REGULARITY ESTIMATES FOR BMO-WEAK SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. This paper studies regularity estimates in Lebesgue spaces for gradients of weak solutions of a class of general quasilinear equations of p-Laplacian type in bounded domains with inhomogeneous conormal boundary conditions. In the considered class of equations, the principals are vector field functions measurable x-variable, and nonlinearly depending on both solution and its gradient. This class of equations consists of the well-known class of degenerate p-Laplace equations for $p > 1$. Under some sufficient conditions, we establish local interior, local boundary, and global $W^{1,q}$-regularity estimates for weak solutions with $q > p$, assuming that the weak solutions are in the John-Nirenberg BMO space. The paper therefore improves available results because it removes the boundedness or continuity assumptions on solutions. Our results also unify and cover known results for equations in which the principals are only allowed to depend on x-variable and gradient of solution variable. More than that, this paper gives a method to treat non-homogeneous boundary value problems directly without using any form of translations that is sometimes complicated due to the nonlinearities.

Keywords: Quasilinear elliptic equations, Quasilinear p-Laplacian type equations, Calderón-Zygmund regularity estimates, Conormal boundary value problems. AMS Subject Classification: 35J92, 35J62, 35J66, 35J60, 35B45.

1. Introduction

This paper establishes local interior, local boundary, and global regularity estimates in Lebesgue spaces for gradients of weak solutions of the following general class of nonlinear degenerate elliptic equations with in-homogeneous conormal boundary condition

$$\left\{ \begin{array}{rcll} \text{div}\,[\mathbf{A}(x,u,\nabla u)] & = & \text{div}\,[|\mathbf{F}(x)|^{p-2}\mathbf{F}(x)] & x \in \Omega, \\ \langle \mathbf{A}(x,u,\nabla u) - |\mathbf{F}|^{p-2}\mathbf{F}, \vec{\nu}\rangle & = & |g(x)|^{p-2}g(x) & x \in \partial\Omega, \end{array} \right. \tag{1.1}$$

where u is an unknown solution, $p \in (1,\infty)$ is fixed, Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$ and with sufficiently smooth boundary $\partial\Omega$, $\vec{\nu}$ is the outward normal vector on $\partial\Omega$, $g : \partial\Omega \to \mathbb{R}$ is a given measurable function, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$, and $\mathbf{F} : \Omega \to \mathbb{R}^n$ is a given measurable vector field function. Moreover, the principal

$$\mathbf{A} = \mathbf{A}(x,z,\xi) : \Omega \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$$

is a given vector field, where $\mathbb{K} \subset \mathbb{R}$ is a given open interval which could be the same as $\mathbb{R}$. We assume that $\mathbf{A}(\cdot,z,\xi)$ is measurable in Ω for every $(z,\xi) \in \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$; $\mathbf{A}(x,\cdot,\xi)$ is Hölder continuous in $\mathbb{K}$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n \setminus \{0\}$; and $\mathbf{A}(x,z,\cdot)$ is differentiable in $\mathbb{R}^n \setminus \{0\}$ for each $z \in \mathbb{K}$ and for a.e. $x \in \Omega$. We assume that there exist constants $\Lambda > 0$ and $\alpha \in (0,1]$ such that $\mathbf{A}$ satisfies the following natural growth conditions

$$\langle \partial_\xi \mathbf{A}(x,z,\xi)\eta, \eta\rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2, \qquad \text{for a.e. } x \in \Omega, \quad \forall\, z \in \mathbb{K}, \quad \forall\, \xi,\eta \in \mathbb{R}^n \setminus \{0\}, \tag{1.2}$$

$$|\mathbf{A}(x,z,\xi)| + |\xi||\partial_\xi \mathbf{A}(x,z,\xi)| \leq \Lambda|\xi|^{p-1}, \qquad \text{for a.e. } x \in \Omega, \quad \forall\, z \in \mathbb{K}, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

$$|\mathbf{A}(x,z_1,\xi) - \mathbf{A}(x,z_2,\xi)| \leq \Lambda|\xi|^{p-1}|z_1 - z_2|^\alpha \qquad \forall\, z_1,z_2 \in \mathbb{K}, \quad \text{for a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

Under the assumptions (1.2)–(1.4), the class of equations (1.1) contains the well-known p-Laplace equations. This general class of equations also appears in many models in applications such as thin film,

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non-Newtonian fluid mechanics and flow in porous media, elasticity and science of "smart materials", and emerging issues in biomathematics and biophysics related to degenerate and/or singular diffusion of molecules on cell surfaces, see for example [10, p. 6-7] for citations, details and discussions regarding these applications.

The most interesting feature in the class of equations (1.1) is that the principal $\mathbf{A}$ is only measurable in $x \in \Omega$, and it depends not only on $\xi \in \mathbb{R}^n \setminus \{0\}$, but also on the z-variable in $\mathbb{K}$. Our goal is to establish the $W^{1,q}$-regularity estimates of Calderón-Zygmund type for weak solutions of this class of equations. In this line of research, we would like to point out that even with $g = \mathbf{F} = 0$, the $W^{1,q}$-regularity estimates for solutions of the class of equation (1.1) are already challenging and not yet well understood. This is due the fact that the Calderón-Zygmund theory relies heavily on the scaling invariance that is not available for (1.1). One can see [45], for example, for the geometric intuition on the scaling invariances in the Calderón-Zygmund theory. In a simpler setting in which $\mathbf{A}$ is independent on $z \in \mathbb{K}$, the equation (1.1) is reduced to the following class equations

$$\operatorname{div}[\mathbf{A}(x, \nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad \text{in} \quad \Omega. \tag{1.5}$$

Though nonlinear, the class of equations (1.5) is invariant under the following natural scalings and dilations

$$u \mapsto u/\lambda, \quad \text{and} \quad u(x) \mapsto \frac{u(rx)}{r}, \quad \text{for all positive numbers} \quad r, \lambda. \tag{1.6}$$

Because of the availability of the homogeneity with respect to (1.6), the $W^{1,q}$-regularity theory for weak solutions of (1.5) is naturally expected. Consequently, though not trivial, this regularity theory has been extensively studied for (1.5), see [3,4,7,12,14,16,17,23,24,32,38]. However, the homogeneity with respect to (1.6) is not available for the class of equations (1.1) in which $\mathbf{A}$ is a vector field function of u-variable in $\mathbb{K}$. This fact presents a serious obstacle in obtaining $W^{1,q}$-estimates for the weak solutions of (1.1) as they do not generate enough estimates to carry out the proof by using existing methods.

In the recent work [22, 35], the $W^{1,q}$-regularity estimates for weak solutions of equations of type (1.1) are addressed. The $W^{1,q}$-regularity estimates are established assuming that the considered solutions are bounded. To overcome the loss of homogeneity that we mentioned, in [22, 35], we introduced some "double-scaling parameter" technique. Essentially, we study an enlarged class of "double parameter" equations of the type (1.1). Then, by some compactness argument, we successfully applied the perturbation method in [7] to tackle the problem. Careful analysis is required to ensure that all intermediate steps in the perturbation process are uniformly with respect to the scaling parameters. See also the work [6, 15] for some related results in which global regularity theory for weak bounded solutions is obtained. In the mentioned papers [6, 22, 35], the boundedness assumption on the solutions is essential to initiate the study. This is because the approach uses maximum principle for the unperturbed equations to implement the perturbation technique. We would like to refer also to [1] for which the $W^{1,q}$-theory for parabolic equations of type (1.1) is also achieved, but only for continuous weak solutions plus some other assumptions on $\mathbf{A}$.

Motivated by [1, 6, 15, 22, 35, 36], in this paper we develop the following significant advancements:

(I) Unlike the mentioned results in [1, 6, 15, 22, 35], this paper develops the $W^{1,q}$-regularity theory for solutions of (1.1) that could be unbounded. In fact, we assume that our solutions are in BMO, the John-Nirenberg space of functions with bounded mean oscillations. Obviously, replacing the L^∞-requirement by the BMO-requirement is valuable in many critical applications in which the a priori L^∞-estimates for solutions are not available. For example, when $p = n$, our weak $W^{1,p}$-solutions are already in BMO. Hence our results are applicable while the known results may be not. See also the work [13] for the BMO-regularity estimates of solutions of equation of type (1.1).

(II) This paper also treats the non-homogeneous boundary condition, i.e. $g \neq 0$. In fact, as one will find in the proof, instead of commonly using some type of translation to reduce the inhomogeneous boundary value problem to the homogeneous one (see [1], for example), this paper perturbs the boundary data g directly. This method seems to be new, and much simpler compared to the traditional translation one. Moreover, it also avoids algebra complications when using the translation due to the nonlinearity in the equations as in (1.1).

The main results in the paper are Theorem 1.1, Theorem 1.2 and Theorem 1.3 below. These results generalize the results in [1,6,22,35,36]. They also recover the results in [3,4,7,12,14,16,17,23,24,32,38] when $\mathbf{A}$ is independent on u-variable. More than that, this paper also simplifies many technical problems in [22,35], and allow the boundary condition to be non-homogeneous. Unlike [6,22,35] we only use "one parameter" in the class of our equations. Precisely, we investigate the following class of equations

$$\begin{cases} \operatorname{div}\left[\mathbf{A}(x,\lambda u,\nabla u)\right] &= \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], \quad \text{in } \Omega,\\ \langle \mathbf{A}(x,\lambda u,\nabla u) - |\mathbf{F}|^{p-2}\mathbf{F},\vec{\nu}\rangle &= |g(x)|^{p-2}g(x), \quad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

with some scaling parameter $\lambda \geq 0$. The class of these equations is the smallest one that is invariant with respect to the scalings and dilations in (1.6), which also includes (1.1). When $\lambda = 0$, the equation (1.7) clearly becomes the equation (1.5). Therefore, Theorem 1.1, Theorem 1.2, and Theorem 1.3 recover known results such as [3,4,7,12,14,16,17,23,24,32,38] regarding (1.5).

Now, some notations are introduced in order to state the main theorems of the paper. For each $\rho > 0, y \in \mathbb{R}^n$, $B_\rho(y)$ denotes the ball in $\mathbb{R}^n$ with radius ρ and centered at $y \in \mathbb{R}^n$. If $y = 0$, we write $B_\rho = B_\rho(0)$. Moreover, with a measurable set $U \subset \mathbb{R}^n$, some $\rho_0 > 0$, and a locally integrable function $f : U \to \mathbb{R}^n$, the semi-norm of bounded mean oscillation of f is defined by

$$[[f]]_{\mathrm{BMO}(U,\rho_0)} = \sup_{y\in U, 0<\rho<\rho_0} \frac{1}{|B_\rho(y)\cap U|}\int_{B_\rho(y)\cap U} |f(x) - \bar{f}_{B_\rho(y)\cap U}|dx, \quad \text{where}$$
$$\bar{f}_{B_\rho(y)\cap U} = \frac{1}{|B_\rho(y)\cap U|}\int_{B_\rho(y)\cap U} f(x)dx.$$

One of our results of the paper is the following global regularity estimate for weak solutions of the (1.7) with the non-homogeneous boundary conditions.

Theorem 1.1. *Let $\Lambda, M > 0, q > p > 1$, and $\alpha \in (0,1]$. Then, there exists a sufficiently small constant $\Upsilon = \Upsilon(\Lambda, M, \alpha, p, q, n) > 0$ such that the following statement holds true. Suppose that Ω is a C^1-domain and $\mathbb{K}$ is an open interval in $\mathbb{R}$. Suppose also that $\mathbf{A} : \Omega \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.4) *on $\Omega\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$ and there is some $\rho_0 \in (0,1)$ so that*

$$[[\mathbf{A}]]_{\mathrm{BMO}(\Omega,\rho_0)} := \sup_{\substack{z\in\mathbb{K}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \sup_{\substack{0<\rho\leq\rho_0\\ y\in\overline{\Omega}}} \frac{1}{|B_\rho(y)\cap\Omega|}\int_{B_\rho(y)\cap\Omega} \frac{|\mathbf{A}(x,z,\xi) - \bar{\mathbf{A}}_{B_\rho(y)\cap\Omega}(z,\xi)|}{|\xi|^{p-1}}dx \leq \Upsilon. \tag{1.8}$$

Then, for every $g \in L^q(\partial\Omega)$, $\mathbf{F} \in L^q(\Omega,\mathbb{R}^n)$, if $u \in W^{1,p}(\Omega)$ is a weak solution of (1.7) *satisfying $[[\lambda u]]_{\mathrm{BMO}(\Omega,\rho_0)} \leq M$ with some $\lambda \geq 0$, the following regularity estimate holds*

$$\int_\Omega |\nabla u(x)|^q dx \leq C\left[\int_{\partial\Omega}|g(x)|^q dS(x) + \int_\Omega |\mathbf{F}(x)|^q dx + \left(\int_\Omega |\nabla u(x)|^p dx\right)^{q/p}\right], \tag{1.9}$$

where C is a constant depending only on q, p, n, Λ, α, $M, \mathbb{K}, \rho_0$ and Ω.

Since local regularity estimates are useful in some problems because they only require information locally. Besides global regularity estimates as Theorem 1.1, the next two theorems on local regularity estimates are also important results of the paper.

Theorem 1.2. *Let $\Lambda > 0, M > 0, q > p > 1$, and $\alpha \in (0,1]$. Then, there exists a sufficiently small constant $\hat{\delta}_0 = \hat{\delta}_0(\Lambda, M, \alpha, p, q, n) > 0$ such that the following statement holds. For $R \in (0,1)$ and some open interval*

$\mathbb{K} \subset \mathbb{R}$, let $\mathbf{A} : B_{2R}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\}) \longrightarrow \mathbb{R}^n$ be a Carathéodory map satisfying (1.2)-(1.4) *on $B_{2R}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$ and*

$$[[\mathbf{A}]]_{\mathrm{BMO}(B_R,R)} := \sup_{\substack{z\in\mathbb{K}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \sup_{\substack{0<\rho\leq R\\ y\in B_R}} \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} \frac{|\mathbf{A}(x,z,\xi) - \bar{\mathbf{A}}_{B_\rho(y)}(z,\xi)|}{|\xi|^{p-1}}\, dx \leq \hat{\delta}_0. \tag{1.10}$$

Then, for every $\mathbf{F} \in L^q(B_{2R},\mathbb{R}^n)$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of

$$\mathrm{div}[\mathbf{A}(x,\lambda u,\nabla u)] = \mathrm{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad in \quad B_{2R}$$

with $[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M$ and some $\lambda \geq 0$, the following regularity estimate holds

$$\fint_{B_R} |\nabla u|^q dx \leq C\left[\fint_{B_{2R}} |\mathbf{F}|^q dx + \left(\fint_{B_{2R}} |\nabla u|^p dx\right)^{q/p}\right],$$

where $\bar{\mathbf{A}}_{B_\rho(y)}(z,\xi) := \fint_{B_\rho(y)} \mathbf{A}(x,z,\xi)\, dx$, and C is a constant depending only on q, q, Λ, α, $M, \mathbb{K}$, and n.

Our next result is the local regularity estimate on the boundary $\partial\Omega$. Instead of working on $\partial\Omega$, we assume that $\partial\Omega$ is sufficiently smooth so that part of $\partial\Omega$ is already flattened. In the next theorem, for $y = (y', y_n) \in \mathbb{R}^n$, and $R > 0$, we denote $B'_R(y')$ the ball in $\mathbb{R}^{n-1}$ of radius R and centered ad $y' \in \mathbb{R}^{n-1}$. Moreover, we write the cylinders in $\mathbb{R}^n$ as

$$D_R(y) = B'_R(y') \times (y_n - R, y_n + R), \quad D^+_R(y) = B'_R(y') \times (\max\{y_n - R, 0\}, y_n + R).$$

When $y = 0$, we write $D^+_R = D^+_R(0)$ and $B'_R = B'_R(0')$. Our local boundary regularity theory is stated in the following theorem

Theorem 1.3. *Let $\Lambda, M > 0, q > p > 1$, and $\alpha \in (0,1]$. Then, there exists a sufficiently small constant $\delta = \delta(\Lambda, M, \alpha, p, q, n) > 0$ such that the following statement holds true. Suppose that for some $R \in (0,1)$ and some open interval $\mathbb{K} \subset \mathbb{R}$, $\mathbf{A} : D^+_{2R}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\}) \longrightarrow \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.4) *on $D^+_{2R}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$ and*

$$[[\mathbf{A}]]_{\mathrm{BMO}(D^+_R,R)} := \sup_{\substack{z\in\mathbb{K}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \sup_{\substack{0<\rho\leq R\\ y\in D^+_R}} \frac{1}{|D_\rho(y)\cap D^+_{2R}|}\int_{D_\rho(y)\cap D^+_{2R}} \frac{|\mathbf{A}(x,z,\xi) - \bar{\mathbf{A}}_{D_\rho(y)\cap D^+_{2R}}(z,\xi)|}{|\xi|^{p-1}}\, dx \leq \delta. \tag{1.11}$$

Then, for every $g \in L^q(B'_{2R})$, $\mathbf{F} \in L^q(D^+_{2R},\mathbb{R}^n)$, if $u \in W^{1,p}(D^+_{2R})$ is a weak solution of

$$\begin{cases} \mathrm{div}[\mathbf{A}(x,\lambda u,\nabla u)] &= \mathrm{div}[|\mathbf{F}(x)|^{p-2}\mathbf{F}(x)] \quad in \quad D^+_{2R}, \\ \langle \mathbf{A}(x,\lambda u,\nabla u) - |\mathbf{F}|^{p-2}\mathbf{F}, \vec{e}_n\rangle &= |g(x')|^{p-2}g(x') \quad on \quad B'_{2R}\times\{0\}, \end{cases}$$

satisfying $[[\lambda u]]_{\mathrm{BMO}(D^+_R,R)} \leq M$ with some $\lambda \geq 0$, the following regularity estimate holds

$$\fint_{D^+_R} |\nabla u(x)|^q dx \leq C\left[\fint_{B'_{2R}} |g(x')|^q dx' + \fint_{D^+_{2R}} |\mathbf{F}(x)|^q dx + \left(\fint_{D^+_{2R}} |\nabla u(x)|^p dx\right)^{q/p}\right], \tag{1.12}$$

where C is a constant depending only on q, p, n, Λ, α, $M, \mathbb{K}$, and where $\vec{e}_n = (0,0,\cdots,0,1) \in \mathbb{R}^n$.

