for each $\xi \in \mathcal{Z}$, $\mathcal{A}_\xi$ is a von Neumann algebra acting on $\mathcal{H}_\xi$. $\xi \to \mathcal{A}_\xi$ is a measurable field of von Neumann algebras if there exist a sequence $\{T_i\}_{i\in\mathbb{Z}}$ of measurable operator fields such that for each $\xi \in \mathcal{Z}$, $\{T_i(\xi)\}_{i\in\mathbb{Z}}$ generates $\mathcal{A}_\xi$. The set of decomposable operators $T = \int_{\xi\in\mathcal{Z}}^{\oplus} T_\xi \, d\mu$ for which $T_\xi \in \mathcal{A}_\xi$ is a von Neumann algebra and it is denoted by $\mathcal{A} = \int_{\xi\in\mathcal{Z}}^{\oplus} \mathcal{A}_\xi \, d\mu$.

3.0. Example. Let G be a discrete group with a 2-cocycle $\nu : G \times G \to A$ where A is a discrete abelian group. The central extension of G with cocycle ν is a group $\tilde{G}$ where $\tilde{G} = A \times G$ as a set and the multiplication is given by $(a_1, g_1)(a_2, g_2) = (a_1 a_2 \nu(g_1, g_2), g_1 g_2)$. Notice that $(a_1, g_1)^{-1} = (a_1^{-1} \nu(g_1, g_1^{-1})^{-1}, g_1^{-1})$. By a result of Serre, if G is a property (T) group and $\nu \neq 0$ in $\mathrm{H}^2(G, A)$ then $\tilde{G}$ also has property (T).

For each character $\alpha \in \hat{A}$ let $L_\alpha(G) = L_{\nu_\alpha}(G)$, where ν_α is the $\mathbb{T}$-valued 2-cocycle given by the formula $\nu_\alpha(g_1, g_2) = \alpha(\nu(g_1, g_2))$.

Let $B = C^*_{\mathrm{red}}(\tilde{G})$ and τ be the natural trace on B defined by $\tau(a, g) = \delta_{(a,g)}^{(e_A, e_G)}$. For each $\alpha \in \hat{A}$ let τ_α be the trace on B defined by $\tau_\alpha(a, g) = \alpha(a)\delta_g^{e_G}$. Let $(\pi, \mathcal{H}_\circ)$ and $(\pi_\alpha, \mathcal{H}_\alpha)$ be the

GNS representations of B with respect to the states τ and τ_α. Then $\mathcal{H}_\circ = \ell^2(\tilde{G})$, $\mathcal{H}_\alpha = \ell^2(G)$, $\pi(B)'' = L(\tilde{G})$ and $\pi_\alpha(B)'' \simeq L_\alpha(G)$. The last equality is easy to check since $\tau_\alpha((a-\alpha(a))(a-\alpha(a))^*) = 0$ so $\pi_\alpha(a) = \alpha(a)I$.

For each $g \in G$ define the vector field x_g to be $x_g(\alpha) = \widehat{(e_A, g)}^{\mathcal{H}_\alpha}$, where for any $b \in B$, $\hat{b}^{\mathcal{H}_\alpha}$ denotes the class of b in $\mathcal{H}_\alpha$. It is clear that for each $\alpha \in \hat{A}$ fixed, the set $\{x_g(\alpha)\}_{g\in G}$ is an orthonormal basis of $\mathcal{H}_\alpha$ and that for each $g_1, g_2 \in G$ the function $\alpha \to \langle x_{g_1}(\alpha), x_{g_2}(\alpha)\rangle_{\mathcal{H}_\alpha}$ is continuous. Then by [8, II.1.4, Proposition 4], there exists a unique structure of measurable Hilbert spaces on $\alpha \mapsto \mathcal{H}_\alpha$ that make the vector fields x_g measurable. Moreover, a vector field x is measurable if and only if $\alpha \to \langle x_g(\alpha), x(\alpha)\rangle_{\mathcal{H}_\alpha}$ is measurable for every $g \in G$.

Let $\theta : \mathcal{H}_\circ \to \int_{\alpha\in\hat{A}}^{\oplus} \mathcal{H}_\alpha \, d\alpha$ be the linear map defined by

$$\theta\widehat{(a, g)}^{\mathcal{H}_\circ} = \left(\widehat{(a, g)}^{\mathcal{H}_\alpha}\right)_{\alpha\in\hat{A}} = \big(\alpha(a)g\big)_{\alpha\in\hat{A}},$$

where the last equality is via the identification $\mathcal{H}_\alpha = l^2(G)$.

