Math 371 Fall 2018. Midterm exam

Show all calculations to receive full credit

1. (20 pts.) Let $x = 3/7$.
   (a) (10 pts.) Find the (infinite) binary representation of x
   (b) (10 pts.) Is x representable on the Marc-32? If not, what is $fl(x)$?

2. (15 pts.) How many terms of the power series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \ldots$ should one take in order to approximate $\ln(1.1)$ with an error guaranteed to be less than 10^{-5}? Hint: Use a result about alternating power series.

3. (20 pts.) The function $f(x) = \cos x - 1$ has a root $r = 0$.
   (a) (10 pts.) What is the multiplicity of this root? Will Newton's method converge quadratically?
   (b) (10 pts.) With $x_0 = .1$, apply two steps of the modified Newton's method designed to recover the quadratic convergence rate.

4. (30 pts.) Consider the matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

   (a) (10 pts.) Perform the Choleski Factorization $A = LL^T$ using direct factorization.
   (b) (10 pts.) Perform the $A = LDL^T$ factorization using the symmetric version of naive Gaussian Elimination. You should use this to check the result of part (a).
   (c) (10 pts.) Using (a) or (b) and the definition show that A is symmetric positive definite. This means you need to show that $\mathbf{x}^T A \mathbf{x} > 0, \ \forall \mathbf{x} \neq \mathbf{0}$.

5. (15 pts.) Suppose

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

   Compute the condition number of A in the $\|\cdot\|_\infty$ norm.

Math 371 Midterm Exam Fall 2018

Solutions

#1 (a)

$$\frac{3}{7} \times 2: \quad 1 > \frac{6}{7} \Rightarrow .0$$

$$\frac{6}{7} \times 2: \quad 1 < \frac{12}{7} \Rightarrow .01, \quad \frac{12}{7} - 1 = \frac{5}{7}$$

$$\frac{5}{7} \times 2 = \frac{10}{7} \Rightarrow .011, \quad \frac{10}{7} - 1 = \frac{3}{7}$$

Repeats

$\frac{3}{7}$ is not representable on any machine since it has an infinite number of nonzero bits

$$\Rightarrow \left(\frac{3}{7}\right)_{10} = .011011011\ldots$$

$$= 1.101101\ldots \times 2^{-2}$$

in normalized form.

b)

23rd bit after .

$$x = 1.101101101101101101101101101\ldots \times 2^{-2}$$

$\Rightarrow$

$$x_- = 1.101\,101\,101\,101\,101\,101\,101\,10110 \times 2^{-2}$$

$$x_+ = 1.101\,101\,101\,101\,101\,101\,101\,10111 \times 2^{-2}$$

Since the first bit discarded $= 1$, $fl(x) = x_+$.

#2

If a series is such that (i) the terms are decreasing and (ii) the signs are alternating, then the error committed by keeping the first n terms is bounded by the absolute value of the first term discarded.

So

Approximation	value of Approx. at $x=.1$	Error = first term discarded
x	$.1$	$(.1)^2/2 = 5 \times 10^{-3}$
$x - \frac{x^2}{2}$	$.1 - (.1)^2/2 = .095$	$(.1)^3/3 = 3.3 \times 10^{-4}$
$x - \frac{x^2}{2} + \frac{x^3}{3}$	$.1 - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} = .095333\bar{3}$	$(.1)^4/4 = 2.5 \times 10^{-5}$
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	$.1 - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} - \frac{(.1)^4}{4} = .0953083\bar{3}$	$(.1)^5/5 = 2 \times 10^{-6}$

"Exact value" $= .09531018$, Error $= 1.846 \times 10^{-6} \le 2 \times 10^{-6}$

#3 (a) $f(x) = \cos x - 1$. Can determine multiplicity of root $= 0$ by looking at derivatives

$f(0) = \cos 0 - 1 = 0$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

Another way is to use Taylor expansion

$\cos x - 1 = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots - 1$

$= x^2\left(-\frac{1}{2} + \cdots\right) \Rightarrow m = 2$

$f''(x) = -\cos x \Rightarrow f''(0) = -1 \neq 0$. $\boxed{\text{Hence multiplicity} = 2}$

Newton's method will not converge quadratically.