Several remarks are emphasized regarding Theorem 1.1, Theorem 1.2, Theorem 1.3 . Firstly, note that these theorems relax and do not require the solutions to be bounded as in [1, 6, 22, 35]. This is significant and it is completely new even for the case $g = 0$, in comparison to the known work for both the Schauder's regularity theory in [11, 18, 20, 27–31, 42–44] and the Sobolev one in [1, 6, 22, 35] regarding weak solutions of equations (1.1). To overcome the loss of boundedness of solutions from the assumption, instead of applying maximum principle during the perturbation process as in [6, 22, 35], we directly derive and carefully use some delicate analysis Hölder's regularity estimates for solutions of the corresponding unperturbed equations, see the estimates (3.4), (4.6), and (4.15) for examples. The well-known reverse Hölder's inequality and John-Nirenberg's inequality also play a central role in our approach. See also similar results in [36] which only treat the interior estimates. Secondly, this paper also treats the non-homogeneous boundary conditions. This is a non-trivial technical issue because it is quite complex due to nonlinearity to reduce the non-homogeneous boundary problem (1.1) to the homogeneous boundary one using some type of translation, see [1] for example. In this paper, instead of using the translation, we perturb the boundary condition directly and treating it as the force terms. This idea is new but natural, and it also avoids all complexity regarding the boundary translation that is due to the nonlinearity of the equations. Thirdly, we note that when $\lambda = 0$ and $g = 0$, Theorem 1.1, Theorem 1.2, and Theorem 1.3 recovers all results in [3, 4, 7, 12, 14, 16, 17, 23, 24, 32, 38] for the case that $\mathbf{A}$ is independent on $z \in \mathbb{K}$. This paper therefore unifies both $W^{1,q}$-theories for (1.1) and (1.5). Lastly, observe that all papers such as [3–6, 38], to cite a few, regarding the $W^{1,q}$-regularity estimates in non-smooth domains only establish globally regularity estimates. Our paper provides not only global regularity estimates but also local interior and boundary ones. Our Theorem 1.2, Theorem 1.3 can be considered as some high regularity estimates of Caccioppoli type which are also extremely important for many practical purposes for which only local information is available. Certainly, our local regularity estimates imply the global ones by flattening the boundary and using partition of unity, see the proof of Theorem 1.1. However, it is generally impossible to derive local estimates directly from the global ones in [3–6, 38]. In this perspective, the contribution of this paper is therefore significant.

We conclude this section by outlining the organization of this paper. Section 2 reviews some definitions, states and proves some preliminary results needed in the paper. Interior regularity estimates and the proof of Theorem 1.2 are given in Section 3. Section 4, is about the regularity estimates near the boundary. The proof of Theorem 1.3 is also given in this section. The proof of Theorem 1.1 is provided in Section 5. This paper is then concluded with some important remarks given in Remark 5.3.

2. Definitions and preliminary results

2.1. **Scaling invariances and weak solutions.** Let $\lambda' \geq 0$, and let us consider a function $u \in W^{1,p}_{loc}(U)$ for some open, bounded set $U \subset \mathbb{R}^n$ satisfying

$$\begin{cases} \operatorname{div}[\mathbf{A}(y, \lambda' u, \nabla u)] & = & \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] & \text{in} \quad U, \\ \langle \mathbf{A}(y, \lambda' u, \nabla u) - |\mathbf{F}|^{p-2}\mathbf{F}, \vec{\nu} \rangle & = & |g(y)|^{p-2} g(y), & \text{on} \quad \partial U \end{cases}$$

in the sense of distribution. Then for some fixed $\lambda > 0$, the rescaled function

$$v(x) = \frac{u(x)}{\lambda} \quad \text{for} \quad x \in U, \quad \lambda > 0 \tag{2.1}$$

solves the equation

$$\begin{cases} \operatorname{div}[\hat{\mathbf{A}}(x, \hat{\lambda} v, \nabla v)] & = & \operatorname{div}[|\hat{\mathbf{F}}|^{p-2}\hat{\mathbf{F}}] & \text{in} \quad U, \\ \langle \hat{\mathbf{A}}(x, \hat{\lambda} v, \nabla v) - |\hat{\mathbf{F}}|^{p-2}\hat{\mathbf{F}}, \vec{\nu} \rangle & = & |\hat{g}|^{p-2}\hat{g}, & \text{on} \quad \partial U \end{cases}$$

in the distributional sense for $\hat{\lambda} = \lambda\lambda' \geq 0$. Here, $\mathbf{A}_\lambda : U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ is defined by

$$\hat{\mathbf{A}}(x, z, \xi) = \frac{\mathbf{A}(x, z, \lambda\xi)}{\lambda^{p-1}} \quad \text{and} \quad \hat{\mathbf{F}}(x) = \frac{\mathbf{F}(x)}{\lambda}, \quad \hat{g}(x) = \frac{g(x)}{\lambda}. \tag{2.2}$$

Remark 2.1. *It is clear that if* $\mathbf{A} : U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$ *satisfies conditions* (1.2)–(1.4) *on* $U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$*, then the rescaled vector field* $\hat{\mathbf{A}} : U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$ *defined in* (2.2) *also satisfies the structural conditions* (1.2)–(1.4) *with the same constants* Λ, p, α.

Let us now give the precise definition of weak solutions that is used throughout the paper.

Definition 2.2. *Let* $\mathbb{K} \subset \mathbb{R}$ *be an interval, let* $\Lambda > 0, p > 1, \alpha \in (0, 1]$. *Also, let* $\Omega \subset \mathbb{R}^n$ *be open, and bounded with sufficiently smooth boundary* $\partial\Omega$*, and* $\mathbf{A} : U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$ *satisfies conditions* (1.2)–(1.4) *on* Ω.

(i) *For every* $\mathbf{F} \in L^p(\Omega;\mathbb{R}^n)$ *and* $\lambda \geq 0$, *a function* $u \in W^{1,p}_{loc}(\Omega)$ *is called a weak solution of*

$$\operatorname{div}\Big[\mathbf{A}(x, \lambda u, \nabla u)\Big] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], \quad \textit{in} \quad \Omega$$

if $\lambda u(x) \in \mathbb{K}$ *for a.e.* $x \in \Omega$, *and*

$$\int_U \Big\langle \mathbf{A}(x, \lambda u, \nabla u), \nabla \varphi \Big\rangle \, dx = \int_U \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla \varphi\rangle \, dx \qquad \forall \, \varphi \in C^\infty_0(\Omega).$$

(ii) *For every* $\mathbf{F} \in L^p(\Omega;\mathbb{R}^n)$, $g \in W^{1,p}(\Omega)$, *and* $\lambda \geq 0$, *a function* $u \in W^{1,p}(\Omega)$ *is a weak solution of* (1.7) *if* $\lambda u(x) \in \mathbb{K}$ *for a.e.* $x \in \Omega$, *and*

$$\int_\Omega \Big\langle \mathbf{A}(x, \lambda u, \nabla u), \nabla \varphi \Big\rangle \, dx = \int_\Omega \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla \varphi\rangle \, dx + \int_{\partial\Omega} |g(x)|^{p-2} g(x)\varphi(x) dS(x). \tag{2.3}$$

for every $\varphi \in W^{1,p}(\Omega)$, *where* $dS(x)$ *is the surface measure of* $\partial\Omega$. *Here,* $C^\infty_0(\Omega)$ *is the set of all smooth compactly supported functions in* Ω, $L^p(\Omega,\mathbb{R}^n)$ *is the Lebesgue space consists all measurable functions* $f : \Omega \to \mathbb{R}^n$ *such that* $|f|^p$ *is integrable on* Ω, *and* $W^{1,p}(\Omega)$ *is the standard Sobolev space on* Ω. *Moreover,* $\langle \cdot, \cdot \rangle$ *is the Euclidean inner product in* $\mathbb{R}^n$.

2.2. **Some simple energy estimates.** We derive some elementary estimates which will be used later.

Lemma 2.3. *Let* $\Lambda > 0, p > 1$, *and let* $U \subset \mathbb{R}^n$ *be a bounded open set, and let* $\mathbb{K}$ *be an interval in* $\mathbb{R}$. *Assume that* $\mathbf{A} : U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n$ *satisfies* (1.2) - (1.3) *on* $U \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$. *Then for any functions* $u, v \in W^{1,p}(U)$ *and any nonnegative function* $\phi \in C(\overline{U})$, *it holds that*

(i) *If* $1 < p < 2$, *then for every* $\tau > 0$,

$$\int_U |\nabla u - \nabla v|^p \phi \, dx \leq \tau \int_U |\nabla u|^p \phi \, dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_U \langle \mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi \, dx.$$

(ii) *If* $p \geq 2$, *then*

$$\int_U |\nabla u - \nabla v|^p \phi \, dx \leq C(\Lambda, p) \int_U \langle \mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi \, dx.$$

Proof. This lemma is well-known, see [42, Lemma 1] and [35, Lemma 3.1]. Observe that from (1.2), the following monotonicity property of $\mathbf{A}$ holds true

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta\rangle \geq \begin{cases} \gamma_0 |\xi - \eta|^p, & \text{if} \quad p \geq 2, \\ \gamma_0(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 & \text{if} \quad 1 < p < 2, \end{cases} \tag{2.4}$$

for all $(x, z) \in U \times \mathbb{K}$ and for all $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, where $\gamma_0 = \gamma_0(\Lambda, p) > 0$ is a constant. From this, the lemma is trivial when $p \geq 2$. On the other hand, if $1 < p < 2$, then the lemma follows directly from [35, Lemma 3.1]. For details of the proof, see [36, Lemma 2.10]. □

Lemma 2.4 (Caccioppoli's type estimates)**.** *Let* $\Lambda > 0, p > 1$ *be fixed, and let* $\mathbf{A} : D^+_r \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ *satisfy* (1.2)-(1.3) *on* $D^+_r \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ *for some* $r > 0$. *Assume that* $v \in W^{1,p}(D^+_r)$ *is a weak solution of*

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, v, \nabla v)] = 0 & \textit{in} \quad D^+_r, \\ \langle \mathbf{A}(x, v, \nabla v), \vec{e}_n\rangle = 0 & \textit{on} \quad B'_r \times \{0\}, \end{cases}$$

it holds that for every $k \in \mathbb{R}$

$$\int_{D^+_r} |\nabla v|^p \phi(x)^p dx \leq C(\Lambda, p) \int_{D^+_r} |v - k|^p |\nabla \phi(x)|^p dx, \quad \forall \, \phi \in C^1_0(D_r), \quad \phi \geq 0.$$

Proof. Using $(v-k)\phi^p$ as a test function, we obtain with some $\epsilon > 0$,

$$\begin{aligned}
&\int_{D_r^+} \langle \mathbf{A}(x,v,\nabla v) - \mathbf{A}(x,v,0), \nabla v\rangle \phi^p dx \\
&= -p\int_{D_r^+} \langle \mathbf{A}(x,v,\nabla v), \nabla\phi\rangle (v-k)\phi^{p-1} dx \leq C(\Lambda,p)\int_{D_r^+} |\nabla v|^{p-1}\phi^{p-1}|\nabla\phi||v-k| dx \\
&\leq \frac{1}{4}\int_{D_r^+} |\nabla v|^p \phi^p(x) dx + C(\Lambda,p)\int_{D_r^+} |v-k|^p |\nabla\phi|^p dx.
\end{aligned}$$

Now, by Lemma 2.3, it follows that

$$\begin{aligned}
\int_{D_r^+} |\nabla u|^p\phi^p dx &\leq \frac{1}{4}\int_{D_r^+} |\nabla u|^p \phi^p dx + C(\Lambda,p)\int_{D_r^+} \langle \mathbf{A}_0(x,\nabla v) - \mathbf{A}_0(x,0), \nabla v\phi^p\rangle dx \\
&\leq \frac{1}{2}\int_{D_r^+} |\nabla v|^p\phi^p dx + C(\Lambda,p)\int_{D_r^+} |v-k|^p|\nabla\phi|^p dx.
\end{aligned}$$

Then, by cancelling similar terms, we obtain

$$\int_{D_r^+} |\nabla v|^p \phi(x)^p dx \leq C(\Lambda,p)\int_{D_r^+} |v-k|^p |\nabla\phi(x)|^p dx.$$

The proof is therefore complete. □

2.3. **Hölder regularity of weak solutions of homogeneous p-Laplacian type equations.** We recall some results on Hölder's regularity for weak solutions of p-Laplacian type equations which will be needed in the paper. These results are consequences of the well-known, and classical De Giorgi-Nash-Möser theory. Our first lemma is about the interior Hölder's regularity estimates, whose proof, for example, can be found in [21, Theorem 7.6] or [27, Theorem 1.1, p. 251].

Lemma 2.5. *Let $\Lambda > 0, p > 1$, and let $\mathbb{A}_0 : B_r \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ be a Carathéodory map satisfying* (1.2)-(1.3) *on $B_r \times (\mathbb{R}^n \setminus \{0\})$ with some $r > 0$. If $v \in W^{1,p}(B_r)$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x,\nabla v)] = 0, \quad \textit{in} \quad B_r$$

Then, there exists $C_0 > 0$ depending only on Λ, n and p such that

$$\|v\|_{L^\infty(B_{5r/6})} \leq C_0 \left[\fint_{B_r} |v|^p dx\right]^{1/p}.$$

Moreover, there exists $\beta \in (0,1)$ depending only on Λ, n, p and on $\|v\|_{L^\infty(B_{5r/6})}$ such that

$$|v(x) - v(y)| \leq C_0 \|v\|_{L^\infty(B_{5r/6})} \left(\frac{|x-y|}{r}\right)^\beta, \quad \forall\, x,y \in \overline{B}_{2r/3}.$$

The following result can be derived from the classical result in [27, Theorem 1.1, p. 251], see also in [21, Theorem 7.8].

Lemma 2.6. *Let $\Lambda > 0, p > 1$ and $r > 0$. Let $\mathbb{A}_0 : D_r^+ \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ be a Carathéodory map and satisfy* (1.2)-(1.3) *on $D_r^+ \times (\mathbb{R}^n \setminus \{0\})$. Assume that $v \in W^{1,p}(D_r^+)$ is a weak solution of the equation*

$$\left\{\begin{array}{rcll} \operatorname{div}[\mathbb{A}_0(x,\nabla v)] &=& 0, & \textit{in} \quad D_r^+, \\ \langle \mathbf{A}_0(x,\nabla v), \vec{e}_n\rangle &=& 0, & \textit{on} \quad B_r' \times \{0\}. \end{array}\right.$$

Then, there exists $C_0 > 0$ depending only on Λ, n and p such that

$$\|v\|_{L^\infty(D^+_{5r/6})} \leq C_0 \left[\fint_{D_r^+} |v|^p dx\right]^{1/p}.$$

Moreover, there exists $\beta \in (0,1)$ depending only on Λ, n, p and on $\|v\|_{L^\infty(D^+_{5r/6})}$ such that

$$|v(x) - v(y)| \leq C_0 \|v\|_{L^\infty(D^+_{5r/6})} \left(\frac{|x-y|}{r}\right)^\beta, \quad \forall\, x, y \in \overline{D}^+_{2r/3}.$$

2.4. **Meyers-Gehring's self-improving regularity estimates.** We need the following classical results on self-improving regularity estimates for weak solutions of p-Laplacian type equations. The following two lemmas are due to N. Meyers and A. Elcrat in [34, Theorem 1, Theorem 2]. These results can be also found in [25, Theorem 1.1].

Lemma 2.7. *Let $\Lambda > 0, p > 1$. Then, there exists $p_0 = p_0(\Lambda, n, p) > p$ such that the following statement holds. Suppose that $\mathbb{A}_0 : B_r \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.3) *on $B_r \times (\mathbb{R}^n \setminus \{0\})$ with some $r > 0$. If $v \in W^{1,p}(B_r)$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0, \quad in \quad B_r$$

then, for every $p_1 \in [p, p_0]$, there exists a constant $C = C(\Lambda, p_1, p, n) > 0$ such that

$$\left(\fint_{B_{2r/3}} |\nabla v|^{p_1} dx\right)^{1/p_1} \leq C \left(\fint_{B_r} |\nabla v|^p dx\right)^{1/p}.$$

The following result is a special case of [34, Theorem 1, Theorem 2].

Lemma 2.8. *For every $\Lambda > 0, p > 1$, there exists $p_0 = p_0(\Lambda, n, p) > p$ such that the following statement holds. Suppose that $\mathbb{A}_0 : D^+_r \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ be a Carathéodory map satisfying* (1.2)-(1.3) *on $D^+_r \times (\mathbb{R}^n \setminus \{0\})$ for some $r > 0$, and suppose that $v \in W^{1,p}(D^+_r)$ is a weak solution of the equation*

$$\begin{cases} \operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0 & in \quad D^+_r, \\ \langle \mathbf{A}_0(x, \nabla v), \vec{e}_n \rangle = 0 & on \quad B'_r \times \{0\}. \end{cases}$$

Then, for every $p_1 \in [p, p_0]$, there exists a constant $C = C(\Lambda, p_1, p, n) > 0$ such that

$$\left(\fint_{D^+_{2r/3}} |\nabla v|^{p_1} dx\right)^{1/p_1} \leq C \left(\fint_{D^+_r} |\nabla v|^p dx\right)^{1/p}.$$

2.5. **Some simple approximation estimates.** We state and prove two simple approximation estimates which are commonly used many papers such as [3–5, 38] for the class of equations of the type (1.5) in which the vector field $\mathbf{A}$ is independent on u-variable. These approximation estimates will be used as intermediate steps for the proof of our main theorems.

