Sample Final

1. Suppose you are given the following data points $(x, f(x))$ for some function $f(x)$:

$$(.750492, .932515), (.785398, 1), (.820305, 1.07237).$$

   Calculate approximations to f' and f'' at $x = .785398$ using appropriate centered formulas.

2. If you enter a number into a hand-held calculator and repeatedly press the cosine button, what number will eventually appear? Provide a proof.

3. Devise a Newton algorithm for computing the fifth root of any positive real number. Discuss choice of the starting value and convergence.

4. Consider the function $f(x) = \cos \pi x$ on the interval [0,1]. With the nodes $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, construct the following

   (i) The Lagrange polynomial interpolant,

   (ii) The Hermite polynomial interpolant,

   (iii The piecewise cubic Hermite interpolant,

   (iv) The cubic spline interpolant with clamped B.C.

   In each case, give an appropriate bound for the error.

5. Calculate the cubic Hermite interpolant of $f(x) = \tan x$ on $[0, \pi/4]$.

6. Set up the system of equations for the natural cubic spline interpolant with 3 subintervals of $f(x) = \tan x$ on $[0, \pi/4]$.

7. Let f be a cubic polynomial on some interval $[a, b]$. Show that the clamped cubic spline interpolant of f on any partition of $[a, b]$ must coincide with f. Show, by a counterexample, that this need not be true for the natural cubic spline interpolant.

8. Use the composite Simpson's rule with $n = 4$ to approximate the integral $\int_0^{48\pi} \sqrt{1 + \cos^2 x}dx$. You should pay particular attention to exploit special features of the integrand!

9. Use Simpson's rule to approximate the improper integral $\int_0^{\pi/2} x^{-1/3} \cos x dx$ as seen in class. Take the first 3 nonzero terms of the Taylor series.

10. Approximate the integral $\int_3^4 \frac{x}{\sqrt{x^2-4}} dx$ using the 2-point Gauss-Legendre quadrature. Compare your result to the exact value.

11. Determine the values for A, B, C that make the formula

$$\int_0^2 xf(x)dx \approx Af(0) + Bf(1) + Cf(2)$$

   exact for all polynomials of degree as high as possible. You need to take $f = 1, x, x^2, \ldots$.

12. Calculate the quadratic least-squares fit for the data

x_i	0.	.25	.5	.75	1.0
y_i	1.	1.2840	1.6487	2.1170	2.7183

   and calculate the error.

13. Calculate the exponential least-squares fit of type $y = be^{ax}$ of the data in the preceding problem and calculate the error.

14. Given the linear multistep method

$$y_{n+1} = y_{n-3} + \frac{8}{3}hf(t_n, y_n) - \frac{4}{3}hf(t_{n-1}, y_{n-1}) + \frac{8}{3}hf(t_{n-2}, y_{n-2}), \quad n = 3, \ldots, N-1.$$

Find the local truncation error and determine the order of the method.

15. Consider the Runge-Kutta method given by the array

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} \\ \hline \frac{1}{4} & 0 & \frac{3}{4} & \end{array}$$

Apply one step of the method with $h = 0.25$ to the IVP

$$y' = \frac{y}{t} - \frac{y^2}{t^2}, \quad 1 < t \leq 2, \quad y(1) = 1.$$

16. Apply two steps of Euler's method with $h = 0.25$ to the IVP

$$y' = te^{3t} - 2y, \quad 0 < t \leq 1, \quad y(0) = 0.$$

Sample Final Solutions

#1) $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$, $x = .785398,\ h = .034907$

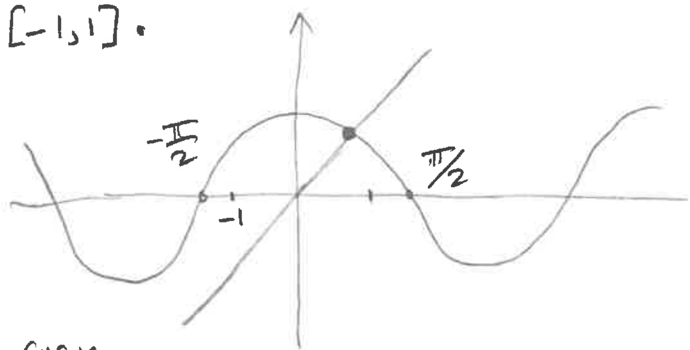
$$\Rightarrow f'(.785398) \approx \frac{1.07237 - .932515}{2(.034907)} = 2.003251$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$\approx \frac{1.07237 - 2(1) + .932515}{(.034907)^2} = 4.00903$$

#2) The result is $x_{n+1} = \cos(x_n)$, $n = 0, 1, \ldots$

For <u>any</u> x_0, $\cos(x_0) \in [-1, 1]$.