We show that θ is an isomorphism of Hilbert spaces and

$$\big(\theta\big(\pi(x)\xi\big)\big)_\alpha = \pi_\alpha(x)\theta(\xi), \quad \forall x \in B,\ \xi \in \mathcal{H}_\circ.$$

Note that

$$\begin{aligned}\left\langle \theta\widehat{(a_1, g_1)}^{\mathcal{H}_\circ}, \theta\widehat{(a_2, g_2)}^{\mathcal{H}_\circ}\right\rangle_{\int^\oplus \mathcal{H}_\alpha} &= \int\limits_{\alpha\in\hat{A}} \left\langle \widehat{(a_1, g_1)}^{\mathcal{H}_\alpha}, \widehat{(a_2, g_2)}^{\mathcal{H}_\alpha}\right\rangle_{\mathcal{H}_\alpha} d\alpha \\ &= \int\limits_{\alpha\in\hat{A}} \tau_\alpha\big(a_1a_2^{-1}\nu\big(g_1, g_2^{-1}\big)\nu\big(g_2, g_2^{-1}\big)^{-1}, g_1g_2^{-1}\big)\, d\alpha,\end{aligned}$$

with the last term being zero whenever $(a_1, g_1) \neq (a_2, g_2)$. Thus

$$\big\langle\theta\widehat{(a_1, g_1)}, \theta\widehat{(a_2, g_2)}\big\rangle_{\int^\oplus \mathcal{H}_\alpha d\alpha} = \delta_{(a_1,g_1)}^{(a_2,g_2)} = \left\langle\widehat{(a_1, g_1)}^{\mathcal{H}_\circ}, \widehat{(a_2, g_2)}^{\mathcal{H}_\circ}\right\rangle_{\mathcal{H}_\circ}$$

showing that θ is an injective morphism of Hilbert spaces.

To check surjectivity, let $\{x(\alpha)\}_\alpha \in \int^\oplus \mathcal{H}_\alpha \, d\alpha$ be a measurable vector field. Since for every fixed $g \in G$ the function $\alpha \mapsto \langle x(\alpha), x_g(\alpha)\rangle$ belongs to $L^2(\hat{A})$, there exist $(d_{a,g})_{a\in A, g\in G}$ such that

$$\sum_{a\in A} d_{a,g}\alpha(a) = \big\langle x(\alpha), x_g(\alpha)\big\rangle, \quad \forall \alpha \in \hat{A},\ g \in G.$$

Moreover, $\sum_{a,g} |d_{a,g}|^2 = \int \|x(\alpha)\|^2 \, d\alpha$ is finite. Define $v = \sum_{a,g} d_{a,g}(a, g)$. Then $v \in \mathcal{H}_\circ$ and $\theta(v) = \{x(\alpha)\}_\alpha$, which shows that θ is an isomorphism.

We now check that $(\theta(\pi(x)\xi))_\alpha = \pi_\alpha(x)\theta(\xi)_\alpha$, $\forall x \in B$, $\xi \in \mathcal{H}_\circ$. This is clear since

$$\big(\theta\big(\pi(x)\xi\big)\big)_\alpha = \widehat{\big(\pi(x)\xi\big)}^{\mathcal{H}_\alpha} = \widehat{(x\xi)}^{\mathcal{H}_\alpha} = \pi_\alpha(x)\widehat{(\xi)}^{\mathcal{H}_\alpha} = \pi_\alpha(x)\theta(\xi)_\alpha.$$

The diagonalizable operator fields on $\int_{\alpha\in\hat{A}}^{\oplus} \mathcal{H}_\alpha\, d\alpha$ correspond, via θ^{-1}, to the elements of the von Neumann algebra $\pi(A)'' \subset B(\mathcal{H}_0)$. Altogether, using [9, 8.4.1] we have thus obtained:

3.1. Proposition. *Let G be a discrete group with a 2-cocycle $\nu : G \times G \to A$, where A is a discrete abelian group. Let $\tilde{G}$ be the central extension of G defined by the cocycle ν. For each $\alpha \in \hat{A}$ let $L_\alpha(G) = L_{\nu_\alpha}(G)$, where ν_α is the $\mathbb{T}$-valued 2-cocycle on G defined as $\nu_\alpha(g_1, g_2) = \alpha(\nu(g_1, g_2))$. Then the von Neumann algebra $L(\tilde{G})$ has the following direct integral decomposition*:

$$L(\tilde{G}) = \int_{\alpha\in\hat{A}}^{\oplus} L_\alpha(G)\, d\alpha.$$

Note that if we let $G = \mathbb{Z}^{2n} \rtimes \Gamma$, $n > 2$, where Γ is a non-amenable subgroup of $Sp(2n, \mathbb{Z})$ and ν the $\mathbb{Z}$-valued 2-cocycle on G defined in Section 2, then from Corollary 2.2 and Proposition 3.1 it follows that the factors in the above direct integral decomposition of $L(\tilde{G})$ are property (T) and they are non-isomorphic modulo countable sets.

But, in fact, one can obtain a general result along these lines, by using an argument similar to Ozawa's proof that there are no "universal" separable II_1 factors [20]. We include the details of the argument, for completeness.

3.2. Theorem. *Let Λ be a discrete property* (T) *group such that the von Neumann algebra $L(\Lambda)$ has diffuse center, and let $L(\Lambda) = \int_{t\in\mathcal{Z}}^{\oplus} M_t\, d\mu$ be its direct integral decomposition. Then there exists a set $\mathcal{Z}_0 \subset \mathcal{Z}$, $\mu(\mathcal{Z}_0) = 0$, such that the factors M_t, $t \in \mathcal{Z} \setminus \mathcal{Z}_0$, are mutually non-stably isomorphic modulo countable sets.*

Proof. Let $B = C^*_{\mathrm{red}}(\Lambda)$, let τ be the canonical trace of B and let $\mathcal{Z} = \widehat{Z(B)}$. The direct integral decomposition of the GNS representation of (B, τ) induces factorial representations $\pi_t : \Lambda \to \mathcal{B}(\mathcal{H}_t)$, $t \in \mathcal{Z}$. The factors in the direct integral decomposition of $L(\Lambda)$ are $M_t = \pi_t(\Lambda)'' \subset \mathcal{B}(\mathcal{H}_t)$, and we may assume $\mathcal{H}_t = L^2(M_t)$. By [9, 8.4.1 and 8.4.2] there exists a measure zero set $\mathcal{Z}_0 \subset \mathcal{Z}$ such that the representations π_t, $t \in \mathcal{Z} \setminus \mathcal{Z}_0$, are mutually non-conjugate.

Assume, by contradiction, that M_t is isomorphic to an amplification $M^{s(t)}$ of the same factor M, for all $t \in S$, where $S \subset \mathcal{Z} \setminus \mathcal{Z}_0$ is uncountable. We may clearly assume $c \leqslant s(t) \leqslant 1$, $\forall t$, for some $c > 0$. To simplify notations, we still denote by π_t the representations of Λ into the unitary group of $p_t M p_t$, induced by the isomorphisms $M_t \simeq p_t M p_t$, where $p_t \in \mathcal{P}(M)$, $\tau(p_t) = s(t)$, $t \in S$.

Let (F_0, δ_0) be property (T) constants for Λ as defined in Section 1. By using a separability argument as in Theorem 1.2 and [11], it follows that there exist $t_1 \neq t_2 \in S$ such that p_{t_1} is close to p_{t_2} and such that if $\pi : \Lambda \to \mathcal{B}(p_{t_1} L^2(M) p_{t_2})$ denotes the representation of Λ given by the formula $\pi(g)\eta = L(\pi_{t_1}(g)) R(\pi_{t_2}(g)^*)\eta$ and ξ is the vector $\xi = \|p_{t_1} p_{t_2}\|^{-1} (p_{t_1} p_{t_2})\hat{}$ then $\|\pi(g)\xi - \xi\|_2 < \delta_0$ for all $g \in F_0$. Since Λ has property (T), there exists a non-zero vector $\eta \in p_{t_1} L^2(M) p_{t_2}$ such that $\pi(g)\eta = \eta$, for all $g \in \Lambda$. Equivalently, if we regard η as a square integrable operator, we have $\pi_{t_1}(g)\eta = \eta\pi_{t_2}(g)$, for all $g \in \Lambda$. By the standard trick, if $v \in M$ is the partial isometry in the polar decomposition of η with the property that the right supports of η and v coincide, then $vv^* \in \pi_{t_1}(\Lambda)' \cap p_{t_1} M p_{t_1} = \mathbb{C} p_{t_1}$, $v^* v \in \pi_{t_2}(\Lambda)' \cap p_{t_2} M p_{t_2} = \mathbb{C} p_{t_2}$ and