Lemma 2.9. *Let $\Lambda > 0, p > 1$ be fixed. Then, for every $\epsilon \in (0,1)$, there exists sufficiently $\delta_0 = \delta_0(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A}_0 : B_{2R} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ such that* (1.2)–(1.4) *hold on $B_{2R} \times (\mathbb{R}^n \setminus \{0\})$, and*

$$[[\mathbf{A}_0]]_{\mathrm{BMO}(B_R,R)} = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \sup_{\substack{x \in B_R \\ 0<\rho<R}} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \frac{|\mathbf{A}_0(y,\xi) - \bar{\mathbf{A}}_{0,B_\rho(x)}(\xi)|}{|\xi|^{p-1}} dy \leq \delta_0.$$

Then, for every $x_0 \in B_R$ and $\rho \in (0, R/2)$, if $w \in W^{1,p}(B_{2\rho}(x_0))$ is a weak solution of

$$\operatorname{div}[\mathbf{A}_0(x, \nabla w)] = 0, \quad in \quad B_{2\rho}(x_0),$$

satisfying

$$\frac{1}{|B_{2\rho}(x_0)|} \int_{B_{2\rho}(x_0)} |\nabla w|^p dx \leq 1,$$

then there is $h \in W^{1,p}(B_{7\rho/4}(x_0))$ such that the following estimate holds

$$\frac{1}{|B_{7\rho/4}(x_0)|} \int_{B_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \leq \epsilon^p, \quad \|\nabla h\|_{L^\infty(B_{3\rho/2}(x_0))} \leq C(\Lambda, n, p).$$

Proof. We skip the proof because it is the same is that of Lemma 2.10 below. □

In the next lemma, for each cylinder $D_\rho(x_0)$ in $\mathbb{R}^n$, the following notation is used

$$T_\rho(x_0) = \partial D_\rho^+(x_0) \cap (\mathbb{R}^{n-1} \times \{0\}).$$

Lemma 2.10. *Let $\Lambda > 0, p > 1$ be fixed. Then, for every $\epsilon \in (0,1)$, there exists a sufficiently small number $\bar{\delta}_0 = \bar{\delta}_0(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A}_0 : D_{2R}^+ \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ such that (1.2)–(1.4) hold on $D_{2R}^+ \times (\mathbb{R}^n \setminus \{0\})$ for some $R > 0$, and*

$$[[\mathbf{A}_0]]_{\mathrm{BMO}(D_R^+, R)} = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \sup_{\substack{x \in D_R^+ \\ 0<\rho<R}} \frac{1}{|D_\rho(x)|} \int_{D_\rho(x) \cap D_{2R}^+} \frac{|\mathbf{A}_0(y,\xi) - \bar{\mathbf{A}}_{0, D_\rho(x) \cap D_{2R}^+}(\xi)|}{|\xi|^{p-1}} dy \leq \bar{\delta}_0.$$

Then, for every $x_0 = (x_0', x_{n0}) \in D_R^+$ and $\rho \in (0, R/2)$, if $w \in W^{1,p}(D_{2\rho}^+(x_0))$ is a weak solution of

$$\begin{cases} \operatorname{div}[\mathbf{A}_0(x, \nabla w)] &= 0 \quad \text{in} \quad D_{2\rho}^+(x_0), \\ \langle \mathbf{A}_0(x, \nabla w), \vec{e}_n \rangle &= 0 \quad \text{on} \quad T_{2\rho}(x_0) \quad \text{if} \quad T_{2\rho}(x_0) \neq \emptyset \end{cases}$$

and if

$$\frac{1}{|D_{2\rho}(x_0)|} \int_{D_{2\rho}^+(x_0)} |\nabla w|^p dx \leq 1,$$

then there is $h \in W^{1,p}(D_{7\rho/4}^+(x_0))$ such that the following estimate holds

$$\frac{1}{|D_{7\rho/4}(x_0)|} \int_{D_{7\rho/4}^+(x_0)} |\nabla w - \nabla h|^p dx \leq \epsilon^p, \quad \|\nabla h\|_{L^\infty(D_{3\rho/2}^+(x_0))} \leq C(\Lambda, n, p). \tag{2.5}$$

Proof. The proof is fundamental and we provide it for completeness. Let us denote

$$\mathbf{a}(\xi) = \frac{1}{|D_{7\rho/4}^+(x_0)|} \int_{D_{7\rho/4}^+(x_0)} \mathbf{A}_0(x,\xi) dx, \quad \Theta(x,\xi) = \frac{|\mathbf{A}_0(x,\xi) - \mathbf{a}(\xi)|}{|\xi|^{p-1}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let $h \in W^{1,p}(D_{7\rho/4}^+(x_0))$ be the weak solution of

$$\begin{cases} \operatorname{div}[\mathbf{a}(\nabla h)] &= 0 \quad \text{in} \quad D_{7\rho/4}^+(x_0), \\ h &= w \quad \text{on} \quad \partial D_{7\rho/4}^+(x_0) \setminus T_{7\rho/4}(x_0) \quad \text{and} \\ \langle \mathbf{a}(\nabla h), \vec{e}_n \rangle &= 0 \quad \text{on} \quad T_{7\rho/4}(x_0) \quad \text{if} \quad T_{7\rho/4}(x_0) \neq \emptyset. \end{cases} \tag{2.6}$$

Observe that the existence of h can be obtained by using standard theory in calculus of variations. Since $w - h \in W^{1,p}(D_{7\rho/4}^+(x_0))$ and $w - h = 0$ on $\partial D_{7\rho/4}^+(x_0) \setminus T_{7\rho/4}(x_0)$ in the sense of trace, we can use $w - h$ as a test function for the equations of w and h to obtain

$$\int_{D_{7\rho/4}^+(x_0)} \langle \mathbf{a}(\nabla w) - \mathbf{a}(\nabla h), \nabla w - \nabla h \rangle dx = \int_{D_{7\rho/4}^+(x_0)} \langle \mathbf{a}(\nabla w) - \mathbf{A}_0(x, \nabla w), \nabla w - \nabla h \rangle dx.$$

We now consider the case $1 < p < 2$ as the case $p \geq 2$ can be done similarly and much simpler. Let $\tau \in (0,1)$ be a number to be determined. By Lemma 2.3, it follows that

$$\begin{aligned} &\int_{D_{7\rho/4}^+(x_0)} |\nabla w - \nabla h|^p dx \\ &\leq \tau \int_{D_{7\rho/4}^+(x_0)} |\nabla w|^p dx + C(\Lambda, p) \tau^{\frac{p-2}{p}} \int_{D_{7\rho/4}^+(x_0)} \langle \mathbf{a}(\nabla w) - \mathbf{a}(\nabla h), \nabla w - \nabla h \rangle dx \\ &\leq \tau \int_{D_{2\rho}^+(x_0)} |\nabla w|^p dx + C(\Lambda, p) \tau^{\frac{p-2}{p}} \int_{D_{7\rho/4}^+(x_0)} \langle \mathbf{a}(\nabla w) - \mathbf{A}_0(x, \nabla w), \nabla w - \nabla h \rangle dx \\ &\leq \tau \int_{D_{2\rho}^+(x_0)} |\nabla w|^p dx + C(\Lambda, p) \tau^{\frac{p-2}{p}} \int_{D_{7\rho/4}^+(x_0)} \Theta(x, \nabla w) |\nabla w|^{p-1} |\nabla w - \nabla h| dx. \end{aligned}$$

Then, by applying the Hölder's inequality, and Young's inequality to the last term on the right hand side of the last estimate, we have

$$\begin{aligned}&\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \\ &\leq \tau \int_{D^+_{2\rho}(x_0)} |\nabla w|^p dx + \frac{1}{2}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p-1}} \int_{D^+_{7\rho/4}(x_0)} \Theta(x, \nabla w)^{\frac{p}{p-1}} |\nabla w|^p dx.\end{aligned}$$

By cancelling similar terms, and using the fact $\Theta(x, \nabla w) \in (0, 2\Lambda)$ and the assumption in the lemma, we infer that

$$\begin{aligned}&\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \\ &\leq C(\Lambda, n, p)\left[\frac{\tau}{|D_{2\rho}(x_0)|}\int_{D^+_{2\rho}(x_0)} |\nabla w|^p dx + \frac{\tau^{\frac{p-2}{p-1}}}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} \Theta(x, \nabla w)|\nabla w|^p dx\right] \\ &\leq C(\Lambda, n, p)\left[\tau + \frac{\tau^{\frac{p-2}{p-1}}}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} \Theta(x, \nabla w)|\nabla w|^p dx\right].\end{aligned}$$

Now, for some number $p_1 > p$ which is sufficiently close to p, and some $\gamma > 1$ such that $\frac{1}{\gamma} + \frac{1}{p_1/p} = 1$. We then apply Hölder's inequality to the last term on the right hand side of the last estimate to obtain

$$\begin{aligned}&\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \\ &\leq C(\Lambda, n, p)\left[\tau + \tau^{\frac{p-2}{p-1}}\left(\frac{1}{|D^+_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} \Theta(x, \nabla w)^\gamma dx\right)^{1/\gamma}\left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w|^{p_1} dx\right)^{\frac{p}{p_1}}\right] \\ &\leq C(\Lambda, n, p)\left[\tau + \tau^{\frac{p-2}{p-1}}[\mathbf{A}_0]^{1/\gamma}_{\mathrm{BMO}(D^+_R, R)}\left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w|^{p_1} dx\right)^{\frac{p}{p_1}}\right].\end{aligned}$$

From this, and Lemma 2.8, it follows that

$$\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \leq C_0(\Lambda, n, p)\left[\tau + \frac{\tau^{\frac{p-2}{p-1}}[[\mathbf{A}_0]]^{1/\gamma}_{\mathrm{BMO}(D^+_R, R)}}{|D_{2\rho}(x_0)|}\int_{D^+_{2\rho}(x_0)} |\nabla w|^p dx\right].$$

Then, by the assumption in the lemma, we obtain

$$\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \leq C_0(\Lambda, n, p)\left[\tau + \tau^{\frac{p-2}{p-1}}[[\mathbf{A}_0]]^{1/\gamma}_{\mathrm{BMO}(D^+_R, R)}\right].$$

From this and by taking τ such that $C_0(\Lambda, n, p)\tau = \epsilon^p/2$ and then choosing $\bar{\delta}_0$ sufficiently small so that

$$C_0(\Lambda, n, p)\tau^{\frac{p-2}{p-1}}\bar{\delta}_0^{\frac{1}{\gamma}} \leq \epsilon^p/2,$$

we have

$$\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)} |\nabla w - \nabla h|^p dx \leq \epsilon^p.$$

This proves the first estimate in (2.5). From this last estimate, the triangle inequality, the assumption in the lemma, and the fact that $\epsilon \in (0,1)$, we see that

$$\begin{aligned}
&\left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)}|\nabla h|^p dx\right)^{1/p}\\
&\leq \left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)}|\nabla h-\nabla w|^p dx\right)^{1/p} + \left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7r/4}(x_0)}|\nabla w|^p dx\right)^{1/p}\\
&\leq \epsilon + C(n,p)\left(\frac{1}{|D_{2\rho}(x_0)|}\int_{D^+_{2\rho}(x_0)}|\nabla w|^p dx\right)^{1/p} \leq C(n,p).
\end{aligned}$$

From this and the classical theory on Lipschitz regularity estimate for weak solution h of the constant coefficient equation (2.6) (see [2, 9–11, 21, 27, 42–44]), it holds that

$$\|\nabla h\|_{L^\infty(D^+_{3\rho/2}(x_0))} \leq C(\Lambda,n,p)\left(\frac{1}{|D_{7\rho/4}(x_0)|}\int_{D^+_{7\rho/4}(x_0)}|\nabla h|^p dx\right)^{1/p} \leq C(\Lambda,n,p),$$

which is the second estimate in (2.5). The proof is therefore complete. □

2.6. **Hardy-Littlewood maximal function and crawling ink-spots lemma.** This section recalls several analysis results, definitions needed in the paper. For completeness, we firstly recall the Hardy-Littlewood maximal operator and its boundedness in L^p-space. For a given locally integrable function f, we define the weighted Hardy-Littlewood maximal function as

$$\mathcal{M}f(x) = \sup_{\rho>0}\fint_{D_\rho(x)}|f(y)|dy, \tag{2.7}$$

For functions f that are defined on a bounded domain, we define

$$\mathcal{M}_\Omega f(x) = \mathcal{M}(f\chi_\Omega)(x).$$

The following classical result is well-known, see for examples [19, 41].

Lemma 2.11. *The followings hold.*

(i) *Strong (q,q): Let $1<q<\infty$, then there exists a constant $C = C(q,n)$ such that*

$$\|\mathcal{M}\|_{L^q(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C.$$

(ii) *Weak $(1,1)$: There exists a constant $C = C(n)$ such that for any $\lambda>0$, we have*

$$\left|\left\{x\in\mathbb{R}^n : \mathcal{M}(f)>\lambda\right\}\right| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n}|f|dx.$$

Secondly, the following standard result in measure theory is also needed in the paper.

Lemma 2.12. *Assume that $g\geq 0$ is a measurable function in a bounded subset $U\subset\mathbb{R}^n$. Let $\theta>0$ and $N>1$ be given constants. Then for any $1\leq p<\infty$*

$$g\in L^p(U) \Leftrightarrow S := \sum_{j\geq 1} N^{pj}|\{x\in U : g(x)>\theta N^j\}| < \infty.$$

Moreover, there exists a constant $C>0$ depending only on θ, N and p. such that

$$C^{-1}S \leq \|g\|^p_{L^p(U)} \leq C(|U|+S).$$

Finally, we recall the following important lemma that is needed in this paper. This lemma is usually referred to "crawling ink-spots" lemma, which is originally due to N. V. Krylov and M. V. Safonov, see [26, 40].

Lemma 2.13 (crawling ink-spots)**.** *Let $R>0$, and assume that C, D are measurable sets satisfying $C\subset D\subset B_R$. Assume also that there are $\kappa_0\in(0,R/2)$, and $0<\epsilon<1$ such that*

(i) $|C| < \epsilon|B_{\kappa_0}|$, *and*
(ii) *for all* $x \in B_R$ *and* $\rho \in (0, \kappa_0)$, *if* $|C \cap B_\rho(x)| \geq \epsilon|B_\rho(x)|$, *then* $B_\rho(x) \cap B_R \subset D$.

Then, there exists $\epsilon_1 = C_0(n)\epsilon$ *for some constant* $C_0(n) > 0$ *such that*

$$|C| \leq \epsilon_1|D|.$$

The same conclusions also hold if we replace balls by cylinders or by upper-half cylinders.

3. Interior regularity gradient estimates and proof of Theorem 1.2

In this section, let $\mathbf{A} : B_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ satisfy (1.2)–(1.4) on $B_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ for some $R > 0$. We study a weak solution $u \in W^{1,p}(B_{2R})$ of the parameter equation

$$\operatorname{div}[\mathbf{A}(x, \lambda u, \nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], \quad \text{in} \quad B_{2R}, \tag{3.1}$$

with the parameter $\lambda \geq 0$. Our goal in this section is to prove Theorem 1.2. Our approach is based on the perturbation technique introduced in [7] together with the "scaling parameter" technique introduced in [22, 35]. The approach is also influenced by the recent developments [3, 4, 6, 37, 45].

3.1. **Interior approximation estimates.**

In our first step, we freeze u in $\mathbf{A}$, and then approximate the solution u of (3.1) by a solution of the corresponding homogeneous equations with frozen u coefficient as in [1, 6].