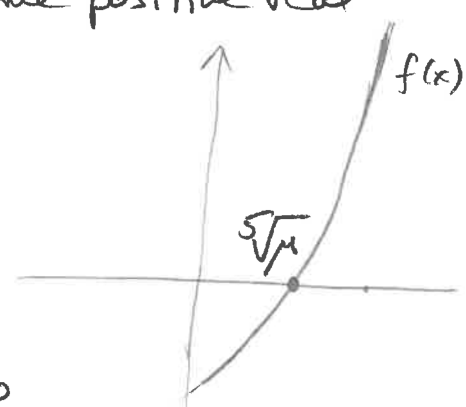


In the interval $[-1,1]$ $\cos x$ is a contraction so $\{x_n\}$ will converge to a fixed point of $\cos x$ which is unique $\approx \boxed{.739}$

#3) let $f(x) = x^5 - \mu$, μ some positive real

<u>Newton's method</u>

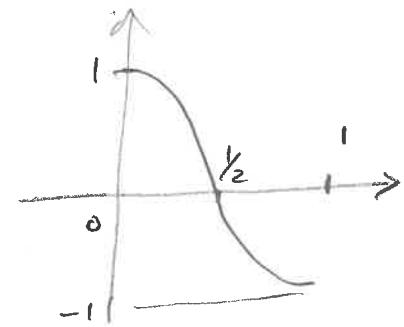
$$\begin{cases} x_{n+1} = x_n - \dfrac{x_n^5 - \mu}{5x_n^4} = \dfrac{4x_n^5 + \mu}{5x_n^4} \\ x_0 = \mu \end{cases}$$



Since f is convex and $f' > 0$

$x_n \to \sqrt[5]{\mu}$.

#4) (i)



$$\begin{array}{cccc}
0 & 1 & & \\
 & & -2 & \\
\frac{1}{2} & 0 & & 0 \\
 & & -2 & \\
1 & -1 & &
\end{array}$$

$$P_2(x) = 1 + (-2)(x-0) + 0(x-0)(x-\tfrac{1}{2})$$
$$= 1 - 2x \qquad \text{reduces to degree } \underline{\underline{1}}$$

Remainder $\dfrac{f^{(3)}(\xi)}{6}(x-0)(x-\tfrac{1}{2})(x-1) = \boxed{\dfrac{\pi^3 \sin \pi\xi}{6}\, x(x-\tfrac{1}{2})(x-1)}$

(ii)

$$\begin{array}{ccccccc}
0 & 1 & & & & & \\
 & & 0 & & & & \\
0 & 1 & & -4 & & & \\
 & & -2 & & -4\pi+16 & & \\
\frac{1}{2} & 0 & & -2\pi+4 & & 8\pi-24 & \\
 & & -\pi & & 4\pi-8 & & -16\pi+48 \\
\frac{1}{2} & 0 & & -4+2\pi & & -8\pi+24 & \\
 & & -2 & & -4\pi+16 & & \\
1 & -1 & & 4 & & & \\
 & & 0 & & & & \\
1 & -1 & & & & &
\end{array}$$

$$H_5(x) = 1 - 4x^2 + (16-4\pi)x^2(x-\tfrac{1}{2}) + (8\pi-24)x^2(x-\tfrac{1}{2})^2$$
$$+ (48-16\pi)x^2(x-\tfrac{1}{2})^2(x-1).$$

