Math 371 — HW #3
November 10, 2018

Exercises on Interpolation and Least-Squares

1. Given the nodes $x_0 < x_1 < \cdots < x_n$ and the associated basis functions $L_{n,k}(x)$, prove the relations
$$\sum_{k=0}^{n} x_k^m L_{n,k}(x) = x^m, \quad , m = 0, \ldots, n.$$
Hint: What is the Lagrange interpolant of x^m, $m = 0, \ldots, n$?

2. Construct the quadratic Lagrange interpolant $p_2(x)$ of $\cos x$ using the nodes $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi$. Observe that the interpolant reduces to an affine function in this case. Use p_2 to approximate $\cos(\frac{\pi}{4})$ and compute the error. Also, using the appropriate remainder term, obtain an estimate (upper bound) for the error. Also, obtain an upper bound for the maximum error $\max_{0\leq x\leq\pi} |\cos x - p_2(x)|$.

3. Construct the cubic Hermite interpolant $H_3(x)$ of $\sin x$ using the nodes $x_0 = 0$, $x_1 = \pi$.Use $H_3(x)$ to approximate $\sin(\frac{\pi}{4})$ and compute the error. Also, using the appropriate remainder term, obtain an estimate (upper bound) for the error. Also, obtain an upper bound for the maximum error $\max_{0\leq x\leq\pi} |\sin x - H_3(x)|$.

4. Construct the cubic Spline interpolant with natural boundary conditions of $\sin x$ using the nodes $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$. Set up the system of linear equations and solve it to find c_0, c_1, c_2, c_3. You may use Matlab or other software to do so. Also compute $\{a_j, b_j, d_j\}_{j=0}^3$. Use the spline to approximate $\sin(\frac{\pi}{3})$. Compute the error.

5. Same problem as above except with Clamped boundary conditions.

6. Consider the data

x	0.2	0.3	0.6	0.9	1.1	1.3	1.4	1.6
y	0.050446	0.098426	0.33277	0.7266	1.0972	1.5697	1.8487	2.5015

   (a) Construct the Least-Squares polynomial approximation $y(x)$ of degree 1.
   (b) Construct the Least-Squares polynomial approximation $y(x)$ of degree 2.
   (c) Construct the Least-Squares approximation of the form $y(x) = be^{ax}$.
   (d) Construct the Least-Squares approximation of the form $y(x) = bx^a$.

   In each case compute the error $E = \left\{\sum_{i=0}^{7}(y_i - y(x_i))^2\right\}^{1/2}$.
   Also compute the maximum error $E = \max_{0\ lei\leq 7} |y_i - y(x_i)|$.

7. Given the data

x	-2	-1	0	1	2
y	-1	1	5	3	0

use the principle/idea of Least-Squares approximation to find the function of the form $y(x) = a + bx^2$ that best fits the data. Note that y is not a complete quadratic, so you cannot treat this as a quadratic case, which has 3 unknown coefficients, but have to develop the system of equations to find a and b.

Exercises on Numerical Differentiation and Integration

1. Use the formula
$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} + O(h^2)$$
to approximate $f'(1.2)$ with $f(x) = \ln x$ using $h = .01, .005, .0025$. Compute the errors in each case. Does the rate of decrease in the errors conform with the $O(h^2)$ estimate? Will the errors decrease at that rate as $h \to 0$?

2. Suppose the error in approximating $f'(a)$ by a certain finite-difference formula is given by
$$E(h) = \frac{u}{h} + \frac{h^2}{6}.$$
What do each of the two terms represent? What is the highest accuracy that can be expected from this formula? Your answer should be in terms of u.

3. Consider the numerical differentiation formula
$$f'(x) = \frac{-f(x+2h) + 8f(a+h) - 8f(a-h) + f(a-2h)}{12h} + E(h).$$
   (a) Use it to approximate $f'(1.2)$ where $f(x) = \ln x$ with $h = .1$.
   (b) Use Taylor's Theorem to express $E(h) = O(h^m)$ for some m. In other words, you need to find the leading term of the error.
   (c) Can you devise an "experimental", and clearly less labor intensive, way to find out what the value of m is?

4. A numerical differentiation formula for approximating the derivative $f'(a)$ is typically of the form
$$f'(a) = \frac{\sum_{k=0}^{n} c_k f(x_k)}{h} + E(h^m),$$
for some coefficients $\{c_k\}_{k=0}^{n}$ and nodes $\{n_k\}_{k=0}^{n}$. Show that if $m \geq 1$, then necessarily $\sum_{k=0}^{n} c_k = 0$.

5. Approximate the integral $\int_0^1 e^{-x^2}\, dx$ using
   (a) The Midpoint rule.
   (b) The Trapezoidal rule.
   (c) Simpson's rule.

   In each case use the remainder term as given in the notes to estimate the error.

6. Repeat the previous problem the composite version of each rule on 4 subintervals.

7. Use Gauss-Legendre quadrature with 3 nodes to approximate $\int_0^2 \frac{dx}{x^2+4} = \frac{\pi}{8}$. Compute the error. Do not forget to map the nodes and weights to the interval $[0, 2]$.

8. Consider the numerical quadrature formula $\sum_{k=0}^{n} w_k f(x_k) \approx \int_a^b f(x)\, dx$ that is based on Lagrange interpolation, i.e. $w_k = \int_a^b L_{n,k}(x)\, dx$. Show that $\sum_{k=0}^{n} w_k = b - a$.

9. Consider a formula as in the preceding problem and assume that its order of accuracy is at least $2n$. Show that the weights w_k must be positive.

10. Consider the quadrature formula

$$\int_a^b f(x)\,dx \approx \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2), \quad \text{where} \quad x_0 = a,\ x_1 = a+h,\ x_2 = b,\ h = \frac{b-a}{3}.$$

    Find the degree of accuracy of the method.

11. Consider the 3-node quadrature formula $w_0 f(a) + w_1 f(x_1) + w_2 f(b) \approx \int_a^b f(x)\,dx$. Such rules are called Gauss-Lobatto rules. Show that the choice of x_1 that gives the highest order of accuracy is $x_1 = \frac{a+b}{2}$. When you calculate x_1 and all 3 weights, you will see that this is indeed Simpson's rule. In other words, if we insist on having two of the nodes of a 3-node rule be the endpoints, then the best choice for the third node is the midpoint of the interval.

12. Similarly, show that if we insist on having one of the nodes of a two-node rule be b, then the best choice of the second node is $\frac{2}{3}a + \frac{1}{3}b$. Compute the weights. This is known as the 2-node Gauss-Radau IIA rule.

13. In the same spirit of the preceding two problems, determine the highest accuracy two-node method with $x_0 = a$.