$\pi_{t_1}(g)v = v\pi_{t_2}(g)$, for all $g \in \Lambda$. This implies that π_{t_1}, π_{t_2} are conjugate representations of Λ, which contradicts $t_1 \neq t_2$. $\square$

4. Conjugacy and isomorphism problems for $M_\alpha(\Gamma)$

We have seen that the cocycle von Neumann algebras $M_\alpha(\Gamma)$ constructed in Section 2 can be regarded as the crossed product von Neumann algebras $R_\alpha \rtimes_{\sigma_\alpha} \Gamma$. Moreover, by [25], when $\alpha \in \mathbb{T}$ is irrational the isomorphism class of the algebras $M_\alpha(\Gamma)$ is completely determined by the cocycle conjugacy class of the actions σ_α of Γ on the hyperfinite II_1 factor $R \simeq R_\alpha$. Thus, the classification of the factors $M_\alpha(\Gamma)$ amounts to the classification up to cocycle conjugacy of the actions (σ_α, Γ). In particular, for a fixed $\Gamma \subset SL(2,\mathbb{Z})$, showing that the factors $M_\alpha(\Gamma)$ are non-isomorphic for different irrational numbers α amounts to showing that the corresponding actions σ_α are non-cocycle conjugate. While we cannot solve this latter problem, we show here that for a large class of subgroups $\Gamma \subset SL(2,\mathbb{Z})$ the conjugacy class of the action σ_α determines the irrational number α.

4.1. Theorem. *Let $\Gamma \subset SL(2,\mathbb{Z})$ be a subgroup of $SL(2,\mathbb{Z})$ containing a parabolic element a and an element b that does not commute with a. If α_1 and α_2 are irrationals in the upper-half torus such that the actions σ_{α_1} and σ_{α_2} of Γ on the hyperfinite II_1 factors $R_{\alpha_j} = L_{\mu_{\alpha_j}}(\mathbb{Z}^2)$ $(j = 1,2)$ are conjugate then $\alpha_1 = \alpha_2$.*

Proof. By replacing Γ with $\gamma\Gamma\gamma^{-1}$ for a certain $\gamma \in SL(2,\mathbb{Z})$, we may assume that a has $(1,0)$ as eigenvector. We may also assume that the corresponding eigenvalue is 1, by substituting a with a^2 if necessary. Thus $a = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, for some $n \in \mathbb{Z}$ non-zero.

For $j = 1,2$ let $\alpha_j = e^{2\pi i t_j}$, $\alpha_j^{1/2} = e^{\pi i t_j}$ with $t_j \in [0,1/2) \setminus \mathbb{Q}$, and let $u_j = \lambda_{\mu_{\alpha_j}}(1,0)$, and $v_j = \lambda_{\mu_{\alpha_j}}(0,1)$ be the unitaries generating $L_{\mu_{\alpha_j}}(\mathbb{Z}^2) = R_{\alpha_j}$. The cocycle relation $u_j v_j = \alpha_j v_j u_j$ implies $u_j^k v_j^l = \alpha_j^{kl} v_j^l u_j^k$ for all $k, l \in \mathbb{Z}$. For $g = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \Gamma$ we have:

$$\sigma_{\alpha_j}(g)\big(u^k v^l\big) = \alpha_j^{\frac{1}{2}(kl-(pk+ql)(rk+sl))} u_j^{pk+kl} v_j^{rk+sl}.$$

Assume σ_{α_1} and σ_{α_2} are conjugate, i.e., there exists an isomorphism $\theta : R_{\alpha_2} \to R_{\alpha_1}$ such that $\theta(\sigma_{\alpha_2}(g)(x)) = \sigma_{\alpha_1}(g)(\theta(x))$ for all $g \in \Gamma$, $x \in R_{\alpha_2}$. We prove $\alpha_1 = \alpha_2$.