Remainder $\dfrac{f^{(6)}(\xi)}{6!}(x-0)^2(x-\tfrac{1}{2})^2(x-1)^2$

$$= \frac{-\pi^6}{6!}(\cos \pi\xi)\, x^2(x-\tfrac{1}{2})^2(x-1)^2$$

(iii) on $[0, 1/2]$

$$
\begin{array}{cccccc}
0 & 1 & & & & \\
 & & 0 & & & \\
0 & 1 & & -4 & & \\
 & & -2 & & -4\pi+16 & \\
\frac{1}{2} & 0 & & -2\pi+4 & & \\
 & & -\pi & & & \\
\frac{1}{2} & 0 & & & &
\end{array}
$$

$$\boxed{1 - 4x^2 + (16 - 4\pi)x^2(x - 1/2)}$$

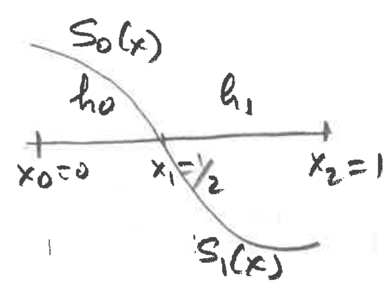
on $[1/2, 1]$

$$
\begin{array}{cccccc}
\frac{1}{2} & 0 & & & & \\
 & & -\pi & & & \\
\frac{1}{2} & 0 & & -4+2\pi & & \\
 & & -2 & & -4\pi+16 & \\
1 & -1 & & 4 & & \\
 & & 0 & & & \\
1 & -1 & & & &
\end{array}
$$

$$\boxed{-\pi(x - 1/2) + (2\pi - 4)(x - 1/2)^2 + (16 - 4\pi)(x - 1/2)^2(x - 1)}$$

Actually this information is contained in the table of $H_5(x)$.

(iv) $\boxed{\begin{array}{l} n = 2, \quad h_0 = h_1 = 1/2 \\ a_0 = 1, \quad a_1 = 0, \quad a_2 = -1 \end{array}}$



System

$$
\begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(0) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ 3f'(1) - \frac{3}{h_1}(a_2 - a_1) \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 2 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix}
\Rightarrow
\boxed{\begin{array}{l} c_0 = -6 \\ c_1 = 0 \\ c_2 = 6 \end{array}}
$$

$a_0 = f(0) = 1$

$b_0 = \dfrac{a_1 - a_0}{h_0} - \dfrac{h_0}{3}(2c_0 + c_1) = 0$

$c_0 = -6$

$d_0 = \dfrac{c_1 - c_0}{3h_0} = 4$

$$\boxed{S_0(x) = 1 - 6x^2 + 4x^3}$$

$a_1 = f(1/2) = 0$

$b_1 = \dfrac{a_2 - a_1}{h_1} - \dfrac{h_1}{3}(2c_1 + c_2) = -3$

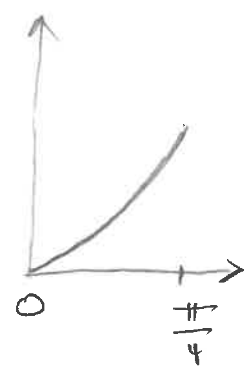
$c_1 = 0$

$d_1 = \dfrac{c_2 - c_1}{3h_1} = 4$

$$\boxed{S_1(x) = -3(x - 1/2) + 4(x - 1/2)^3}$$

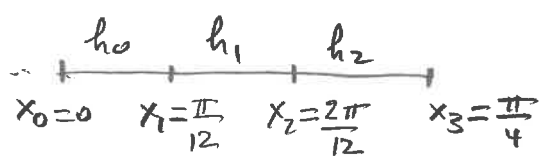
#5) $f(x) = \tan x$, $f'(x) = \sec^2 x$

0	0			
		1		
0	0		$\frac{16}{\pi^2} - \frac{4}{\pi}$	
		$\frac{4}{\pi}$		$\frac{48}{\pi^2} - \frac{128}{\pi^3}$
$\frac{\pi}{4}$	1		$\frac{8}{\pi} - \frac{16}{\pi^2}$	
		2		
$\frac{\pi}{4}$	1			



$$\boxed{H_3(x) = x + \left(\frac{16}{\pi^2} - \frac{4}{\pi}\right)x^2 + \left(\frac{48}{\pi^2} - \frac{128}{\pi^3}\right)x^2\left(x - \frac{\pi}{4}\right)}, \quad R = \frac{f^{(4)}(\xi)}{4!}x^2\left(x - \frac{\pi}{4}\right)^2$$

#6). $n = 3$, $h_0 = h_1 = h_2 = \frac{\pi}{12}$



$a_0 = \tan 0 = 0$, $a_1 = \tan\frac{\pi}{12} = .26795$; $a_2 = \tan\frac{\pi}{6} = .57735$, $a_3 = \tan\frac{\pi}{4} = 1$

system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ 0 \end{bmatrix}$$

only need to set up the system, not to solve it.