(b) $x_{n+1} = x_n - m\dfrac{f(x_n)}{f'(x_n)}$ is appropriate modification

$x_{n+1} = x_n - 2\dfrac{\cos x_n - 1}{-\sin x_n}$

n	x_n
0	.1
1	-8.341×10^{-5}
2	6.79×10^{-11}

#4 (a)

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & \frac{1}{3} & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

row1 · col1 $\Rightarrow l_{11}^2 = 1 \Rightarrow \boxed{l_{11} = 1}$

row2 · col1 $\Rightarrow l_{21} l_{11} = \frac{1}{2} \Rightarrow \boxed{l_{21} = 1/2}$

row3 · col1 $\Rightarrow l_{31} l_{11} = \frac{1}{3} \Rightarrow \boxed{l_{31} = 1/3}$

row2 · col2 $\Rightarrow l_{21}^2 + l_{22}^2 = \frac{1}{3} \Rightarrow l_{22} = \sqrt{\frac{1}{3} - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{12}} = \boxed{\frac{1}{2\sqrt{3}}} = l_{22}$

row3 · col2 $\Rightarrow l_{31} \cdot l_{21} + l_{32} l_{22} = \frac{1}{4} \Rightarrow l_{32} = \left(\frac{1}{4} - \frac{1}{3} \cdot \frac{1}{2}\right) \Big/ \frac{1}{2\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}} = l_{32}}$

row3 · col3 $\Rightarrow l_{31}^2 + l_{32}^2 + l_{33}^2 = \frac{1}{5} \Rightarrow$

$l_{33} = \sqrt{\frac{1}{5} - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{2\sqrt{3}}\right)^2} = \frac{1}{6\sqrt{5}}$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ .5 & .28868 & 0 \\ .\bar{3} & .28868 & .07454 \end{bmatrix}$$

set to zero

(b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \xrightarrow[m_{31}=\frac{1}{3}]{m_{21}=\frac{1}{2}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \boxed{*} \\ 0 & \frac{1}{12} & \frac{4}{45} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{4}{45} \end{bmatrix}$

$*$ do not compute since we know it will be $\frac{1}{12}$ by symmetry

$\xrightarrow{m_{32}=1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{bmatrix} \xrightarrow{\text{set to zero}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{180} \end{bmatrix} \equiv D$

$\frac{4}{45} - \frac{1}{12} = \frac{16-15}{180}$.

$$\begin{array}{ccccccc} & A & = & L & D & L^T & \\ \Rightarrow & \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{180} \end{bmatrix} & \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} & . \end{array}$$

Now

$$L D^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{6\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix}$$

which is exactly the L found by Choleski

(c) For any $x \in \mathbb{R}^3$, $x^T A x = x^T L L^T x$ using Choleski
$= (L^T x)^T (L^T x)$

$= \|L^T x\|_2^2 \geq 0, \quad \forall x.$

If $x^T A x = 0$, Then $\|L^T x\|_2 = 0 \Rightarrow L^T x = 0$ since $\|\cdot\|_2$ is a norm

$\Rightarrow x = 0$ since L is invertible.

Thus, we have shown that for any $x \neq 0$ $x^T A x > 0 \Rightarrow A$ is s.p.d.

#5

$$\|A\|_\infty = \max\left\{\overbrace{1+\frac{1}{2}+\frac{1}{3}}^{\text{This is clearly the largest term}}, \frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right\}$$

$$= 1+\frac{1}{2}+\frac{1}{3} = \frac{11}{6}$$

$$\|A^{-1}\|_\infty = \max\left\{9+36+30, 36+192+180, 30+180+180\right\}$$

$$= \max\left\{75, 408, 390\right\} = 408$$

$\Rightarrow$

$$\mathrm{Cond}_{\|\cdot\|_\infty}(A) = \|A\|_\infty \, \|A^{-1}\|_\infty = \frac{11}{6} \times 408 = 748$$