Lemma 3.1. *Let* $\Lambda, M > 0$, $p > 1$ *and* $\alpha \in (0, 1]$ *be fixed. Then, for every* $\epsilon \in (0, 1)$, *there exists a sufficiently small number* $\delta_1 = \delta_1(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ *such that the following holds. Assume that* $\mathbf{A} : B_{2R}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\}) \to \mathbb{R}^n$ *satisfies*(1.2)–(1.4) *on* $B_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ *for some* $R > 0$ *and some open set* $\mathbb{K} \subset \mathbb{R}$, *and assume that* $\mathbf{F} \in L^p(B_{2R}, \mathbb{R}^n)$ *satisfies*

$$\fint_{B_r(x_0)} |\mathbf{F}|^p dx \leq \delta_1^p,$$

with some $x_0 \in B_R$ *and some* $r \in (0, R)$. *Suppose also that* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3.1) *satisfying*

$$\fint_{B_r(x_0)} |\nabla u|^p dx \leq 1, \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M,$$

for some $\lambda \geq 0$. *Then,*

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p dx \leq \epsilon^p, \tag{3.2}$$

where $v \in W^{1,p}(B_r)$ *is the weak solution of*

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, \lambda u, \nabla v)] = 0, & \text{in} \quad B_r(x_0), \\ v = u, & \text{on} \quad \partial B_r(x_0). \end{cases} \tag{3.3}$$

Moreover, it also holds that

$$\lambda\left(\fint_{B_r(x_0)} |v - \bar{u}_{B_r(x_0)}|^p dx\right)^{1/p} \leq C(n, p)[M + r\lambda\epsilon], \quad \text{and} \quad \left(\fint_{B_r(x_0)} |\nabla v|^p dx\right)^{1/p} \leq 2. \tag{3.4}$$

Proof. We first note that the existence of weak solution v of (3.3) follows from the standard theory in calculus of variations as $\mathbf{A}_0(x, \xi) := \mathbf{A}(x, \lambda u(x), \xi)$ satisfies all assumptions in (1.2)–(1.4), see [21] for example. Therefore, we only need to prove the estimates (3.2), and (3.4). Take $v - u \in W_0^{1,p}(B_r(x_0))$ as a test function for the equation (3.3), we obtain

$$\int_{B_r(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla v), \nabla u - \nabla v\rangle dx = 0.$$

Similarly, we can use $v - u$ as a test function for the equation for (3.1) to see that

$$\int_{B_r(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla u), \nabla u - \nabla v\rangle dx = \int_{B_r(x_0)} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v\rangle dx.$$

Therefore,

$$\int_{B_r(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla u) - \mathbf{A}(x, \lambda u, \nabla v), \nabla v - \nabla u\rangle dx = \int_{B_r(x_0)} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v\rangle dx. \tag{3.5}$$

Then, it follows from Lemma 2.3, and (3.5), that for each $\tau \in (0, 1)$,

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla u - \nabla v|^p dx \\
&\leq \tau \int_{B_r(x_0)} |\nabla u|^p dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla u) - \mathbf{A}(x, \lambda u, \nabla v), \nabla v - \nabla u\rangle dx \\
&\leq \tau \int_{B_r(x_0)} |\nabla u|^p dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} |\langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v\rangle| dx \\
&\leq \tau \int_{B_r(x_0)} |\nabla u|^p dx + \frac{1}{2} \int_{B_r(x_0)} |\nabla u - \nabla v|^p dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} |\mathbf{F}(x)|^p dx,
\end{aligned}$$

where we have used Hölder's inequality and Young's inequality in the last estimate. By cancelling similar terms in the last estimate, we obtain

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p dx \leq 2\tau \fint_{B_r(x_0)} |\nabla u|^p dx + C(\Lambda, \tau, p) \fint_{B_r(x_0)} |\mathbf{F}(x)|^p dx. \tag{3.6}$$

Now, choose $\tau = \epsilon^p/4$, and then choose $\delta_1 = \delta_1(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ sufficiently small such that $C(\Lambda, \tau, p)\delta^p < \epsilon^p/2$, the estimate (3.2) follows. It remains to prove (3.4). By the Poincaré's inequality, we see that

$$\begin{aligned}
\left(\fint_{B_r(x_0)} |v - \bar{u}_{B_r(x_0)}|^p dx\right)^{1/p} &\leq \left[\left(\fint_{B_r(x_0)} |v - u|^p dx\right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p dx\right)^{1/p}\right] \\
&\leq \left[C(n, p) r \left(\fint_{B_r(x_0)} |\nabla v - \nabla u|^p dx\right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p dx\right)^{1/p}\right].
\end{aligned}$$

Therefore,

$$\lambda \left(\fint_{B_r(x_0)} |v - \bar{u}_{B_r(x_0)}|^p dx\right)^{1/p} \leq C(n, p)[M + r\lambda\epsilon],$$

which is the first estimate in (3.4). Meanwhile, the second estimate in (3.4) follow directly from (3.2), the assumption in the lemma, and the triangle inequality. The proof of the lemma is therefore complete. □

Our next step is the most delicate one. We approximate the solution u by the solution w of

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, \lambda \bar{u}_{B_{\bar{\kappa}r}(x_0)}, \nabla w)] = 0, & \text{in } B_{\bar{\kappa}r}(x_0), \\ w = v, & \text{on } \partial B_{\bar{\kappa}r}(x_0), \end{cases} \tag{3.7}$$

where v is the weak solution of (3.3) and $\bar{\kappa} \in (0, 1/3)$ is a sufficiently small number that will be determined. The following lemma is the most important one in our approach.

Lemma 3.2. *Let $\Lambda, M > 0, p > 1$ and $\alpha \in (0, 1]$ be fixed, and let $\epsilon \in (0, 1)$. There exist sufficiently small numbers $\bar{\kappa} = \bar{\kappa}(\Lambda, M, p, n, \alpha, \epsilon) \in (0, 1/3)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : B_{2R} \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ satisfies* (1.2)–(1.4)*, and assume that $\mathbf{F} \in L^p(B_{2R}, \mathbb{R}^n)$ and*

$$\fint_{B_r(x_0)} |\mathbf{F}|^p dx \leq \delta_2^p,$$

some $x_0 \in B_1$*, some* $r \in (0, R)$*. Then, for every* $\lambda \geq 0$*, if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3.1) *satisfying*

$$\fint_{B_r(x_0)} |\nabla u|^p dx \leq 1, \quad \fint_{B_{2\bar{\kappa}r}(x_0)} |\nabla u|^p dx \leq 1, \quad \text{and} \quad \lambda[[u]]_{\mathrm{BMO}(B_R,R)} \leq M,$$

it holds that

$$\fint_{B_{\bar{\kappa}r}(x_0)} |\nabla v - \nabla w|^p dx \leq \epsilon^p, \quad \text{and} \quad \left(\fint_{B_{\bar{\kappa}r}(x_0)} |\nabla w|^p dx\right)^{1/p} \leq C_0(n,p).$$

where w *is the weak solution of* (3.7).

Proof. We skip the proof of this lemma as it is similar and much simpler than that of Lemma 4.2 in the next section. □

Summarizing our efforts, we can prove the following proposition which is the main result of the subsection.

Proposition 3.3. *Let* $\Lambda, M > 0, p > 1$ *and* $\alpha \in (0,1]$ *be fixed. Then, for every* $\epsilon \in (0,1)$*, there exist sufficiently small numbers* $\bar{\kappa} = \bar{\kappa}(\Lambda, M, p, n, \alpha, \epsilon) > 0$ *and* $\delta' = \delta'(\epsilon, \Lambda, M, \alpha, n, p) \in (0, \epsilon)$ *such that the following holds. Assume that* $\mathbf{A} : B_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ *such that* (1.2)–(1.4) *hold and* (1.10) *holds with* $\hat{\delta}_0$ *replaced by* δ'*, and assume that*

$$\frac{1}{|B_{4r}(x_0)|} \int_{B_{4r}(x_0)} |\mathbf{F}|^p dx \leq (\delta')^p,$$

for some $x_0 \in \overline{B}_R$ *and some* $r \in (0, R/4)$*. Then, for every* $\lambda \geq 0$*, if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3.1) *satisfying*

$$\frac{1}{|B_{4r}(x_0)|} \int_{B_{4r}(x_0)} |\nabla u|^p dx \leq 1, \quad \frac{1}{|B_{8\bar{\kappa}r}(x_0)|} \int_{B_{4\bar{\kappa}r}(x_0)} |\nabla u|^p dx \leq 1, \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M,$$

then there is $h \in W^{1,p}(B_{7\bar{\kappa}r/2}(x_0))$ *such that the following estimate holds*

$$\frac{1}{|B_{7\bar{\kappa}r/2}(x_0)|} \int_{B_{7\bar{\kappa}r/2}(x_0)} |\nabla u - \nabla h|^p dx \leq \epsilon^p, \quad \|\nabla h\|_{L^\infty(B_{3\bar{\kappa}r}(x_0))} \leq C(\Lambda, n, p). \tag{3.8}$$

Proof. The proof is the same as that of Proposition 4.3 in the next section, using Lemma 2.9 and Lemma 3.2. We skip it. □

Proof of Theorem 1.2. Once Proposition 3.3 is established, the proof of Theorem 1.2 becomes routine, using Lemma 2.13, and some iteration technique. Indeed, the proof is similar to that of Theorem 1.3 which will be given in Subsection 4.3. We therefore skip the proof. □

4. Boundary regularity gradient estimates and proof of Theorem 1.3

This section proves Theorem 1.3. We recall that for some $x_0 = (x_0', x_n^0) \in \mathbb{R}^n$, and some fixed $R > 0$, we denote the upper-half cylinder in $\mathbb{R}^n$ as

$$D_{2R}^+(x_0) = B_{2R}'(x_0') \times (\max\{x_{n0} - 2R, 0\}, x_{n0} + 2R),$$

where $B_{2R}'(x_0') = \{x' \in \mathbb{R}^{n-1} : |x' - x_0'| < 2R\}$ is the ball in $\mathbb{R}^{n-1}$ centered at x_0' with radius $2R$. We also write

$$T_{2R}(x_0) = \partial D_{2R}^+(x_0) \cap \left(\mathbb{R}^{n-1} \times \{0\}\right).$$

Note that if $T_{2R}(x_0) \neq \emptyset$, then $T_{2R}(x_0) = B_{2R}'(x') \times \{0\}$. When $x_0 = 0$, we also write $D_{2R}^+ = D_{2R}^+(0), B_{2R}' = B_{2R}'(0')$ and $T_{2R} = T_{2R}(0)$ for simplicity.

For every $\lambda \geq 0$, we investigate a weak solution $u \in W^{1,p}(D_{2R}^+)$ of the following equation in upper-half cylinder

$$\begin{cases} \mathrm{div}\,[\mathbf{A}(x, \lambda u, \nabla u)] = \mathrm{div}\,[|\mathbf{F}|^{p-2}\mathbf{F}], & \text{in } D_{2R}^+, \\ \langle \mathbf{A}(x, \lambda u, \nabla u) - |\mathbf{F}|^{p-2}\mathbf{F}, \vec{e}_n \rangle = |g(x')|^{p-2} g(x'), & \text{on } T_{2R}. \end{cases} \tag{4.1}$$

By weak solution of (4.1), we mean that $u \in W^{1,p}(D_{2R}^+)$, $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in D_{2R}^+$ and

$$\int_{D_{2R}^+} \langle \mathbf{A}(x, \lambda u, \nabla u), \nabla \varphi \rangle dx = \int_{B'_{2R}} |g(x')|^{p-2} g(x') \varphi(x', 0) dx' + \int_{D_{2R}^+} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla \varphi \rangle dx, \tag{4.2}$$

for all $\varphi \in C_0^\infty(D_{2R})$. Our approach is based on the perturbation technique using freezing coefficient equations. To this end, we employ the "scaling parameter" technique introduced in [22, 35, 36]. The approach is also influenced by the recent developments [3, 4, 6, 36, 37, 45].

4.1. **Boundary approximation estimates.** We begin with the following lemma which perturbs the force terms and provides comparison estimates in energy spaces for our solution u of (4.1) with the solution v of its corresponding homogeneous equation, i. .e equation (4.5) below.

Lemma 4.1. *Let $\Lambda, M > 0, p > 1$ and $\alpha \in (0, 1]$ be fixed. Then, for every $\epsilon \in (0, 1)$, there exists $\bar{\delta}_1 = \bar{\delta}_1(\epsilon, \Lambda, n, p) > 0$ and sufficiently small such that the following statement holds. Assume that $\mathbf{A} : D_{2R}^+ \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ satisfies (1.2)–(1.4) on $D_{2R}^+ \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$, and assume that*

$$\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |\mathbf{F}|^p dx + \frac{1}{|B'_r(x'_0)|} \int_{B'_r(x'_0)} |g(x')|^p dx' \leq \delta_1,$$

for some $x_0 = (x'_0, x_{n0}) \in D_R^+$ and $r \in (0, R)$. Then, for every $\lambda \geq 0$, if $u \in W^{1,p}(D_{2R}^+)$ is a weak solution of (4.1) satisfying

$$\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |\nabla u|^p dx \leq 1, \quad \text{and} \quad \lambda \left(\frac{1}{|D_r^+(x_0)|} \int_{D_r^+(x_0)} |u - \bar{u}_{D_r^+(x_0)}|^p dx \right)^{1/p} \leq M, \tag{4.3}$$

then

$$\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |\nabla u - \nabla v|^p dx \leq \epsilon^p, \tag{4.4}$$

where $v \in W^{1,p}(D_r^+(x_0))$ is the weak solution of

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, \lambda u, \nabla v)] = 0 & \text{in } D_r^+(x_0), \\ v = u - \bar{u}_{D_r^+(x_0)} & \text{on } \partial D_r^+(x_0) \setminus T_r(x_0), \quad \text{and} \\ \langle \mathbf{A}(x, \lambda u, \nabla v), \vec{e}_n \rangle = 0 & \text{on } T_r(x_0) \quad \text{if} \quad T_r(x_0) \neq \emptyset. \end{cases} \tag{4.5}$$

Moreover, it holds that

$$\lambda \left(\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |v|^p dx \right)^{1/p} \leq C(n, p)[\epsilon r \lambda + M], \quad \text{and} \quad \left(\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |\nabla v|^p dx \right)^{1/p} \leq 2. \tag{4.6}$$

Proof. If $T_r(x_0) = \emptyset$, the lemma follows directly from Lemma 3.1 with cylinders replacing balls. Therefore, we only need to consider the case that $T_r(x_0) \neq \emptyset$. By taking $\mathbf{A}_0(x, \xi) = \mathbf{A}(x, \lambda u(x), \xi)$, we see that $\mathbf{A}_0$ is independent on $z \in \mathbb{K}$ and satisfies all conditions in (1.2)–(1.3) on $D_{2R}^+ \times (\mathbb{R}^n \setminus \{0\})$. Therefore, the existence of the weak solution v of (4.6) follows from the standard theory in calculus of variations. Moreover, since $v - [u - \bar{u}_{D_r^+(x_0)}] \in W^{1,p}(D_r^+(x_0))$ and $v - [u - \bar{u}_{D_r^+(x_0)}] = 0$ on $\partial D_r^+(x_0) \setminus T_r(x_0)$ in the sense of trace, we can use $v - [u - \bar{u}_{D_r^+(x_0)}]$ as a test function for the equation (4.5) and then obtain

$$\int_{D_r^+(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla v), \nabla u - \nabla v \rangle dx = 0. \tag{4.7}$$

Similarly, we can also use $v - [u - \bar{u}_{D_r^+(x_0)}]$ as a test function for the equation (4.1) and infer that

$$\begin{aligned} &\int_{D_r^+(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla u), \nabla u - \nabla v \rangle dx \\ &= \int_{B'_r(x'_0)} |g(x')|^{p-2} g(x')[v(x', 0) - u(x', 0) + \bar{u}_{D_r^+(x_0)}] dx' + \int_{D_r^+(x_0)} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v \rangle dx, \end{aligned}$$

The last estimate and (4.7) together yield

$$\begin{aligned}
&\int_{D_r^+(x_0)} \langle \mathbf{A}(x,\lambda u,\nabla u) - \mathbf{A}(x,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\
&= \int_{B_r'(x_0')} |g(x')|^{p-2} g(x')[v(x',0) - u(x',0) + \bar{u}_{D_r^+(x_0)}]dx' + \int_{D_r^+(x_0)} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v\rangle dx.
\end{aligned}$$

From this, we can proceed as in the proof of Lemma 3.1 with some modification due to the availability of the boundary term g. We only need to consider the case $1 < p < 2$ as the case $p \geq 2$ can be done similarly but much simpler. For some $\tau \in (0,1)$ to be determined, from Lemma 2.3, and by using the conditions (1.2)–(1.4), we infer that

$$\begin{aligned}
&\int_{D_r^+(x_0)} |\nabla u - \nabla v|^p dx \\
&\leq \tau \int_{D_r^+(x_0)} |\nabla u|^p dx + C(\Lambda,\tau) \int_{D_r^+(x_0)} \langle \mathbf{A}(x,\lambda u,\nabla u) - \mathbf{A}(x,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\
&= \tau \int_{D_r^+(x_0)} |\nabla u|^p dx + C(\Lambda,\tau) \Bigg[\int_{B_r'(x_0')} |g(x')|^{p-2} g(x')[v(x',0) - u(x',0) + \bar{u}_{D_r^+(x_0)}]dx' \\
&\qquad + \int_{D_r^+(x_0)} \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla u - \nabla v\rangle dx\Bigg]. \\
&\leq \tau \int_{D_r^+(x_0)} |\nabla u|^p dx + C(\Lambda,\tau) \Bigg[\int_{B_r'(x_0')} |g(x')|^{p-1} |v(x',0) - u(x',0) + \bar{u}_{D_r^+(x_0)}| dx' \\
&\qquad + \int_{D_r^+(x_0)} |\mathbf{F}|^{p-1} |\nabla u - \nabla v| dx\Bigg] \\
&\leq \frac{1}{2} \int_{D_r^+(x_0)} |\nabla u - \nabla v|^p dx + \tau \int_{D_r^+(x_0)} |\nabla u|^p dx \\
&\qquad + C(\Lambda,\tau) \Bigg[\int_{D_r^+(x_0)} |\mathbf{F}|^p dx + \int_{B_r'(x_0')} |g(x')|^{p-1} |v(x',0) - u(x',0) + \bar{u}_{D_r^+(x_0)}| dx'\Bigg],
\end{aligned}$$

where we have used Hölder's inequality and Young's inequality in our last step. This estimate and the first assumption in (4.3) imply that

$$\begin{aligned}
&\frac{1}{|D_r(x_0)|} \int_{D_r^+(x_0)} |\nabla u - \nabla v|^p dx \\
&\leq 2\tau + \frac{C(\Lambda,\tau)}{|D_r(x_0)|} \Bigg[\int_{D_r^+(x_0)} |\mathbf{F}|^p dx + \int_{B_r'(x_0')} |g(x')|^{p-1} |[v(x',0) - u(x',0) + \bar{u}_{D_r^+(x_0)}| dx'\Bigg].
\end{aligned}$$

We now control the last term in the right hand side of the above estimate. With some sufficiently small $\tau' > 0$ such that $\tau' C(n,p) = 1/2$ for some universal constant $C(n,p) > 0$ to be determined, we can use the Hölder's inequality, Young's inequality, Poincaré-Sobolev's inequality, and the Sobolev trace inequality to see that

$$\begin{aligned}
&\frac{C(\Lambda,\tau)}{|D_r(x_0)|}\int_{B'_r(x'_0)}|g(x')|^{p-1}|v(x',0)-u(x',0)+\bar{u}_{B^+_r(x_0)}|dx'\\
&=\frac{C(\Lambda,n,\tau)}{r}\fint_{B'_r(x'_0)}|g(x')|^{p-1}|v(x',0)-u(x',0)+\bar{u}_{B^+_r(x_0)}|dx'\\
&\leq C(\Lambda,p,n,\tau,\tau')\fint_{B'_r(x'_0)}|g(x')|^p dx'+\frac{\tau'}{r^p}\fint_{B'_r(x'_0)}|v(x',0)-u(x',0)+\bar{u}_{B^+_r(x_0)}|^p dx'\\
&\leq C(\Lambda,p,n,\tau,\tau')\fint_{B'_r(x'_0)}|g(x')|^p dx'+\frac{C(n,p)\tau'}{|D_r(x_0)|}\int_{D^+_r(x_0)}|\nabla u(x)-\nabla v(x)|^p dx\\
&\leq C(\Lambda,p,n,\tau)\fint_{B'_r(x'_0)}|g(x')|^p dx'+\frac{1}{2|D_r(x_0)|}\int_{D^+_r(x_0)}|\nabla u(x)-\nabla v(x)|^p dx.
\end{aligned}$$