#7 Clamped cubic spline. We know that given a function f, its clamped cubic spline is uniquely defined and is given as the solution of a linear system. If f is a cubic polynomial it satisfies the linear system; in particular $f'(a)=S'(a)$, $f'(b)=S'(b)$. Therefore, by uniqueness, $S=f$.

Natural cubic spline In this case $S''(a)=S''(b)=0$ so unless $f''(a)=f''(b)=0$, S cannot be equal to f.

#8 The idea here is to make maximum use of periodicity. $\cos x$ is periodic with period 2π and so is $f(x)=\sqrt{1+\cos^2 x}$

$$\int_0^{48\pi}\sqrt{1+\cos^2 x}\,dx = 24\int_0^{2\pi}\sqrt{1+\cos^2 x}\,dx$$
$$= 96\int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 x}\,dx.$$

Hence we use the composite Simpson's rule with $\boxed{n=4}$ on $\int_0^{\pi/2}\sqrt{1+\cos^2 x}\,dx$ and multiply by 96.

$$h=\frac{\frac{\pi}{2}-0}{4}=\frac{\pi}{8}$$

h

$0 \quad \frac{\pi}{8} \quad \frac{\pi}{4} \quad \frac{3\pi}{8} \quad \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 x}\,dx \cong \frac{h}{3}\left[f(0)+4f\left(\tfrac{\pi}{8}\right)+2f\left(\tfrac{\pi}{4}\right)+4f\left(\tfrac{3\pi}{8}\right)+f\left(\tfrac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{24}\Big[1.414213 + 4(1.361452 + 1.070722)$$

$$+ 2(1.2247448) + 1\Big] \approx 1.9101$$

$$\Rightarrow \int_0^{48\pi} \sqrt{1+\cos^2 x}\, dx \approx 96(1.9101) = 183.3696$$

\#9 $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + R_4(x)$

$$x^{-1/3}\cos x = \left[x^{-1/3} - \frac{1}{2}x^{5/3} + \frac{1}{24}x^{11/3}\right] + x^{-1/3}R_4(x)$$

$$\int_0^{\frac{\pi}{3}} \left[x^{-1/3} - \frac{1}{2}x^{5/3} + \frac{1}{24}x^{11/3}\right]dx = \frac{3}{2}x^{2/3} - \frac{3}{16}x^{8/3} + \frac{1}{112}x^{14/3}\Big]_0^{\pi/3}$$

$$\approx \boxed{1.144626}$$

let $G(x) = \begin{cases} x^{-1/3}\cos x - \left[x^{-1/3} - \frac{1}{2}x^{5/3} + \frac{1}{24}x^{11/3}\right], & x \neq 0 \\ 0 & x = 0. \end{cases}$

Apply Simpson's rule to G,

$$\int_0^{\frac{\pi}{3}} G(x)\,dx \cong \frac{h}{18}\left[G(0) + 4G\left(\frac{\pi}{6}\right) + G\left(\frac{\pi}{3}\right)\right]$$

$$\approx \boxed{.1826161}$$

$$\Rightarrow \int_0^{\pi/3} x^{-1/3}\cos x\,dx \approx 1.144626 + .1826161$$

$$\approx 1.3272421$$

#10

$$\int_3^4 \frac{x}{\sqrt{x^2-4}}\,dx = \int_{u=5}^{u=12} \frac{du/2}{u^{1/2}} \qquad u = x^2-4, \quad du = 2x\,dx$$

$$= \frac{1}{2}\cdot 2\, u^{1/2}\Big|_5^{12} = \sqrt{12}-\sqrt{5} \approx 1.22803$$

$\hat{x}_1 = -\frac{1}{\sqrt{3}}$, $\hat{x}_2 = \frac{1}{\sqrt{3}}$ — we need to map $\hat{x}_1, \hat{x}_2$ to the interval $[3,4]$

