MATH 472 SAMPLE MIDTERM EXAM

March 6, 2018

(1) Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1.297 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \qquad b = \begin{pmatrix} 0.8644 \\ 0.1440 \end{pmatrix}$$

Solve the system using Gauss elimination. Do all calculations using 4 decimal digit rounding arithmetic. Explain what happens.

(2) Let $\bar{x}$ be an approximate solution of the linear system $Ax = b$ and let $r = A\bar{x} - b$ denote the residual.

(i) Derive the estimate

$$\|x - \bar{x}\| \le \|A^{-1}\| \, \|r\|.$$

(ii) The linear system $Ax = b$ with

$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \qquad b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix},$$

has exact solution $x = (2, -2)^T$. Given the approximate solution $\bar{x} = (0.9911, -0.4870)^T$, calculate the residual and obtain an estimate for $\|A^{-1}\|_\infty$, and thus one for $\kappa_\infty(A)$. Compare the estimate with the directly calculated value of the condition number.

(iii) In the light of the above, comment on the statement: *A small residual means small error in the solution of a linear system.*

(3) Show that the matrix A

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is symmetric positive definite. Find its Cholesky decomposition.

(4) Find the LDL^T factorization of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 0 & -6 \\ 2 & -6 & -2 \end{pmatrix}.$$

Is this matrix positive definite? Justify your answer.

(5) Prove that an orthogonal triangular matrix is diagonal.

(6) Find the QR factorization of the matrix

$$A = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$$

using Householder transformations.

(7) Compute the QR factorization of the matrix

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 4 & -1 \\ 3 & 7 & 2 \end{pmatrix},$$

and use it to solve the linear system $Ax = b$ with $b = (1, 3, 12)^T$.

(8) Consider the overdetermined linear system

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(i) Calculate the least-squares solution using the method of normal equations.

(ii) Calculate the least-squares solution using the QR method.