Therefore,

$$\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|\nabla u-\nabla v|^p dx\leq 4\tau+C(\Lambda,n,p,\tau)\left[\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|\mathbf{F}|^p dx+\fint_{B'_r(x'_0)}|g(x')|^p dx'\right].$$

Hence, by choosing $\tau=\epsilon^p/8$, and $\bar{\delta}_1>0$ sufficiently small such that $C(\Lambda,n,p,\tau)\bar{\delta}_1<\epsilon^p/2$, we obtain

$$\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|\nabla u-\nabla v|^p dx\leq\epsilon^p,$$

and this proves (4.4). We finally need to prove (4.6). Observe that

$$\begin{aligned}
&\left(\frac{1}{|D_r(x_0)|}\int_{D_r(x_0)}|v|^p dx\right)^{1/p}\\
&\leq\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|v-[u-\bar{u}_{D^+_r(x_0)}]|^p dx\right)^{1/p}+\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|u-\bar{u}_{D^+_r(x_0)}|^p dx\right)^{1/p}.
\end{aligned}$$

Since $v-[u-\bar{u}_{D^+_r(x_0)}]=0$ on $\partial D^+_r(x_0)\setminus T_r(x_0)$ in the sense of trace, we can use the Poincaré's inequality for the first term in the right hand side of the above inequality to obtain

$$\begin{aligned}
&\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|v|^p dx\right)^{1/p}\\
&\leq C(n,p)\left[r\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|\nabla u-\nabla v|^p dx\right)^{1/p}+\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|u-\bar{u}_{D^+_r(x_0)}|^p dx\right)^{1/p}\right]\\
&\leq C(n,p)\left[r\epsilon+\left(\frac{1}{|D^+_r(x_0)|}\int_{D^+_r(x_0)}|u-\bar{u}_{D^+_r(x_0)}|^p dx\right)^{1/p}\right],
\end{aligned}$$

where in the last step, we used the fact that $\frac{1}{2}|D_r(x_0)|\leq|D^+_r(x_0)|\leq|D_r(x_0)|$. From this and the second assumption in (4.3), it follows that

$$\lambda\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)}|v|^p dx\right)^{1/p}\leq C(n,p)[r\epsilon\lambda+M].$$

This proves the first estimate in (4.6). To prove the second estimate, we use the triangle inequality, (4.4) and the assumption in the lemma to obtain

$$\begin{aligned}&\left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla v|^p dx\right)^{1/p}\\ &\leq \left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla v-\nabla u|^p dx\right)^{1/p}+\left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla u|^p dx\right)^{1/p}\leq \epsilon+1.\end{aligned}$$

From this and as $\epsilon \in (0,1)$, the second estimate in (4.6) follows. The proof is therefore complete. □

Our next step is the most delicate one in the approach. In this step, we will compare the solution u of (4.1) by a solution w of the following equation

$$\left\{\begin{array}{rcll} \text{div}\,[\mathbf{A}(x,\lambda\bar{u}_{D_{\kappa r}^+(x_0)},\nabla w)] &=& 0, & \text{in}\quad D_{\kappa r}^+(x_0),\\ w &=& v, & \text{on}\quad \partial D_{\kappa r}^+(x_0)\setminus T_{\kappa r}(x_0),\\ \langle \mathbf{A}(x,\lambda\bar{u}_{B_{\kappa r}^+(x_0)},\nabla w),\vec{e}_n\rangle &=& 0, & \text{on}\quad T_{\kappa r}(x_0)\quad \text{if}\quad T_{\kappa r}(x_0)\neq\emptyset. \end{array}\right. \tag{4.8}$$

where v is defined in Lemma 4.1 and $\kappa \in (0,1/3)$ is a sufficiently small constant. The following lemma is the most important one in the approach.

Lemma 4.2. *Let $\Lambda, M>0$, $p>1$ and $\alpha\in(0,1]$ be fixed. Then, for every $\epsilon\in(0,1)$, there exist sufficiently small numbers $\kappa=\kappa(\Lambda,M,\alpha,n,p,\epsilon)\in(0,1/3)$ and $\bar{\delta}_2=\bar{\delta}_2(\epsilon,\Lambda,M,\alpha,n,p)\in(0,\epsilon)$ such that the following holds. Assume that $\mathbf{A}: D_{2R}^+\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{R}^n$ such that (1.2)–(1.4) hold on $D_{2R}^+\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$, and assume that*

$$\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\mathbf{F}(x)|^p dx+\frac{1}{|B_r'(x_0')|}\int_{B_r'(x_0')}|g(x')|^p dx'\leq\delta_2,$$

for some $x_0=(x',x_{n0})\in D_R^+$ and $r\in(0,R)$. Then, for every $\lambda\geq 0$, if $u\in W^{1,p}(D_{2R}^+)$ is a weak solution of (4.1) satisfying

$$\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla u|^p dx\leq 1,\quad \frac{1}{|D_{2\kappa r}(x_0)|}\int_{D_{2\kappa r}^+(x_0)}|\nabla u|^p dx\leq 1,\quad \textit{and}\quad [[\lambda u]]_{\text{BMO}(D_R^+,R)}\leq M,$$

then the following estimate holds

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla u-\nabla w|^p dx\leq\epsilon^p,\quad \textit{and}\quad \left(\frac{1}{|D_{\kappa r}(x_0)}\int_{D_{\kappa r}(x_0)}|\nabla w|^p\right)^{1/p}\leq C_0(n,p). \tag{4.9}$$

where $w\in W^{1,p}(D_{\kappa r}^+(x_0))$ is the weak solution of (4.8).

Proof. For a given sufficiently small $\epsilon>0$, let $\epsilon'\in(0,\epsilon/2)$ and $\kappa\in(0,1/3)$ be sufficiently small numbers depending on $\epsilon,\Lambda,M,n,\alpha,p$ which will be determined. Then, let $\bar{\delta}_2=\bar{\delta}_1(\epsilon'\kappa^{\frac{n}{p}},\Lambda,n,p)$, where $\bar{\delta}_1$ is defined in Lemma 4.1. By applying Lemma 4.1, there is $v\in W^{1,p}(D_r^+(x_0))$ such that

$$\begin{aligned}&\left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla u-\nabla v|^p dx\right)^{1/p}\leq\epsilon'\kappa^{\frac{n}{p}},\quad\text{and}\\ &\lambda\left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|v|^p dx\right)^{1/p}\leq C(n,p)[r\epsilon'\kappa^{\frac{n}{p}}\lambda+M].\end{aligned} \tag{4.10}$$

Moreover,

$$\left(\frac{1}{|D_r(x_0)|}\int_{D_r^+(x_0)}|\nabla v|^p dx\right)^{1/p}\leq 2.$$

In the above estimates, we would like to note that if $x_{n0} > r$, then $D_r^+(x_0) = D_r(x_0)$. From the first estimate in (4.10), the triangle inequality and the assumption of the lemma, it follows that

$$
\begin{aligned}
&\left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla v|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla v - \nabla u|^p dx\right)^{1/p} + \left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla u|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{(2\kappa)^n|D_r(x_0)|}\int_{D^+_{r}(x_0)} |\nabla v - \nabla u|^p dx\right)^{1/p} + 1 \leq \frac{\epsilon'}{2^{n/p}} + 1 \leq 2.
\end{aligned}
$$

Consequently, we have obtained the following important estimates

$$
\left(\frac{1}{|D_r(x_0)|}\int_{D^+_{r}(x_0)} |\nabla v|^p dx\right)^{1/p} \leq 2, \quad \text{and} \quad \left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla v|^p dx\right)^{1/p} \leq 2. \tag{4.11}
$$

We may assume now that $\lambda > 0$ as the case $\lambda = 0$ is much simpler. From the standard Caccioppli's type estimates (see Lemma 2.4) and the second estimate in (4.10), we see that

$$
\begin{aligned}
\left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla v|^p dx\right)^{1/p} &\leq \frac{C(\Lambda, n, p)}{(1-2\kappa)\kappa^{\frac{n}{p}} r}\left(\frac{1}{|D_r(x_0)|}\int_{D^+_{r}(x_0)} |v|^p dx\right)^{1/p} \\
&\leq C(\Lambda, n, p)\Big[\epsilon' + M(\lambda r \kappa^{\frac{n}{p}})^{-1}\Big],
\end{aligned} \tag{4.12}
$$

where in the last estimate, we used the fact that $\kappa \in (0, 1/3)$ to control the factor $1 - 2\kappa$. Now, let w be the weak solution of (4.8), whose existence follows from standard theory in calculus of variations. It remains to prove the estimate (4.9). We only need to consider the case $1 < p < 2$, since the case $p \geq 2$ is similar, and simpler. Since $w - v \in W^{1,p}(D^+_{\kappa r}(x_0))$ and $w - v = 0$ on $\partial D^+_{\kappa r}(x_0)) \setminus T_{\kappa r}(x_0)$ in the sense of trace, we can take $w - v$ as a test function for the equation (4.8) and the equations (4.5) to obtain

$$
\int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla v), \nabla w - \nabla v\rangle dx = \int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla w), \nabla w - \nabla v\rangle dx = 0. \tag{4.13}
$$

From this and Lemma 2.3, we infer that

$$
\begin{aligned}
&\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\leq \frac{1}{4}\int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(n,p)\int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla v) - \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla w), \nabla v - \nabla w\rangle dx \\
&= \frac{1}{4}\int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(n,p)\int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla v), \nabla v - \nabla w\rangle dx \\
&\leq \frac{1}{4}\int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, n,p)\int_{D^+_{\kappa r}(x_0)} |\nabla v|^{p-1}|\nabla v - \nabla w| dx \\
&\leq C(\Lambda, n, p)\int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + \frac{1}{2}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx.
\end{aligned}
$$

Then, by cancelling similar terms, we obtain

$$
\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \leq C(\Lambda, n, p)\int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx.
$$

This last estimate together with (4.12) imply that

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p} \leq C(\Lambda,n,p)\left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D_{2\kappa r}^+(x_0)}|\nabla v|^p dx\right)^{1/p}$$
$$\leq C_1(\Lambda,n,p)\Big[\epsilon' + M(\kappa^{\frac{n}{p}} r\lambda)^{-1}\Big].$$

Therefore, if $\frac{MC_1(\Lambda,n,p)}{\kappa^{\frac{n}{p}} r\lambda} \leq \epsilon/4$, we choose $\epsilon' \in (0,\epsilon/2)$ and sufficiently small such that

$$C_1(\Lambda,n,p)(\epsilon')^p < \epsilon/2,$$

and then obtain

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p} \leq \epsilon/2,$$

From this, the first estimate in (4.10), and the triangle inequality, it follows that

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla u-\nabla w|^p dx\right)^{1/p}$$
$$\leq \left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla u-\nabla v|^p dx\right)^{1/p} + \left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D_{\kappa r}^+(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p}$$
$$\leq \left(\frac{1}{\kappa^n|D_{r}(x_0)|}\int_{D_{r}^+(x_0)}|\nabla u-\nabla v|^p dx\right)^{1/p} + \epsilon/2 \leq \epsilon' + \epsilon/2 \leq \epsilon,$$

which gives the first estimate in (4.9). Therefore, it remains to consider the case

$$\lambda r\kappa^{\frac{n}{p}}\epsilon \leq 4C_1(\Lambda,n,p)M. \tag{4.14}$$

In this case, we note that from our choice that $\epsilon' \in (0,\epsilon/2)$, we particularly have

$$\lambda\epsilon'\kappa^{\frac{n}{p}} r \leq C(\Lambda,M,n,p).$$

Then, it follows from (4.10) that

$$\lambda\left(\frac{1}{|D_{r}(x_0)|}\int_{D_{r}^+(x_0)}|v|^p dx\right)^{1/p} \leq C(\Lambda,M,n,p).$$

From this, with Remark 2.1, and with some suitable scaling of the equation (4.5), we can apply the Hölder's regularity theory (Lemma 2.5 and Lemma 2.6) for the solution $\hat{v} := \lambda v$. We then find that there is $\beta \in (0,1)$ depending only on Λ, M, n, p such that

$$|\hat{v}(x)-\hat{v}(y)| \leq C(\Lambda,M,p,n)\left(\frac{|x-y|}{r}\right)^{\beta}, \quad \forall\ x,y \in \overline{D_{2r/3}^+}(x_0). \tag{4.15}$$

From now on, for simplicity, we write $\hat{u} = u - \bar{u}_{D^+_{\kappa r}(x_0)}$ where $\bar{u}_{D^+_{\kappa r}(x_0)} = \fint_{D^+_{\kappa r}(x_0)} u(x)dx$. We can use Lemma 2.3, the condition (1.4) of the vector field $\mathbf{A}$, and (4.13) to obtain with some $\tau > 0$ sufficiently small,

$$\begin{aligned}
&\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\le \tau \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla v) - \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla w), \nabla v - \nabla w\rangle dx \\
&\le \tau \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{D^+_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{D^+_{\kappa r}(x_0)}, \nabla v) - \mathbf{A}(x, \lambda u, \nabla v), \nabla v - \nabla w\rangle dx \\
&\le \tau \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{D^+_{\kappa r}(x_0)} [\lambda \hat{u}]^{\alpha} |\nabla v|^{p-1} |\nabla v - \nabla w| dx \\
&\le \frac{1}{2} \int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx + \tau \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p-1}} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{\frac{\alpha p}{p-1}} |\nabla v|^p dx,
\end{aligned}$$

where we have used Hölder's inequality and Young's inequality in our last step of the above estimates. By cancelling similar terms in the last estimate, we infer that

$$\begin{aligned}
&\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\le \frac{2\tau}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx + \frac{C(\Lambda, p)\tau^{\frac{p-2}{p-1}}}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{\frac{\alpha p}{p-1}} |\nabla v|^p dx.
\end{aligned}$$

For $q_1 > p$ and sufficiently close to p depending only on Λ, n and p, we write $q_0 = \frac{\alpha p p_1}{(p-1)(p_1-p)} > p$. Then, using the Hölder's inequality, and the self-improving regularity estimates, Lemma 2.7 and Lemma 2.8, we obtain

$$\begin{aligned}
&\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\le \frac{2\tau}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\nabla v|^p dx \\
&\quad + C(\Lambda, p)\tau^{\frac{p-2}{p-1}} \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_0} \right)^{\frac{p_1-p}{p_1}} \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\nabla v|^{p_1} dx \right)^{\frac{p}{p_1}} \\
&\le C(\Lambda, n, p) \left[\tau + \tau^{\frac{p-2}{p-1}} \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_0} dx \right)^{\frac{p_1-p}{p_1}} \right] \left(\frac{1}{|D_{2\kappa r}(x_0)|} \int_{D^+_{2\kappa r}(x_0)} |\nabla v|^p dx \right)
\end{aligned}$$

Now, by applying Hölder's inequality and the well-known John-Nirenberg's inequality, we see that

$$\begin{aligned}
\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_0} dx &= \frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\hat{u}|^{p/2} |\lambda \hat{u}|^{q_0 - p/2} dx \\
&\le \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx \right)^{1/2} \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^{2q_0 - p} dx \right)^{1/2} \\
&\le C(n, p, q_0) [[\lambda u]]^{q_0 - \frac{p}{2}}_{\mathrm{BMO}(D^+_R, R)} \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx \right)^{1/2} \\
&\le C(\Lambda, M, \alpha, n, p) \left(\frac{1}{|D_{\kappa r}(x_0)|} \int_{D^+_{\kappa r}(x_0)} \left|\lambda \hat{u}\right|^p dx \right)^{1/2}.
\end{aligned}$$

Therefore,

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx$$
$$\leq C(\Lambda, M, n, \alpha, p)\left[\tau + \tau^{\frac{p-2}{p-1}}\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx\right)^{\frac{p_1-p}{2p_1}}\right]\left(\frac{1}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla v|^p dx\right).$$

This and the second estimate in (4.11) imply that

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \leq C(\Lambda, M, n, \alpha, p)\left[\tau + \tau^{\frac{p-2}{p-1}}\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx\right)^{\frac{p_1-p}{2p_1}}\right]. \tag{4.16}$$

On the other hand, note that with $\hat{u} = u - \bar{u}_{D^+_{\kappa r}(x_0)}$, we can write

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx = \frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda(\hat{u} - \bar{\hat{u}}_{D^+_{\kappa r}(x_0)})|^p dx$$
$$\leq C(p)\left[\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda(\hat{u} - v)|^p dx + \frac{1}{|D_{r\kappa}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda(v - \bar{v}_{D^+_{\kappa r}(x_0)})|^p dx\right.$$
$$\left. + \frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda(\bar{\hat{u}}_{D^+_{\kappa r}(x_0)} - \bar{v}_{D^+_{\kappa r}(x_0)})|^p dx\right.$$
$$\leq C(n, p)\left[\frac{1}{\kappa^n |D_r(x_0)|}\int_{D^+_r(x_0)} |\lambda(\hat{u} - v)|^p dx + \frac{1}{|D^+_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\hat{v} - \bar{\hat{v}}_{D^+_{\kappa r}(x_0)})|^p dx\right].$$

Then, by using the Poincaré's inequality to the first term on the right hand side of the last estimate, we obtain