$\hat{w}_1 = 1$, $\hat{w}_2 = 1$

$$x = \ell(\hat{x}) = \frac{b-a}{2}\hat{x} + \frac{b+a}{2} = \frac{\hat{x}}{2} + \frac{7}{2}$$

$$\Rightarrow \int_3^4 \frac{x\,dx}{\sqrt{x^2-4}} \approx \underbrace{\frac{1}{2}}_{\frac{b-a}{2}}\left[1\cdot\frac{-\frac{1}{2\sqrt{3}}+\frac{7}{2}}{\sqrt{\left(-\frac{1}{2\sqrt{3}}+\frac{7}{2}\right)^2-4}} + 1\cdot\frac{\frac{1}{2\sqrt{3}}+\frac{7}{2}}{\sqrt{\left(\frac{1}{2\sqrt{3}}+\frac{7}{2}\right)^2-4}}\right]$$

$$\approx \boxed{1.22788}$$

#11

Exact value $\int_0^2 x\,f(x)\,dx$	Approximation $Af(0)+Bf(1)+Cf(2)$
$f(x)=1 \Rightarrow \int_0^2 x\cdot 1\,dx = \frac{x^2}{2}\Big\vert_0^2 = 2$	$A\cdot 1 + B\cdot 1 + C\cdot 1$
$f(x)=x \Rightarrow \int_0^2 x\cdot x\,dx = \frac{x^3}{3}\Big\vert_0^2 = 8/3$	$A\cdot 0 + B\cdot 1 + C\cdot 2$
$f(x)=x^2 \Rightarrow \int_0^2 x\cdot x^2\,dx = \frac{x^4}{4}\Big\vert_0^2 = 4$	$A\cdot 0 + B\cdot 1 + C\cdot 4$
$f(x)=x^3 \Rightarrow \int_0^2 x\cdot x^3\,dx = \frac{x^5}{5}\Big\vert_0^2 = \frac{32}{5}$	$A\cdot 0 + B\cdot 1 + C\cdot 8$
⋮	⋮

we want to see how far the agreement between the exact value and the approximation can be carried out.

From the 2nd and 3rd Equations

$$\begin{aligned} A\cdot 0 + B + 2C &= 8/3 \\ A\cdot 0 + B + 4C &= 4 \end{aligned} \Rightarrow \boxed{B = \frac{4}{3},\ C = 2/3}$$

From the 1st Equ.: $\boxed{A = 2 - B - C = 0}$

we next see if the 4-th Equ. is satisfied

$$\frac{32}{5} \overset{?}{=} 0\cdot 0 + \frac{4}{3} + 8\cdot\frac{2}{3} = \frac{20}{3}. \quad \underline{\underline{NO}}$$

Hence formula with $A=0$, $B=4/3$, $C=2/3$ is exact for polynomials of degree ≤ 2.

#12 $\boxed{m=2}$ 5 data pts. $\Rightarrow$ $n=4$

$$\begin{bmatrix} \sum_{i=0}^{n} x_i^4 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^2 \\ \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i \\ \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} y_i x_i^2 \\ \sum_{i=0}^{n} y_i x_i \\ \sum_{i=0}^{n} y_i \end{bmatrix}$$

$$\begin{bmatrix} 1.38281 & 1.56250 & 1.87500 \\ 1.56250 & 1.87500 & 2.50 \\ 1.87500 & 2.50 & 5 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 4.40154 \\ 5.45140 \\ 8.7680 \end{bmatrix}$$

$a_2 = 0.843741$, $a_1 = 0.864099$, $a_0 = 1.005147$

$\boxed{P_2(x) = a_2 x^2 + a_1 x + a_0}$ $\quad$ $\text{error} = \sum_{i=0}^{4} (y_i - P_2(x_i))^2 \doteq .0027$

#13

$$\begin{bmatrix} \sum_{i=0}^{4} x_i^2 & \sum_{i=0}^{4} x_i \\ \sum_{i=0}^{4} x_i & \sum_{i=0}^{4} 1 \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{4} x_i \ln y_i \\ \sum_{i=0}^{4} \ln y_i \end{bmatrix}$$

$$\begin{bmatrix} 1.875 & 2.50 \\ 2.5 & 5 \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} 1.8749953 \\ 2.499974 \end{bmatrix} \Rightarrow \begin{matrix} a = 1.0000133 \\ \ln b = 10^{-5} \to 0 \end{matrix}$$

