MATH 472 SAMPLE FINAL EXAM

April 27, 2018

(1) Show that the matrix A

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is symmetric positive definite. Find its Cholesky decomposition.

(2) Find the LDL^T factorization of the matrix

$$A = \begin{pmatrix} 4 & -4 & -12 \\ -4 & 3 & 10 \\ -12 & 10 & 37 \end{pmatrix}.$$

Is this matrix positive definite? Justify your answer.

(3) Consider the elementary permutation matrix

$$P_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Show that P_{24} can be expressed as a Householder matrix $H(u)$ for some u.

(4) We say that a matrix A is *convergent* if $\lim_{n\to\infty} A^n = 0$. Decide whether the following statement is true or false: "If any element of a matrix A is greater than or equal to 1, then A is divergent". *True* requires a proof while *False* requires a counterexample.

(5) Show that $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ is not convergent and that $B = \begin{bmatrix} \frac{1}{2} & 0 \\ 16 & \frac{1}{2} \end{bmatrix}$ is.

(6) Find the QR factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & 0 & 1 \\ -2 & 1 & 4 \end{pmatrix}$$

using Householder transformations. Then, use the factors to solve the linear system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (9, 5, 12)^T$.

(7) Let $\mathbf{x} = (4, 1, -2, 2)^T$. Find a Householder matrix H such that

$$H(\mathbf{v})\mathbf{x} = (z, 0, 0, 0)^T.$$

You need to determine $\mathbf{v}$, H and z.

(8) Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(i) Compute the spectral radius of the Jacobi matrix $J = D^{-1}(L+U)$.
(ii) Compute the spectral radius of the Gauss-Seidel matrix $\mathcal{L}_1 = (D-L)^{-1}U$.

(9) Consider the matrix A

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

We know that 4 is an eigenvalue.
(i) Find the eigenvector corresponding to $\lambda = 4$.
(ii) Apply Wielandt deflation to find the remaining eigenvalues.

(10) Consider the overdetermined system

$$\begin{pmatrix} 2 & 1 \\ 4 & 1 \\ 6 & 1 \\ 8 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 28 \\ 40 \end{bmatrix}.$$

In general, overdetermined systems cannot be solved in the classical sense.
(i) Show that the above system does not have a solution.
(ii) Obtain the QR factorization of A of the above example and write

$$R = \begin{bmatrix} R_{11} \\ \mathbf{0} \end{bmatrix}, \qquad Q^T b = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix},$$

where R_{11} is a 2×2 matrix and $\tilde{b}_1$ is a 2-vector.
(iii) Show that the least-squares solution is given by $x = R_{11}^{-1}\tilde{b}_1$.
(iv) Find the least-squares solution using the normal equations.

(11) Consider the nonlinear system

$$\begin{aligned} 3x^2 - y^2 &= 0 \\ 3xy^2 - x^3 - 1 &= 0. \end{aligned}$$

Starting with $\mathbf{x}^{(\mathbf{0})} = (0.6, 0.8)^T$ and $A_0 = F'(\mathbf{x}^{(\mathbf{0})})$, apply two iterations of Broyden's method. Use the Sherman-Morrison formula.

(12) Suppose you knew that the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

had an eigenvalue near 1.9. How would you use this information to obtain a better approximation of the eigenvalue?