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx\right)^{1/p}$$
$$\leq C(\Lambda, n, p)\left[\frac{\lambda r}{\kappa^{\frac{n}{p}}}\left(\frac{1}{|D_r(x_0)|}\int_{D^+_r(x_0)} |\nabla u - \nabla v|^p dx\right)^{1/p} + \sup_{x,y \in \overline{D^+_{\kappa r}}(x_0)} |\hat{v}(x) - \hat{v}(y)|\right].$$

This last estimate, the first estimate in (4.10), and (4.15) together imply that

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\lambda \hat{u}|^p dx\right)^{1/p} \leq C(\Lambda, p, n)\Big[(\lambda r)\epsilon' + \kappa^\beta\Big].$$

From this last estimate, the estimate (4.16) can be written as

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \leq C(\Lambda, M, n, \alpha, p)\left[\tau + \tau^{\frac{p-2}{p-1}}\left(\lambda r \epsilon' + \kappa^\beta\right)^{\frac{p(p_1-p)}{2p_1}}\right].$$

From this estimate and (4.14), we can further imply that

$$\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \leq C_2(\Lambda, M, \alpha, p, n)\left(\tau + \tau^{\frac{2-p}{p-1}}\Big[\frac{\epsilon'}{\epsilon \kappa^{\frac{n}{p}}} + \kappa^\beta\Big]^{\frac{p(p_1-p)}{2p_1}}\right).$$

Then, we first choose $\tau > 0$ such that

$$C_2(\Lambda, M, n, \alpha, p)\tau = \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p.$$

Then, we chose $\kappa \in (0, 1/3)$ sufficiently small such that

$$\kappa^{\beta} \leq \frac{1}{2}\left(\frac{(\epsilon/2)^p}{2C_2(\Lambda, M, n, \alpha, p)\tau^{\frac{p-2}{p-1}}}\right)^{\frac{2p_1}{p(p_1-p)}}$$

and then choose $\epsilon' \in (0, \epsilon/2)$ so small such that

$$\frac{\epsilon'}{\epsilon\kappa^{\frac{n}{p}}} \leq \frac{1}{2}\left(\frac{(\epsilon/2)^p}{2C_2(\Lambda, M, n, \alpha, p)\tau^{\frac{p-2}{p-1}}}\right)^{\frac{2p_1}{p(p_1-p)}}.$$

From these choices, we obtain

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx\right)^{1/p} \leq \epsilon/2.$$

From this, we then use the triangle inequality and the first estimate in (4.10) to derive the estimate

$$\left(\frac{1}{|D_{\kappa r}(x_0)|}\int_{D^+_{\kappa r}(x_0)} |\nabla u - \nabla w|^p dx\right)^{1/p} \leq \epsilon.$$

This completes the proof of the first estimate in (4.9). To prove the second estimate in (4.9), we use triangle inequality and the assumption in the lemma to see that

$$\begin{aligned}\left(\frac{1}{|D_{\kappa r}(x_0)}\int_{D^+_{\kappa r}(x_0)} |\nabla w|^p dx\right)^{1/p} &\leq \left(\frac{1}{|D_{\kappa r}(x_0)}\int_{D^+_{\kappa r}(x_0)} |\nabla w - \nabla u|^p dx\right)^{1/p} + \left(\frac{1}{|D_{\kappa r}(x_0)}\int_{D^+_{\kappa r}(x_0)} |\nabla u|^p dx\right)^{1/p}\\ &\leq \epsilon + \left(\frac{2^n}{|D_{2\kappa r}(x_0)|}\int_{D^+_{2\kappa r}(x_0)} |\nabla u|^p dx\right)^{1/p} \leq 1 + 2^{n/p} = C_0(n,p).\end{aligned}$$

The proof of the lemma is now complete. □

Now, we can state and prove the following important result, which is the main result of the subsection.

Proposition 4.3. *Let $\Lambda, M > 0, p > 1$ be fixed. Then, for every $\epsilon \in (0, 1)$, there exist sufficiently small numbers $\kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon) \in (0, 1/3)$ and $\delta = \delta(\epsilon, \Lambda, M, \alpha, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : D^+_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.11) *hold, and*

$$\frac{1}{|D_{4r}(x_0)|}\int_{D^+_{4r}(x_0)} |\mathbf{F}(x)|^p dx + \frac{1}{|B'_{4r}(x'_0)|}\int_{B'_{4r}(x'_0)} |g(x')|^p dx' \leq \delta^p,$$

for some $x_0 = (x'_0, x_{n0}) \in D^+_R$ and some $r \in (0, R/4)$. Then, for every $\lambda \geq 0$, if $u \in W^{1,p}(D^+_{2R})$ is a weak solution of (4.1) *satisfying*

$$\frac{1}{|D_{4r}(x_0)|}\int_{D^+_{4r}(x_0)} |\nabla u|^p dx \leq 1, \quad \frac{1}{|D_{8\kappa r}(x_0)|}\int_{D^+_{8\kappa r}(x_0)} |\nabla u|^p dx \leq 1, \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(D^+_R, R)} \leq M,$$

then there is $h \in W^{1,p}(D^+_{7\kappa r/2}(x_0))$ such that the following estimate holds

$$\left(\frac{1}{|D_{7\kappa r/2}(x_0)|}\int_{D^+_{7\kappa r/2}(x_0)} |\nabla u - \nabla h|^p dx\right)^{1/p} \leq \epsilon, \quad \|\nabla h\|_{L^\infty(D^+_{3\kappa r}(x_0))} \leq C(\Lambda, n, p). \tag{4.17}$$

Proof. For a given $\epsilon \in (0, 1)$, let

$$\delta = \min\left\{\bar{\delta}_0(\epsilon/[2C_0(n,p)], \Lambda, n, p), \bar{\delta}_2(\epsilon/[2(8/7)^{n/p}], \Lambda, M, \alpha, p)\right\},$$

where $\bar{\delta}_0$ and $\bar{\delta}_2$ are the numbers defined in Lemma 2.10, and Lemma 4.2, respectively. Moreover, $C_0(n, p)$ is the number defined in (4.9). Also, let $\kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon) \in (0, 1/3)$ be the number defined in Lemma

4.2. We now prove our result with this choice of δ and κ. From the assumptions we can apply Lemma 4.2 with r replaced by $4r$ to find $w \in W^{1,p}(D^+_{4\kappa r}(x_0))$, which is the weak solution of (4.8) satisfying

$$
\begin{aligned}
&\left(\frac{1}{|D_{4\kappa r}(x_0)|}\int_{D^+_{4\kappa r}(x_0)} |\nabla u - \nabla w|^p dx\right)^{1/p} \leq \frac{\epsilon}{2(8/7)^{n/p}}, \quad \text{and} \\
&\left(\frac{1}{|D_{4\kappa r}(x_0)}\int_{D_{4\kappa r}(x_0)} |\nabla w|^p\right)^{1/p} \leq C_0(n,p).
\end{aligned} \tag{4.18}
$$

Now, let $\mathbf{A}_0(x,\xi) = \mathbf{A}(x, \bar{u}_{D^+_{4\kappa r}(x_0)}, \xi)$ for all $x \in D^+_{2R}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We then apply Lemma 2.10 with $\rho = 2\kappa r$ and some suitable scaling to find a function $h \in W^{1,p}(D^+_{7\kappa r/2}(x_0))$ satisfying

$$
\left(\frac{1}{|D_{7\kappa r/2}(x_0)|}\int_{D^+_{7\kappa r/2}(x_0)} |\nabla w - \nabla h|^p dx\right)^{1/p} \leq \epsilon/2, \quad \text{and} \quad \|\nabla h\|_{L^\infty(D^+_{3\kappa r}(x_0))} \leq C(\Lambda, n, p). \tag{4.19}
$$

Then, from (4.18), (4.19), and the triangle inequality, we obtain

$$
\begin{aligned}
&\left(\frac{1}{|D_{7\kappa r/2}(x_0)|}\int_{D^+_{7\kappa r/2}(x_0)} |\nabla u - \nabla h|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{|D_{7\kappa r/2}(x_0)|}\int_{D^+_{7\kappa r/2}(x_0)} |\nabla u - \nabla w|^p dx\right)^{1/p} + \left(\frac{1}{|D_{7\kappa r/2}(x_0)|}\int_{D^+_{7\kappa r/2}(x_0)} |\nabla w - \nabla h|^p dx\right)^{1/p} \\
&\leq (8/7)^{n/p}\left(\frac{1}{|D_{4\kappa r}(x_0)|}\int_{D^+_{4\kappa r}(x_0)} |\nabla u - \nabla w|^p dx\right)^{1/p} + \epsilon/2 \leq \epsilon.
\end{aligned}
$$

The proof is therefore complete. □

4.2. **Boundary level set estimates.** Recall that the Hardy-Littlewood maximal function $\mathcal{M}(f)$ is defined in (2.7). Moreover, if f is defined only in U, we write $\mathcal{M}_U(f) = \mathcal{M}(f\chi_U)$, where χ_U is the characteristic function of a measurable set $U \subset \mathbb{R}^n$. Also recall that $B'_\rho(x')$ is the ball in $\mathbb{R}^{n-1}$ centered at $x' \in \mathbb{R}^{n-1}$ and with radius $\rho > 0$. Our first result of this subsection is the following important lemma on the density of the level sets of $|\nabla u|$ for a weak solution u of (4.1).

Lemma 4.4. *Let $\Lambda > 0, M > 0$, $p > 1$ and $\alpha \in (0,1]$ be fixed, and let $\epsilon \in (0,1)$. Then there exist sufficiently large number $N = N(\Lambda, n, p) \geq 1$ and two sufficiently small numbers $\kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon) \in (0, 1/3)$ and $\delta = \delta(\epsilon, \Lambda, M, p, \alpha, n) \in (0,\epsilon)$ such that the following statement holds. Suppose that $\mathbf{A} : D^+_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.11) *hold for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. Suppose that $u \in W^{1,p}(D^+_{2R})$ is a weak solution of* (4.1) *satisfying $[[\lambda u]]_{\mathrm{BMO}(D^+_R, R)} \leq M$ with some $\lambda \geq 0$. If $y = (y', y_n) \in D^+_R$ and $\rho \in (0, \kappa_0)$ such that*

$$
\begin{aligned}
&D_\rho(y) \cap \left\{x \in D^+_R : \mathcal{M}_{D^+_{2R}}(|\nabla u|^p)(x) \leq 1\right\} \cap \\
&\cap \left\{x = (x', x_n) \in D^+_R : \mathcal{M}_{D^+_{2R}}(|\mathbf{F}|^p)(x) + \mathcal{M}_{B'_{2R}}(g)(x') \leq \delta^p\right\} \neq \emptyset,
\end{aligned}
$$

for $\kappa_0 = \min\left\{1, R\right\}\kappa/6$, then

$$
\left|\left\{x \in D^+_R : \mathcal{M}_{D^+_{2R}}(|\nabla u|^p) > N\right\} \cap D_\rho(y)\right| \leq \epsilon |D_\rho(y)|. \tag{4.20}
$$

Proof. The proof is standard using Proposition 4.3. However, as Proposition 4.3 is stated differently compared to other similar available approximation estimates in the literature, details of the proof of this lemma is required. For a given $\epsilon \in (0,1)$, let $\epsilon' > 0$ be sufficiently small to be determined depending only on ϵ, Λ, n and p. Then, let $\kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon')$ and $\delta = \delta(\epsilon', \Lambda, M, p, n, \alpha)$ be the numbers defined in Proposition

4.3. We prove the lemma with this choice of δ, κ. By the assumption, we can find $x_0 = (x_0', x_{n0}) \in D_\rho(y) \cap D_R^+$ such that

$$\mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p)(x_0) + \mathcal{M}_{B_{2R}'}(|g|^p)(x_0') \leq \delta^p. \tag{4.21}$$

Let $r = \kappa^{-1}\rho \in (0, R/6)$, and we plan to use Proposition 4.3 with this r. Therefore, we need to show that all conditions in Proposition 4.3 hold. Since $\rho \in (0, \kappa_0)$ and κ is sufficiently small, we see that

$$D_{4r}^+(y) \subset D_{5r}^+(x_0) \subset D_{2R}^+, \quad \text{and} \quad B_{4r}'(y') \subset B_{5r}'(x_0') \subset B_{2R}'.$$

From this and (4.21), it follows that

$$\frac{1}{|D_{4r}(y)|}\int_{D_{4r}^+(y)} |\nabla u|^p dx \leq \frac{|D_{5r}(x_0)|}{|D_{4r}(y)|}\frac{1}{|D_{5r}(x_0)|}\int_{D_{5r}^+(x_0)} |\nabla u|^p dx \leq \left(\frac{5}{4}\right)^n,$$

and

$$\begin{aligned}
&\frac{1}{|D_{4r}(y)|}\int_{D_{4r}^+(y)} |\mathbf{F}|^p dx + \frac{1}{|B_{4r}'(y')|}\int_{B_{4r}'(y')} |g(x')|^p dx' \\
&\leq \frac{|D_{5r}(x_0)|}{|D_{4r}(y)|}\frac{1}{|D_{5r}(x_0)|}\int_{D_{5r}^+(x_0)} |\mathbf{F}|^p dx + \frac{|B_{5r}'(x_0')|}{|B_{4r}'(y')|}\frac{1}{|B_{5r}'(x_0')|}\int_{B_{5r}'(x_0')} |g(x')|^p dx' \\
&\leq \left(\frac{5}{4}\right)^n \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x_0) + \left(\frac{5}{4}\right)^{n-1} \mathcal{M}_{B_{2R}'}(|g|^p)(x_0') \leq \left(\frac{5}{4}\right)^n \delta^p,
\end{aligned}$$

Moreover, since $\rho = \kappa r \in (1, R/18)$, we can check that $D_{8\rho}^+(y) \subset D_{9\rho}^+(x_0) \subset D_{2R}^+$ and therefore it follows from (4.21) that

$$\begin{aligned}
&\frac{1}{|D_{8\kappa r}(y)|}\int_{D_{8\kappa r}^+(y)} |\nabla u|^p dx = \frac{1}{|D_{8\rho}(y)|}\int_{D_{8\rho}^+(y)} |\nabla u|^p dx \\
&\leq \frac{|D_{9\rho}(x_0)|}{|D_{8\rho}(y)|}\frac{1}{|D_{9\rho}(x_0)|}\int_{D_{9\rho}(x_0)\cap D_{2R}^+} |\nabla u|^p dx \leq \left(\frac{9}{8}\right)^n.
\end{aligned}$$

Hence, all conditions in Proposition 4.3 are satisfied when taking the scaling $u \mapsto \tilde{u} := u/(5/4)^n$ into consideration. From this, Lemma 2.1, and our choice of κ, δ, we can apply Proposition 4.3 for $\tilde{u}$ and then scale back to u find a function $h \in W^{1,p}(D_{\frac{7\rho}{2}}^+(y))$ satisfying

$$\frac{1}{|D_{\frac{7\rho}{2}}(y)|}\int_{D_{\frac{7\rho}{2}}^+(y)} |\nabla u - \nabla h|^p dx \leq (\epsilon')^p \left(\frac{5}{4}\right)^n, \quad \text{and} \quad \|\nabla h\|_{L^\infty(D_{3\rho}^+(y))} \leq C_*(\Lambda, n, p).$$

Let us now denote $N = \max\left\{2^p C_*^p, 2^n\right\}$, and we will prove (4.20) with this choice of N. To this end, we will firstly prove that

$$\left\{x \in D_\rho(y) \cap D_R^+ : \mathcal{M}_{D_{\frac{7\rho}{2}}^+(y)}(|\nabla u - \nabla h|^p)(x) \leq C_*^p\right\} \subset \left\{x \in D_\rho(y) \cap D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x) \leq N\right\}. \tag{4.22}$$

To prove this statement, let x be a point in the set on the left side of (4.22), and we shall verify that

$$\mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x) \leq N. \tag{4.23}$$

Let $\rho' > 0$ be any number. If $\rho' < 2\rho$, then $D_{\rho'}(x) \cap D_{2R}^+ \subset D_{3\rho}(y) \cap D_{2R}^+ \subset D_{2R}^+$, and it follows that

$$\begin{aligned}
&\left(\frac{1}{|D_{\rho'}(x)|}\int_{D_{\rho'}(x)\cap D_{2R}^+} |\nabla u(z)|^p dz\right)^{1/p} = \left(\frac{1}{|D_{\rho'}(x)|}\int_{D_{\rho'}^+(x)} |\nabla u(z)|^p dz\right)^{1/p} \\
&\le \left(\frac{1}{|D_{\rho'}(x)|}\int_{D_{\rho'}(x)\cap D_{7\rho/2}^+(y)} |\nabla u(z) - \nabla h(z)|^p dz\right)^{1/p} + \left(\frac{1}{|D_{\rho'}(x)|}\int_{D_{\rho'}^+(x)} |\nabla h(z)|^p dz\right)^{1/p} \\
&\le \left(\mathcal{M}_{D_{7\rho/2}^+(y)}(|\nabla u - \nabla h|^p)(x)\right)^{1/p} + \|\nabla h\|_{L^\infty(D_{3\rho}^+(y))} \le 2C_* \le N^{1/p}.
\end{aligned}$$

On the other hand, if $\rho' \ge 2\rho$, we note that $D_{\rho'}(x) \cap D_{2R}^+ \subset D_{2\rho'}(x_0) \cap D_{2R}^+$, and it follows from this and (4.21) that

$$\frac{1}{|D_{\rho'}(x)|}\int_{D_{\rho'}(x)\cap D_{2R}^+} |\nabla u(z)|^p dz \le \frac{|D_{2\rho'}(x_0)|}{|D_{\rho'}(x)|}\frac{1}{|D_{2\rho'}^+(x_0)|}\int_{D_{2\rho'}^+(x_0)\cap D_{2R}^+} |\nabla u(z)|^p dz \le 2^n \le N.$$

Hence, (4.23) is verified and therefore (4.22) is proved. Observe that (4.22) is in fact equivalent to

$$\left\{x \in D_\rho(y) \cap D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x) > N\right\} \subset E := \left\{x \in D_\rho(y) \cap D_R^+ : \mathcal{M}_{D_{7\rho/2}^+(y)}(|\nabla u - \nabla h|^p)(x) > C_*^p\right\}. \tag{4.24}$$

On the other hand, from the weak type (1,1) estimate of Hardy-Littlewood maximal function, see Lemma 2.11, it is true that

$$\frac{|E|}{|D_\rho(y)|} \le \frac{C(n)}{C_*^p |D_{7\rho/2}(y)|}\int_{D_{7\rho/2}^+(y)} |\nabla u - \nabla h|^p dz \le C_1(\Lambda, n, p)(\epsilon')^p.$$

Then, by choosing ϵ' depending on ϵ, Λ, n, p such that $C_1(\Lambda, n, p)(\epsilon')^p = \epsilon$, we obtain $|E| \le \epsilon |D_\rho(y)|$. From this estimate and the definition of E in (4.24), the estimate (4.20) follows and the proof is complete. □

Our level set estimate is the following result, which is the main result of the section.