$$\Rightarrow b \approx 1$$

$$err = \sum_{i=0}^{4} (y_i - b e^{a x_i})^2 \approx 2.7 \times 10^{-9}$$

#14

$$LTE = y(t_{n+1}) - y(t_{n-3}) - \frac{8}{3} h y'(t_n) + \frac{4}{3} h y'(t_{n-1}) - \frac{8}{3} h y'(t_{n-2})$$

$$= y(t_n + h) - y(t_n - 3h) - \frac{8}{3} h y'(t_n) + \frac{4}{3} h y'(t_n - h) - \frac{8}{3} h y'(t_n - 2h)$$

$$=$$

$$y(t_n+h) \to \left[y + hy' + \frac{h^2}{2} y'' + \frac{h^3}{6} y''' + \frac{h^4}{24} y^{(4)} + \frac{h^5}{120} y^{(5)} + O(h^6) \right]$$

$$-y(t_n - 3h) \to -\left[y - 3hy' + \frac{9h^2}{2} y'' - \frac{27h^3}{6} y''' + \frac{81 h^4}{24} y^{(4)} - \frac{243}{120} h^5 y^{(5)} + O(h^6) \right]$$

$$-\frac{8}{3} h y'(t_n) \to -\frac{8}{3} h y'$$

$$+$$

$$\frac{4}{3} h y'(t_n - h) \to \frac{4}{3} h \left[y' - hy'' + \frac{h^2}{2} y''' - \frac{h^3}{6} y^{(4)} + \frac{h^4}{24} y^{(5)} + O(h^5) \right]$$

$$-\frac{8}{3} h y'(t_n - 2h) \to -\frac{8}{3} h \left[y' - 2hy'' + \frac{4h^2}{2} y''' - \frac{8h^3}{6} y^{(4)} + \frac{16 h^4}{24} y^{(5)} + O(h^5) \right]$$

collecting terms:

$$LTE = y\left[1 \overset{0}{\nearrow} 1\right] + hy'\left[1 + 3 - \frac{8}{3} \overset{0}{\nearrow} + \frac{4}{3} - \frac{8}{3}\right]$$

$$+ h^2 y''\left[\frac{1}{2} - \frac{9}{2} \overset{0}{\nearrow} - \frac{4}{3} + \frac{16}{3}\right] + h^3 y'''\left[\frac{1}{6} + \frac{9}{2} \overset{0}{\nearrow} + \frac{2}{3} - \frac{16}{3}\right]$$

$$+ h^4 y^{(4)}\left[\frac{1}{24} - \frac{27}{8} \overset{0}{\nearrow} - \frac{2}{9} + \frac{32}{9}\right]$$

$$+ h^5 y^{(5)}\underbrace{\left[\frac{1}{120} - \frac{243}{120} + \frac{1}{18} - \frac{16}{9}\right]}_{= \frac{-673}{180} \neq 0,} + O(h^6)$$

Hence $LTE = -\frac{673}{180} h^5 y^{(5)} + O(h^6)$

$\Rightarrow$ order of the method is $4 = 5 - 1$.

#15 $y_0 = 1$, $t_0 = 1$, $h = 0.25$

$$k_1 = hf(t_0, y_0) = hf(1,1) = 0$$

$$k_2 = h f\left(t_0 + \frac{h}{3}, y_0 + \frac{k_1}{3}\right) = hf\left(1 + \frac{.25}{3}, 1 + \frac{1}{3}\cdot 0\right) = .0177515$$

$$k_3 = h f\left(t_0 + \frac{2h}{3}, y_0 + \frac{2}{3}k_2\right) = hf\left(1 + \frac{.5}{3}, 1 + \frac{2}{3}(.0177515)\right)$$

$$= .02877514$$

$$y_1 = y_0 + \frac{k_1}{4} + 0 \cdot k_2 + \frac{3}{4}k_2 = 1.0215813$$

$$y_1 \approx \frac{t_1}{1 + \ln t_1} = \frac{1.25}{1 + \ln 1.25} \approx 1.021956$$

#16) $y_0 = 0, \quad h = .25, \quad t_0 = 0$

$$y_1 = y_0 + h\, f(t_0, y_0) = 0 + (.25)\left[0e^{3(0)} - 2y_0\right] = 0$$

$$t_1 = t_0 + h = .25$$

$$y_2 = y_1 + h\, f(t_1, y_1) = 0 + (.25)\left[(.25)e^{3(.25)} - 2\overset{0}{\cancel{y_1}}\right]$$

$$\cong .13231$$