Denote $u = u_1$, $v = v_1$, $u' = \theta(u_2)$, $v' = \theta(v_2)$. To simplify notations, we identify $x \in R_{\alpha_1}$ with its image $\hat{x}$ in $L^2(R_{\alpha_1})$. Thus $(u^k v^l)_{(k,l)\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2(R_{\alpha_1}, \tau)$ and R_{α_1} is identified with the set of "Fourier expansions" $\sum_{(k,l)\in\mathbb{Z}^2} \lambda_{k,l} u^k v^l$ in $L^2(R_{\alpha_1}, \tau)$, that are (twisted) left convolvers on $L^2(R_{\alpha_1}, \tau)$. Let

$$u' = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l} u^k v^l, \qquad v' = \sum_{(k,l)\in\mathbb{Z}^2} d_{k,l} u^k v^l$$

for some $c_{k,l}, d_{k,l} \in \mathbb{C}$ such that

$$\sum_{(k,l)\in\mathbb{Z}^2} |c_{k,l}|^2 < \infty, \qquad \sum_{(k,l)\in\mathbb{Z}^2} |d_{k,l}|^2 < \infty.$$

Since the actions α_1, α_2 are conjugate via θ, we have

$$\sigma_{\alpha_1}(g)\big((u')^k(v')^l\big) = \alpha_2^{\frac{1}{2}(kl-(pk+ql)(rk+sl))}(u')^{pk+ql}(v')^{rk+sl}.$$

Choosing $g = a$, $k = 1$, $l = 0$, we obtain $\sigma_{\alpha_1}(a)(u') = u'$. Thus

$$\sigma_{\alpha_1}(a)\Bigg(\sum_{(k,l)\in\mathbb{Z}^2} c_{k,l}u^k v^l\Bigg) = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l}\alpha_1^{-\frac{1}{2}nl^2}u^{k+nl}v^l = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l}u^k v^l$$

which implies $\alpha_1^{-\frac{1}{2}nl^2}c_{k-nl,l} = c_{k,l}$, $\forall k, l \in \mathbb{Z}$. Thus, for l non-zero $|c_{k,l}| = |c_{k-nl,l}| = |c_{k-2nl,l}| = \cdots$ have to be all zero since $\sum_{(k,l)\in\mathbb{Z}^2}|c_{k,l}|^2 < \infty$. Denote $c_k = c_{k,0}$. Then $u' = \sum_{k\in\mathbb{Z}} c_k u^k$.

Let $b = \big(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\big)$, $m_1m_4 - m_2m_3 = 1$. Then $ab \neq ba$ is equivalent to $m_3 \neq 0$. Using the formula for σ_{α_1} for $g = b$, $k = 1$, $l = 0$, we obtain $\sigma_{\alpha_1}(b)(u') = \alpha_2^{-\frac{1}{2}m_1m_3}(u')^{m_1}(v')^{m_3}$.

This implies

$$u'\sigma_{\alpha_1}(b)(u') = \alpha_2^{-\frac{1}{2}m_1m_3}(u')^{m_1}u'(v')^{m_3} = \alpha_2^{m_3}\sigma_{\alpha_1}(b)(u')u',$$

and thus

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_k u^k &\Bigg(\sum_{j\in\mathbb{Z}} c_j\alpha_1^{-\frac{1}{2}m_1m_3j^2}u^{m_1j}v^{m_3j}\Bigg)\\ &= \alpha_2^{m_3}\Bigg(\sum_{j\in\mathbb{Z}} c_j\alpha_1^{-\frac{1}{2}m_1m_3j^2}u^{m_1j}v^{m_3j}\Bigg)\sum_{k\in\mathbb{Z}} c_k u^k.\end{aligned}$$

Hence we obtain:

$$\sum_{k,j\in\mathbb{Z}} c_kc_j\alpha_1^{-\frac{1}{2}m_1m_3j^2}\big(1 - \alpha_2^{m_3}\alpha_1^{-m_3kj}\big)u^{k+m_1j}v^{m_3j} = 0.$$

Since the function $(k, j) \to (k + m_1 j, m_3 j)$ is injective for $m_3 \neq 0$, it follows:

$$c_kc_j\big(\alpha_1^{m_3kj} - \alpha_2^{m_3}\big) = 0, \quad \forall k, j \in \mathbb{Z}.$$

Letting $k = j$ we obtain $c_k = 0$, for all k except possibly two values $k_0, -k_0$. Indeed, since α_1 is not a root of unity there exists at most one $N = m_3k^2$ such that $\alpha_1^N = \alpha_2^{m_3}$.