Lemma 4.5. *Let $\Lambda > 0, M > 0$, $p > 1$ and $\alpha \in (0,1]$ be fixed, and let $\epsilon \in (0,1)$. Then there exist sufficiently large number $N = N(\Lambda, n, p) \ge 1$ and two sufficiently small numbers $\kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon) \in (0, 1/3)$ and $\delta = \delta(\epsilon, \Lambda, M, p, \alpha, n) \in (0, \epsilon)$ such that the following statement holds. For some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$, assume that $\mathbf{A} : D_{2R}^+ \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.11) *hold. Then, for any $\lambda \ge 0$, if $u \in W^{1,p}(D_{2R}^+)$ is a weak solution of* (4.1) *satisfying*

$$[[\lambda u]]_{\mathrm{BMO}(D_R^+, R)} \le M, \quad \left|\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^2) > N\}\right| \le \epsilon |D_{\kappa_0}|, \tag{4.25}$$

for $\kappa_0 = \min\{1, R\}\kappa/6$, then with ϵ_1 defined in Lemma 2.13,

$$\begin{aligned}
&\left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p > N\right\}\right| \\
&\le \epsilon_1 \Big[\left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > 1\right\}\right| \\
&\quad + \left|\left\{x = (x', x_n) \in D_R^+ : \mathcal{M}_{B'_{2R}}(|g|^p)(x') > \delta^p\right\}\right| + \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) > \delta^p\right\}\right|\Big].
\end{aligned} \tag{4.26}$$

Proof. Let N, κ, δ be the numbers defined in Lemma 4.4. We plan to apply Lemma 2.13 for

$$C = \left\{x \in D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x) > N\right\},$$

and

$$\begin{aligned}
D = &\left\{x \in D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x) > 1\right\} \\
&\cup \left\{x = (x', x_n) \in D_R^+ : \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p)(x) + \mathcal{M}_{B'_{2R}}(|g|^p)(x') > \delta^p\right\}.
\end{aligned}$$

By the second assumption in (4.25), we see that (i) of Lemma 2.13 holds. Moreover, by Lemma 4.4, we see that (ii) of Lemma 2.13 is satisfied. Therefore, all conditions in Lemma 2.13 hold and (4.26) follows from Lemma 2.13. □

4.3. **Proof of the $W^{1,q}$-regularity estimates on flat domains.** From the Lemma 4.5 and an iterating procedure, we obtain the following lemma

Lemma 4.6. *Let $\Lambda, M, p, \alpha, \epsilon, N, \delta, \kappa_0$ be as in Lemma 4.5. Also, let $\mathbf{A}$ be as in Lemma 4.5. Then, for any $\lambda \geq 0$, if $u \in W^{1,p}(D_{2R}^+)$ is a weak solution of* (4.1) *satisfying*

$$[[\lambda u]]_{\mathrm{BMO}(D_R^+,R)} \leq M, \quad \textit{and} \quad \left|\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > N\}\right| \leq \epsilon |D_{\kappa_0}|,$$

then with ϵ_1 defined in Lemma 2.13, and for any $k \in \mathbb{N}$,

$$\begin{aligned} &\left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p > N^k\right\}\right| \leq \epsilon_1^k \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > 1\right\}\right| \\ &\quad + \sum_{i=1}^k \epsilon_1^i \left[\left|\left\{D_R^+ : \mathcal{M}_{B'_{2R}}(|g|^p) > \delta N^{k-i}\right\}\right| + \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) > \delta N^{k-i}\right\}\right|\right]. \end{aligned} \tag{4.27}$$

Proof. The proof is based induction on k using the iteration of Lemma 4.5. We skip the proof. One can see, for example, [37, Lemma 4.10] for details. □

We now can complete the proof of Theorem 1.3.

Proof of Theorem 1.3. For given $\Lambda > 0, M > 0, \alpha \in (0,1]$ and $q > p > 1$, let $N = N(\Lambda, p, n)$ be the number defined in Lemma 4.5, and let $q' = q/p > 1$. Let $\epsilon \in (0,1)$ be a sufficiently small number and depending only on Λ, M, n, p, q such that

$$\epsilon_1 N^{q'} = 1/2,$$

where ϵ_1 is defined in Lemma 4.6. With this ϵ, we choose

$$\delta = \delta(\epsilon, \Lambda, M, p, \alpha, n), \quad \kappa = \kappa(\Lambda, M, p, \alpha, n, \epsilon), \quad \kappa_0 = \kappa/6$$

defined as in Lemma 4.5. Assume that (1.11) holds with this choice of δ and we will prove Theorem 1.3. For $\lambda \geq 0$ and $R \in (0,1]$, let u be a weak solution of (4.1) satisfying $[[\lambda u]]_{\mathrm{BMO}(D_R^+,R)} \leq M$, and let

$$E = E(\lambda, N) = \left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > N\right\}. \tag{4.28}$$

We assume for a moment that the following extra condition

$$|E| \leq \epsilon |D_{\kappa_0}|. \tag{4.29}$$

Let us now consider the sum

$$S = \sum_{k=1}^\infty N^{q'k} \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > N^k\right\}\right|.$$

From (4.29), we can apply Lemma 4.6 to obtain

$$\begin{aligned} S \leq \sum_{k=1}^\infty N^{kq'} &\left[\sum_{i=1}^k \epsilon_1^i \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) > \delta N^{k-i}\right\}\right|\right. \\ &\quad \left. + \sum_{i=1}^k \epsilon_1^i \left|\left\{D_R^+ : \mathcal{M}_{B'_{2R}}(|g|^p) > \delta N^{k-i}\right\}\right|\right] \\ &\quad + \sum_{k=1}^\infty \left(N^{q'}\epsilon_1\right)^k \left|\left\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > 1\right\}\right). \end{aligned}$$

By Fubini's theorem, the above estimate can be rewritten as

$$
\begin{aligned}
S \le \sum_{j=1}^{\infty} (N^{q'}\epsilon_1)^j \Bigg[&\sum_{k=j}^{\infty} N^{q'(k-j)} \Big| \Big\{ D_R^+ : \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) > \delta N^{k-j} \Big\} \Big| \\
&+ \sum_{k=j}^{\infty} N^{q'(k-j)} \Big| \Big\{ D_R^+ : \mathcal{M}_{B'_{2R}}(|g|^p) > \delta N^{k-j} \Big\} \Big| \Bigg] \\
&+ \sum_{k=1}^{\infty} \Big(N^{q'}\epsilon_1\Big)^k \Big| \Big\{ D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > 1 \Big\} \Big|.
\end{aligned} \tag{4.30}
$$

Observe that

$$
\Big| \Big\{ D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > 1 \Big\} \Big| \le |D_R^+|.
$$

From this, the choice of ϵ, and Lemma 2.12, and (4.30) it follows that

$$
S \le C \left[\left\| \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) \right\|_{L^{q'}(D_R^+)}^{q'} + \left\| \mathcal{M}_{B'_{2R}}(|g|^p) \right\|_{L^{q'}(D_R^+)}^{q'} + |D_R^+| \right].
$$

Applying the Lemma 2.12 again, we see that

$$
\left\| \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) \right\|_{L^{q'}(D_R^+)}^{q'} \le C \left[\left\| \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) \right\|_{L^{q}(D_R^+)}^{q'} + \left\| \mathcal{M}_{B'_{2R}}(|g|^p) \right\|_{L^{q'}(D_R^+)}^{q'} + |D_R^+| \right].
$$

By the Lesbegue's differentiation theorem, we observe that

$$
|\nabla u(x)|^p \le \mathcal{M}_{D_{2R}^+}(|\nabla u|^p)(x), \quad \text{a.e } x \in D_R^+.
$$

Moreover, observe that

$$
\left\| \mathcal{M}_{B'_{2R}}(|g|^p) \right\|_{L^{q'}(D_R^+)}^{q'} \le 2R \left\| \mathcal{M}_{B'_R}(|g|^p) \right\|_{L^{q'}(B'_{2R})}^{q'}
$$

Therefore,

$$
\|\nabla u\|_{L^q(D_R^+)}^q \le C \left[\left\| \mathcal{M}_{D_{2R}^+}(|\mathbf{F}|^p) \right\|_{L^{q'}(D_R^+)}^{q} + R \left\| \mathcal{M}_{B'_{2R}}(|g|^p) \right\|_{L^{q'}(B'_R)}^{q'} + |D_R^+| \right].
$$

Then, by Lemma 2.11, it follows

$$
\|\nabla u\|_{L^q(D_R^+)} \le C \left[\|\mathbf{F}\|_{L^q(D_{2R}^+)} + R\, \|g\|_{L^q(B'_{2R})} + |D_R^+|^{1/q} \right].
$$

This implies that

$$
\fint_{D_R^+} |\nabla u|^q dx \le C \left[\fint_{D_{2R}^+} |\mathbf{F}(x)|^q dx + \fint_{B'_{2R}} |g(x')|^q dx' + 1 \right]. \tag{4.31}
$$

In conclusion, we have proved (4.31) as long as u is a weak solution of (4.1) for all $\lambda \ge 0$ and (4.29) holds.

We now remove the extra condition (4.29). Assuming that u is a weak solution of (4.1) with some $\lambda \ge 0$. Let $\gamma > 1$ sufficiently large to be determined. Let $\lambda' = \lambda\gamma \ge 0$, $u_\gamma = u/\gamma$, $\mathbf{F}_\gamma = \mathbf{F}/\gamma$ and $g_\gamma = g/\gamma$. We note that u_γ is a weak solution of

$$
\left\{
\begin{array}{rcll}
\text{div}[\hat{\mathbf{A}}(x, \lambda' u_\gamma, \nabla u_\gamma)] & = & \text{div}[|\mathbf{F}_\gamma|^{p-2}\mathbf{F}_\gamma], & \text{in} \quad D_{2R}^+, \\
\langle \hat{\mathbf{A}}(x, \lambda' u_\gamma, \nabla u_\gamma) - |\mathbf{F}_\gamma|^{p-2}\mathbf{F}_\gamma, \vec{e}_n \rangle & = & |g_\gamma|^{p-2} g_\gamma, & \text{on} \quad T_{2R},
\end{array}
\right. \tag{4.32}
$$

where

$$
\hat{\mathbf{A}}(x, z, \xi) = \frac{\mathbf{A}(x, z, \gamma\xi)}{\gamma^{p-1}}.
$$

Note that by Remark 2.1, $\hat{\mathbf{A}}$ satisfies all (1.2)-(1.4) with the same constants Λ, p, α. Moreover, it is simple to check that

$$
[[\hat{\mathbf{A}}]]_{\text{BMO}(D_R^+, R)} = [[\mathbf{A}]]_{\text{BMO}(D_R^+, R)} < \delta.
$$

We denote

$$E_\gamma = \Big\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u_\gamma|^2) > N\Big\}.$$

and we assume without loss of generality that

$$K_0 = \left(\frac{1}{|D_{2R}|}\int_{D_{2R}^+} |\nabla u|^p dx\right)^{1/p} > 0. \tag{4.33}$$

We claim that we can choose $\gamma = C_0^* K_0$ with some sufficiently large constant C_0^* depending only on Λ, M, p, q, n, α such that

$$|E_\gamma| \leq \epsilon |D_{\kappa_0}|. \tag{4.34}$$

If this holds, we can apply (4.31) for u_γ which is a weak solution of (4.32) to obtain

$$\fint_{D_R^+} |\nabla u_\gamma|^q dx \leq C\left[\fint_{D_{2R}^+} |\mathbf{F}_\gamma(x)|^q dx + \fint_{B'_{2R}} |g_\gamma(x')|^q dx' + 1\right]$$

Then, by multiplying this equality with γ^q, we obtain

$$\fint_{D_R^+} |\nabla u|^q dx \leq C\left[\fint_{D_{2R}^+} |\mathbf{F}(x)|^q dx + \fint_{B'_{2R}} |g(x')|^q dx' + \left(\fint_{D_{2R}^+} |\nabla u|^p\right)^{q/p}\right]$$

and this is our desired estimate (1.12). The proof of Theorem 1.3 is therefore complete if we can prove (4.34). To this end, we observe from the definition of E_γ, and the weak type (1-1) estimate for maximal function, we see that

$$\begin{aligned}\frac{|E_\gamma|}{|D_{\kappa_0}|} &= C(n)\Big(\frac{R}{\kappa_0}\Big)^n \frac{\Big|\Big\{D_R^+ : \mathcal{M}_{D_{2R}^+}(|\nabla u|^p) > N\gamma^p\Big\}\Big|}{|D_{2R}|} \\ &= \frac{C(n,p)}{N\gamma^p}\Big(\frac{R}{\kappa_0}\Big)^n \frac{1}{|D_{2R}|}\int_{D_{2R}^+} |\nabla u|^p dx \leq \frac{C(p,n)K_0^p}{N\gamma^p}\Big(\frac{R}{\kappa_0}\Big)^n,\end{aligned}$$

where K_0 is defined in (4.33). From this, and since $\kappa_0 = \kappa_0(\Lambda, M, p, q, \alpha, n)$, $R \in (0,1)$, we conclude that

$$\frac{|E_\gamma|}{|D_{\kappa_0}|} \leq C^*(\Lambda, M, p, q, \alpha, n)\left(\frac{K_0}{\gamma}\right)^p$$

Now, we choose γ such that

$$\gamma = K_0\left[\epsilon^{-1} C^*(\Lambda, M, p, q, \alpha, n)\right]^{1/p} = C_0^*(\Lambda, M, p, q, \alpha, n) K_0$$

then, it follows

$$|E_\gamma| \leq \epsilon |D_{\kappa_0}|.$$

This proves (4.34) and completes the proof. □

5. Global regularity estimates and proof of Theorem 1.1

This section proves Theorem 1.1. The proof is standard using Theorem 1.2, Theorem 1.3, and the compactness of Ω. To be self-contained, we provide most details of the proof in dealing with the boundary terms. In this section, for any $n \times n$ matrix Q, we denote Q^* the transposed matrix of Q. We begin with the following elementary lemma that is needed for the proof.

Lemma 5.1. *Assume* $\mathbf{A} : \Omega \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ *such that* (1.2)-(1.4) *hold on* $\Omega \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$. *Then, for every* $n \times n$ *rotation matrix* Q, *and for* $x_0 \in \mathbb{R}^n$, *we define*

$$\tilde{\mathbf{A}}(x, z, \xi) = \mathbf{A}(Q(x - x_0), z, \xi Q) \cdot Q^*.$$

Then, $\tilde{\mathbf{A}}$ *also satisfies* (1.2)-(1.4) *on* $\tilde{\Omega}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$ *with the same constant* Λ, α, p*, where* $\tilde{\Omega} = Q^*\Omega + x_0$*. Moreover,*

$$[[\tilde{\mathbf{A}}]]_{\mathrm{BMO}(\tilde{\Omega},\rho_0)} = [[\mathbf{A}]]_{\mathrm{BMO}(\Omega,\rho_0)}.$$

Proof. Since Q is a rotation matrix, $|Q\xi| = |\xi|$ for all $\xi \in \mathbb{R}^n$. Therefore, it follows that $|\tilde{\mathbf{A}}(x,z,\xi)| = |\mathbf{A}(Q(x-x_0), z, \xi Q)|$ for all $(x,z,\xi) \in \tilde{\Omega}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\})$. Moreover, note that

$$D_\xi\tilde{\mathbf{A}}(x,z,\xi) = D_\xi\mathbf{A}(x-x_0, z, \xi Q), \quad \forall\ (x,z,\xi) \in \tilde{\Omega}\times\mathbb{K}\times(\mathbb{R}^n\setminus\{0\}).$$

Hence, it is simple to verify that $\tilde{\mathbf{A}}$ satisfies (1.2)-(1.4) with the same constant Λ, α, p. On the other hand, since Q is measure-preserving, it is a simple calculation using integration by substitution to check that $[[\tilde{\mathbf{A}}]]_{\mathrm{BMO}(\tilde{\Omega},\rho_0)} = [[\mathbf{A}]]_{\mathrm{BMO}(\Omega,\rho_0)}$. The proof of the lemma is therefore complete. □

Proof of Theorem 1.1. For given numbers $\Lambda > 0, M > 0, q > p > 1$ and $\alpha \in (0,1]$, let

$$\Upsilon = \min\Big\{\hat{\delta}_0(\Lambda, M, n, p, q), \delta(C_*\Lambda, M, n, p, q)/[2C_0]\Big\}, \tag{5.1}$$

where $\hat{\delta}_0$ is defined in Theorem 1.2, δ is defined in Theorem 1.3, and the constants $C_* = C_*(n,p), C_0 = C_0(\Lambda, n, p)$ are defined in Lemma 5.2 below. We prove Theorem 1.1 with this choice of Υ.