Since u' is not a scalar, we know $k_0 \neq 0$. Taking $j = -k_0$ and using $\alpha_1^{-m_3k_0^2} \neq \alpha_1^{m_3k_0^2} = \alpha_2^{m_3}$ we obtain $c_{k_0}c_{-k_0} = 0$. Thus only one coefficient of the Fourier expansion of u' is non-zero. So far we have showed then that

$$u' = cu^{k_0} \quad \text{and} \quad \alpha_1^{k_0^2} = \alpha_2.$$

Now substituting u' in the relation $u'v' = \alpha_2 v'u'$ we obtain

$$cu^{k_0}\biggl(\sum_{(k,l)\in\mathbb{Z}^2} d_{k,l}u^k v^l\biggr) = \alpha_2\biggl(\sum_{(k,l)\in\mathbb{Z}^2} d_{k,l}u^k v^l\biggr)cu^{k_0}.$$

Thus

$$\sum_{(k,l)\in\mathbb{Z}^2} d_{k,l}u^{k_0+k}v^l = \sum_{(k,l)\in\mathbb{Z}^2} d_{k,l}\alpha_2\alpha_1^{-k_0 l}u^{k+k_0}v^l$$

which yields

$$d_{k,l}\bigl(1-\alpha_2\alpha_1^{-k_0 l}\bigr) = 0, \quad \forall k, l \in \mathbb{Z}.$$

Since $\alpha_1^{k_0 l} \neq \alpha_2$ unless $l = k_0$ we obtain that $d_{k,l} = 0$, for all $k \in \mathbb{Z}$ and $l \neq k_0$. Denote $d_k = d_{k,k_0}$. Thus we have $u' = cu^{k_0}$ and $v' = (\sum_k d_k u^k)v^{k_0}$. This implies that for every $j \geqslant 1$ there exists $w_j \in W^*(1,u)$ such that $(v')^j = w_j v^{jk_0}$. Using the formula for σ_{α_1} one more time for $g = b$, $k \neq 0$ arbitrary and $l = 1$, we have

$$\begin{aligned}\sigma_{\alpha_1}(b)\bigl((u')^k v'\bigr) &= \alpha_2^{\frac{1}{2}(k-(m_1k+m_2)(m_3k+m_4))}(u')^{m_1k+m_2}(v')^{m_3k+m_4}\\ &= \alpha_2^{\frac{1}{2}(k-(m_1k+m_2)(m_3k+m_4))}c^{m_1k+m_2}u^{k_0(m_1k+m_2)}w_{m_3k+m_4}v^{k_0(m_3k+m_4)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_{\alpha_1}(b)\bigl((u')^k v'\bigr) &= \sigma_{\alpha_1}(b)\biggl(\sum_l c^k d_l u^{kk_0+l}v^{k_0}\biggr)\\ &= \sum_l c^k d_l \alpha_1^{\frac{1}{2}[(kk_0+l)k_0-(m_1(kk_0+l)+m_2k_0)(m_3(kk_0+l)+m_4k_0)]}\\ &\quad\times u^{m_1(kk_0+l)+m_2k_0}v^{m_3(kk_0+l)+m_4k_0}.\end{aligned}$$

Identifying the corresponding coefficients, for every l we must have either $d_l = 0$ or $m_3(kk_0 + l) + m_4k_0 = k_0(m_3k + m_4)$, which implies $l = 0$. Thus $d_l = 0$, $\forall l \neq 0$ and $v' = dv^{k_0}$ for some $d \in \mathbb{C}$. Altogether, $u' = cu^{k_0}$, $v' = dv^{k_0}$ for some $c, d \in \mathbb{C}$. Since u', v' generate R_{α_1}, this implies $k_0 = 1$ or $k_0 = -1$. But $\alpha_1^{-1} \neq \alpha_2$ because α_1, α_2 belong to the upper half torus. Thus $k_0 = 1$ and $\alpha_1 = \alpha_2$. □

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Appendix A. A general result on fundamental groups

We give here a short proof of a result in [23], showing that the HT factors $M_\alpha(\Gamma)$, $\alpha \in \mathbb{T}$, $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable, have at most countable fundamental group. The result we prove is in fact much more general, covering all results of this type in [22,23], as particular cases:

A.1. Theorem. *Let M be a separable II_1 factor. Assume there exists a non-zero projection $p \in M$ such that pMp contains a von Neumann subalgebra B such that $B \subset pMp$ is a rigid inclusion and $B' \cap pMp \subset B$. Then $\mathcal{F}(M)$ is countable.*

Proof. Recall from [23, 4.2] that $B \subset M$ rigid implies there exist $F \subset M$ finite and $\delta > 0$ such that if $\phi : M \to M$ is a subunital, subtracial completely positive map which satisfies $\|\phi(x) - x\|_2 \leqslant \delta$, $\forall x \in F$, then $\|\phi(u) - u\|_2 \leqslant 1/2$, $\forall u \in \mathcal{U}(B)$.

Since the fundamental groups of M and pMp coincide, it is clearly sufficient to prove the statement in the case $p = 1$. For each $t \in (0,1) \cap \mathcal{F}(M)$ choose a projection $p_t \in \mathcal{P}(B)$ and an isomorphism $\theta_t : M \simeq p_t M p_t$. Since B is diffuse we can make the choice so that, in addition, we have $p_t \leqslant p_{t'}$ whenever $t \leqslant t'$.

Assume $\mathcal{F}(M)$ is uncountable. Thus, $[c,1) \cap \mathcal{F}(M)$ is uncountable for some $0 < c < 1$. By the separability of M, this implies there exist $t, s \in \mathcal{F}(M) \cap [c,1)$, $t < s$, such that $\|\theta_s(x) - \theta_t(x)\|_2 \leqslant \delta c$, $\forall x \in F$.

Thus, if we denote $\theta = \theta_s^{-1} \circ \theta_t$ then θ is an isomorphism of M onto qMq, where $q = \theta_s^{-1}(\theta_t(1)) = \theta_s^{-1}(p_t)$, and we have $\theta(1) \leqslant 1$, $\tau(q) \geqslant c$, $\tau \circ \theta \leqslant \tau$, $\|\theta(x) - x\|_2 \leqslant \delta$, $\forall x \in F$. Consequently, we have $\|\theta(u) - u\|_2 \leqslant 1/2$, $\forall u \in \mathcal{U}(B)$.

Let k denote the unique element of the minimal norm $\| \ \|_2$ in $K = \overline{\mathrm{co}}^w\{\theta(u)u^* \mid u \in \mathcal{U}(B)\}$. Then $\|k - 1\|_2 \leqslant 1/2$ and thus $k \neq 0$. Also, since $\theta(u)Ku^* \subset K$ and $\|\theta(u)ku^*\|_2 = \|k\|_2$, $\forall u \in \mathcal{U}(B)$, by the uniqueness of k it follows that $\theta(u)ku^* = k$, or equivalently $\theta(u)k = ku$, for all $u \in \mathcal{U}(B)$. By a standard trick, if $v \in M$ is the (non-zero) partial isometry in the polar decomposition of k and if we express any element in B as linear combination of unitaries, then we get $\theta(b)v = vb$, $\forall b \in B$, $v^*v \in B' \cap M = \mathcal{Z}(B)$, $vv^* \in \theta(B)' \cap qMq = \mathcal{Z}(\theta(B)q)$.

Since, in particular, $v^*v \in B$, we can apply the above to $b = v^*v$ to get $\theta(v^*v)v = vv^*v$. But this implies $\theta(v^*v)vv^* = vv^*$, so that $\theta(v^*v) \geqslant vv^*$. This is a contradiction, since θ shrinks the trace of any elements by $\tau(q) < 1$, while $\tau(vv^*) = \tau(v^*v)$. $\square$

A.2. Corollary. *For each $\Gamma \subset SL(2,\mathbb{Z})$ non-amenable and $\alpha \in \mathbb{T}$, the factor $M_\alpha(\Gamma)$, as defined in Sections 0 and 2, has countable fundamental group.*

Proof. Since $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$ has the relative property (T) (cf. [2]), the inclusion of von Neumann algebras $R_\alpha = L_\alpha(\mathbb{Z}^2) \subset L_\alpha(\mathbb{Z}^2 \rtimes \Gamma) = M_\alpha(\Gamma)$ has the relative property (T) and $R'_\alpha \cap M_\alpha(\Gamma) \subset R_\alpha$. Thus A.1 applies. $\square$

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