Step I: In this step, we estimate $|\nabla u|$ in a neighborhood of $\partial\Omega$. Consider some $x_0 \in \partial\Omega$. By Lemma 5.1, we can use a translation, and a rotation to assume without loss of generality that $x_0 = 0$, and there is a function $\varphi : \overline{B}'_{2R} \to \mathbb{R}$ independent our choice of x_0 such that $\varphi \in C^1(\overline{B}'_{2R})$, $\varphi(0) = 0$, $\quad \nabla_{x'}\varphi(0) = 0$,

$$\Omega_{2R} := \Big\{(x', x_n) \in \overline{B}'_{2R}\times\mathbb{R} : \varphi(x') < x_n < \varphi(x') + 2R\Big\} \subset \Omega, \quad \text{and}$$
$$\Gamma_{2R} := \Big\{(x', \varphi(x')) \in \mathbb{R}^n : x' \in \overline{B}'_{2R}\Big\} \subset \partial\Omega,$$

for some $R \in (0,1)$. Geometrically, the hyperplane $\{x_n = 0\}$ is tangent to $\partial\Omega$ at $x_0 = 0 \in \partial\Omega$. Then, by the continuity of $\nabla_{x'}\varphi$ and since $\nabla_{x'}\varphi(0) = 0$, we can take $R \in (0, \rho_0)$ sufficiently small so that

$$|\nabla\varphi(x')| \le \Upsilon, \quad \forall\ x' \in \bar{B}'_{2R}. \tag{5.2}$$

Observe also that the outward normal vector $\vec{\nu}(x)$ on Γ_{2R} is

$$\vec{\nu}(x) = \frac{1}{\sqrt{1 + |\nabla\varphi(x')|^2}}(\nabla\varphi(x'), -1), \quad \text{for all} \quad x = (x', x_n) \in \Gamma_{2R}. \tag{5.3}$$

Let us denote

$$\hat{\Omega}_{2R} = \Big\{x = (x', x_n) \in \mathbb{R}^n : x' \in B'_{2R} \text{ and } \varphi(x') - 2R < x_n < \varphi(x') + 2R\Big\}.$$

Then, let $\Phi : \hat{\Omega}_{2R} \to D_{2R}$ and $\Psi : D_{2R} \to \hat{\Omega}_{2R}$ be the homomorphisms defined by

$$\Phi(x', x_n) = (x', x_n - \varphi(x')), \quad \Psi(y', y_n) = (y', y_n + \varphi(y')).$$

Observe that from (5.2), $\varphi(0') = 0$, and since Υ is sufficiently small we see that Φ and Ψ are almost like an identity map. In particular, from their explicit formulas, we see that

$$D\Phi(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -D_1\varphi(x') & -D_2\varphi(x') & -D_3\varphi(x') & \cdots & -D_{n-1}\varphi(x') & 1 \end{bmatrix}, \tag{5.4}$$

and

$$D\Psi(y) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ D_1\varphi(y') & D_2\varphi(y') & D_3\varphi(y') & \cdots & D_{n-1}\varphi(y') & 1 \end{bmatrix}. \tag{5.5}$$

From the explicit formulas (5.4) and (5.5), it follows that

$$\Phi = \Psi^{-1}, \quad D\Psi(y) = [D\Phi(\Psi(y))]^{-1}, \quad \text{and} \quad \det(D\Psi) = \det(D\Phi) = 1. \tag{5.6}$$

Moreover, from (5.2), (5.4) and (5.5), we also have

$$\|D\Phi\|^2_{L^\infty(\Omega_{2R})} \leq n + \|\nabla\varphi\|^2_{L^\infty} \leq n+1, \quad \|D\Psi\|^2_{L^\infty(D^+_{2R})} \leq n + \|\nabla\varphi\|^2_{L^\infty} \leq n+1. \tag{5.7}$$

Now, for each $y = (y', y_n) \in D^+_{2R}$, let us denote

$$\hat{u}(y) = u(\Psi(y)), \quad \hat{g}(y') = |g(\Psi(y',0))|^{p-2} g(\Psi(y',0))\sqrt{1 + |\nabla\varphi(y')|^2}, \quad \text{and}$$
$$\mathbf{G}(y) = |\hat{\mathbf{F}}(\Psi(y))|^{p-2}\mathbf{F}(\Psi(y)) \cdot [D\Phi(\Psi(y))]^*.$$

Moreover, define $\hat{\mathbf{A}} : D^+_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ by

$$\hat{\mathbf{A}}(y,z,\xi) = \mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y))]) \cdot [D\Phi(\Psi(y))]^*, \quad \forall\, (y,z,\xi) \in D^+_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}), \tag{5.8}$$

where $[\nabla\Phi(\Psi(y))]^*$ is the transposed matrix of the matrix $[\nabla\Phi(\Psi(y))]$. For every $\phi \in C^1_0(\hat{\Omega}_{2R})$, we denote $\hat{\phi}(y) = \phi(\Psi(y))$ for all $y \in D_{2R}$. Then, note that $\hat{\phi} \in C^1_0(D_{2R})$, and

$$\begin{aligned}\int_{B'_{2R}} \hat{g}(y')\hat{\phi}(y',0)dy' &= \int_{B'_{2R}} |g(y',\varphi(y')))|^{p-2} g(y',\varphi(y'))\phi(y',\varphi(y'))\sqrt{1+|\nabla\varphi(y')|^2}dy' \\ &= \int_{\Gamma_{2R}} |g(x)|^{p-2} g(x)\phi(x) dS(x).\end{aligned}$$

Similarly, by writing the solution u in the weak form (2.3) and the solution $\hat{u}$ in the weak form (4.2), and then using integration by substitution for the other terms, we see that $\hat{u} \in W^{1,p}(D^+_{2R})$ is a weak solution of

$$\left\{\begin{array}{rcll} \operatorname{div}[\hat{\mathbf{A}}(y,\lambda\hat{u},\nabla\hat{u})] &=& \operatorname{div}[\mathbf{G}], & \text{in} \quad D^+_{2R}, \\ \langle \hat{\mathbf{A}}(y,\lambda\hat{u},\nabla\hat{u}) - \mathbf{G}, \vec{e}_n\rangle &=& \hat{g}, & \text{on} \quad T_{2R}. \end{array}\right. \tag{5.9}$$

By Lemma 5.2 below, and our choice of Υ in (5.1), we see all conditions required in Theorem 1.3 hold for our equation (5.9). Therefore, it follows from Theorem 1.3 that

$$\|\nabla\hat{u}\|_{L^q(D^+_R)} \leq C\left[\left\||\mathbf{G}|^{\frac{1}{p-1}}\right\|_{L^q(D^+_{2R})} + R^{1/q}\left\||\hat{g}|^{\frac{1}{p-1}}\right\|_{L^q(B'_{2R})} + \|\nabla\hat{u}\|_{L^p(D^+_{2R})}\right]. \tag{5.10}$$

Now, by (5.2) we see that $\|\nabla\varphi\|_{L^\infty(B'_{2R})} \ll 1$. Then, it follows that

$$\int_{\Gamma_{2R}} |g(x)|^q dS(x) = \int_{B'_{2R}} |g(x',\varphi(x'))|^q\sqrt{1+|\nabla\varphi(x')|^2}dx' \sim \left\||\hat{g}|^{\frac{1}{p-1}}\right\|^q_{L^q(B'_{2R})}. \tag{5.11}$$

On the other hand, observe that by using the substitution $y = \Phi(x)$ and with (5.6), we have

$$\|\nabla\hat{u}\|^q_{L^q(D^+_R)} = \int_{D^+_R} |\nabla\hat{u}(y)|^q dy = \int_{\Psi(D^+_R)} |\nabla\hat{u}(\Phi(x))|^q dx = \int_{\Omega_R} |\nabla\hat{u}(\Phi(x))|^q dx.$$

Note that with $x = \Psi(y) \in \Omega_R$, $y = \Phi(x) \in D^+_{2R}$, and $\hat{u}(y) = u(\Psi(y))$, it follows from (5.5) that

$$|\nabla\hat{u}(y)|^2 = |[D\Psi(\Phi(x))]\nabla u(x)|^2 = |\nabla_{x'}u(x)|^2 + |\nabla_{x'}\varphi(x')\cdot\nabla_{x'}u(x) + D_n u(x)|^2$$

Again, from (5.2), we see that $\|\nabla\varphi\|_{L^\infty(B'_{2R})}$ is sufficiently small, and therefore

$$|\nabla\hat{u}(\Phi(x))| \sim |\nabla u(x)|.$$

Hence, $\|\nabla\hat{u}\|_{L^q(D^+_R)} \sim \|\nabla u\|_{L^q(\Omega_R)}$. Similarly, $\left\||\mathbf{G}|^{\frac{1}{p-1}}\right\|_{L^q(D^+_{2R})} \sim \|\mathbf{F}\|_{L^q(\Omega_{2R})}$. From these last two estimates, and from (5.10)-(5.11), we conclude that

$$\int_{\Omega_R(x_0)} |\nabla u|^q dx \leq C\left[\int_\Omega |\mathbf{F}|^q dx + \int_{\partial\Omega} |g(x)|^q dS(x) + \left(\int_\Omega |\nabla u|^p dx\right)^{q/p}\right]. \tag{5.12}$$

Step II: In this step, we estimate $|\nabla u|$ in the interior points in Ω. Consider a point $y_0 \in \Omega$. Let $R \in (0, \rho_0)$ sufficiently small such that $B_{2R}(y_0) \subset \Omega$. Then, from our choice of Υ as in (5.1), we can apply Theorem 1.2 to obtain

$$\int_{B_R(y_0)} |\nabla u(x)|^q dx \leq C \left[\int_\Omega |\mathbf{F}|^q dx + \left(\int_\Omega |\nabla u|^p dx \right)^{q/p} \right]. \tag{5.13}$$

Step III: In this step, we combine **Step I** and **Step II** to derive our global estimate (1.9). Indeed, because $\overline{\Omega}$ is compact, we can cover it by a family of finite number of balls of two types: balls in Ω with centers in Ω and balls with centers on the boundary $\partial\Omega$. Because of this, and from (5.12), (5.13), we see that (1.9) follows. Our proof is therefore complete. □

Next, we state and prove the following fundamental lemma that is used in the above proof.

Lemma 5.2. *There exist two constants $C_* = C_*(n, p) > 0$ and $C_0 = C_0(\Lambda, n, p) > 0$ such that the vector field $\hat{\mathbf{A}}$ defined in* (5.8) *satisfies the conditions* (1.2)-(1.4) *on $D^+_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ with Λ replaced by $C_*\Lambda$. Moreover,*

$$[[\hat{\mathbf{A}}]]_{\mathrm{BMO}(D^+_R, R)} \leq C_0(\Lambda, n, p)\Big([[\mathbf{A}]]_{\mathrm{BMO}(\Omega, \rho_0)} + \|\nabla \varphi\|_{L^\infty(B'_{2R})} \Big). \tag{5.14}$$

Proof. The proof that $\hat{\mathbf{A}}$ satifies (1.2)-(1.4) on $D^+_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ with Λ replaced by $C_*(n, p)\Lambda$ for some constant C_* follows from some elementary calculation. In this proof, we use the definition of $\hat{\mathbf{A}}$ in (5.8), the fact that $\mathbf{A}$ satisfies (1.2)-(1.4) on $\Omega \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$, the estimates (5.2)-(5.4), and the fact that Υ is sufficiently small. We skip the details of this proof.

Next, we prove the estimate (5.14). Note that for each $(y, z, \xi) \in D_{2R} \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$, we can write

$$\mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y))]) = \mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y)) - I_n] + \xi)$$

Then, by using the mean value theorem, we obtain

$$\mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y))]) = \mathbf{A}(\Psi(y), z, \xi) + \mathbf{A}_\xi(\Psi(y), z, \eta) \cdot [D\Phi(\Psi(y)) - I_n]\xi.$$

where $\mathbf{A}_\xi(\Psi(y), z, \eta)$ is the matrix of partial derivatives of $\mathbf{A}$ in ξ-variable and

$$\eta = \Big(s[D\Phi(\Psi(y))] + (1 - s)I_n \Big)\xi, \tag{5.15}$$

with some $s \in (0, 1)$. From this, we then can decompose the vector field $\hat{\mathbf{A}}$ as

$$\hat{\mathbf{A}}(y, z, \xi) = \mathbf{B}(y, z, \xi) + \mathbf{D}(y, z, \xi), \tag{5.16}$$

where

$$\begin{aligned} \mathbf{B}(y, z, \xi) &= \mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y))]) \cdot [D\Phi(\Psi(y)) - I_n]^* + \mathbf{A}_\xi(\Psi(y), z, \eta) \cdot [D\Phi(\Psi(y)) - I_n]\xi, \\ \mathbf{D}(y, z, \xi) &= \mathbf{A}(\Psi(y), z, \xi). \end{aligned} \tag{5.17}$$

We now estimate $[[\mathbf{B}]]_{\mathrm{BMO}(D^+_{2R}, R)}$ and $[[\mathbf{D}]]_{\mathrm{BMO}(D^+_{2R}, R)}$ with respect to their definitions in (1.11). Observe that from the explicit definition of $\mathbf{B}$ in (5.17) and the conditions (1.2)-(1.4), we see that

$$\begin{aligned} |\mathbf{B}(y, z, \xi)| &\leq |\mathbf{A}(\Psi(y), z, \xi[D\Phi(\Psi(y))])||D\Phi(\Psi(y)) - I_n| + |\mathbf{A}_\xi(\Psi(y), z, \eta)||D\Phi(\Psi(y)) - I_n||\xi| \\ &\leq \Lambda\Big[|\xi|^{p-1} \|D\Phi\|_{L^\infty(\Omega^+_{2R})} + |\eta|^{p-2}|\xi| \Big] \|D\Phi - I_n\|_{L^\infty(\Omega_{2R})} \\ &\leq \Lambda\Big[|\xi|^{p-1}\sqrt{n+1} + |\eta|^{p-2}|\xi| \Big] \|\nabla\varphi\|_{L^\infty(B'_{2R})}, \end{aligned} \tag{5.18}$$

where we have used (5.4) in the last estimate. On the other hand, as Υ is sufficiently small, it follows from (5.2) and (5.4) that $\|D\Phi - I_n\|_{L^\infty(\Omega_{2R})} \ll 1$. Then, from (5.15), we infer that

$$\frac{1}{2}|\xi| \leq |\eta| \leq 2|\xi|.$$

By pluging this last estimate into (5.18), we can find $C = C(\Lambda, n, p) > 0$ such that

$$|\mathbf{B}(y, z, \xi)| \leq C|\xi|^{p-1} \|\nabla\varphi\|_{L^\infty(B'_{2R})}.$$

From this and from the definition of the BMO-semi norm of the principal vector fields as in (1.11), we obtain

$$[[\mathbf{B}]]_{\mathrm{BMO}(D_R^+, R)} \leq 2 \sup_{z\in\mathbb{K}} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \|\mathbf{B}(\cdot, z, \xi)\|_{L^\infty(D_{2R}^+)} \leq C_1(\Lambda, n, p) \|\nabla\varphi\|_{L^\infty(B'_{2R})}, \tag{5.19}$$

for some constant $C_1 = C_1(\Lambda, n, p) > 0$. On the other hand, by (5.6) and (5.7), we see that $\Psi : D_{2R}^+ \to \Omega_{2R}$ is a homomorphism that is measure-preserving. Then, with some standard calculation using (5.6) and (5.7) and the definition of $\mathbf{D}$ in (5.17), we obtain

$$[[\mathbf{D}]]_{\mathrm{BMO}(D_R^+, R)} \leq C_2(n)[[\mathbf{A}]]_{\mathrm{BMO}(\Omega_R, R)} \leq C_2(n)[[\mathbf{A}]]_{\mathrm{BMO}(\Omega, \rho_0)}.$$

One can find in [22, p. 2164], for instance, the details of the proof of the above estimate. From the last estimate, (5.16), and (5.19), we infer that

$$[[\hat{\mathbf{A}}]]_{\mathrm{BMO}(D_R^+, R)} \leq C_0(\Lambda, n, p)\Big([[\mathbf{A}]]_{\mathrm{BMO}(\Omega, \rho_0)} + \|\nabla\varphi\|_{L^\infty(B'_{2R})}\Big).$$

This is our desired estimate (5.14). The proof of the lemma is therefore complete. □

We finally conclude the paper with a few remarks regarding the main results.

Remark 5.3. *We would like to point out the following important remarks on Theorem 1.1, Theorem 1.2, and Theorem 1.3.*

(i) *We do not require g to have an extension on Ω. Moreover, by the conormal boundary condition in* (1.1)*, and the conditions of the principal vector field* $\mathbf{A}$*, it can be seen that $|\nabla u| \sim g$. Therefore, the estimates in Theorem 1.1 and Theorem 1.3 seem to be natural and optimal regarding the regularity for g.*

(ii) *When $\lambda = 1$, Theorem 1.1 provides regularity estimates for weak solutions of* (1.1) *provided that $u \in \mathrm{BMO}$ with $M = [[u]]_{\mathrm{BMO}(\Omega, \rho_0)}$. If u is assumed to be in* VMO*, the Sarason's space of functions of vanishing mean oscillation defined in [39], then the condition $[[\lambda u]]_{\mathrm{BMO}(\Omega, \rho_0)} \leq M$ holds by taking $M = 1$ and ρ_0 is sufficiently small. This regularity condition on u is automatically satisfied if $p = n$.*

(iii) *It is well-known from [33] that the smallness conditions* (1.8)*,* (1.10)*, and* (1.11) *on $[[\mathbf{A}]]_{\mathrm{BMO}}$ are necessary. Moreover, if* $\mathbf{A}$ *is assumed to satisfy the Sarason's* VMO *vanishing mean oscillation in x-variable* (*see [1, 8, 14, 32]*)*, then* (1.8)*,* (1.10)*, and* (1.11) *hold for any given M and q by choosing R, ρ_0 sufficiently small.*

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