ON MASUDA UNIQUENESS THEOREM FOR LERAY-HOPF WEAK SOLUTIONS IN MIXED-NORM SPACES

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Abstract. We revisit the well-known work of K. Masuda in 1984 on the weak-strong uniqueness of $L^\infty L^3$ Leray-Hopf weak solutions of Navier-Stokes equation. We modify the argument, and extend the uniqueness result to the scaling critical anisotropic Lebesgue space with mixed-norms. As a consequence, our results cover the class of initial data and solutions which may be singular or decay with different rates along different spatial variables. The result relies on the establishment of several refined properties of solutions of the Stokes and Navier-Stokes equations in mixed-norm Lebesgue spaces which seem to be of independent interest.

1. Introduction

Consider an incompressible fluid moving in $\mathbb{R}^d$ with velocity $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ and pressure $P : \mathbb{R}^d \times (0,T) \to \mathbb{R}$, the Cauchy problem for the Navier-Stokes equations is written by

$$\left\{ \begin{array}{rclll} u_t - \Delta u + (u\cdot\nabla)u + \nabla P & = & 0, & \text{in} & \mathbb{R}^d \times (0,T), \\ \text{div}(u) & = & 0 & \text{in} & \mathbb{R}^d \times (0,T), \\ u(0,\cdot) & = & a_0(\cdot) & \text{in} & \mathbb{R}, \end{array} \right. \tag{1.1}$$

where $T \in (0,\infty], d \geq 3$ and $a_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a given divergence-free vector field representing the initial velocity. This paper revisits the well-known work of K. Masuda [28] and extends its uniqueness result for Leray-Hopf weak solutions of the Navier-Stokes equation (1.1) to the setting of critical anisotropic Lebesgue spaces. For the reader's convenience, let us recall the definition of Leray-Hopf weak solutions of (1.1).

Definition 1.1. Let $T \in (0,\infty]$, $a_0 \in L^2_\sigma(\mathbb{R}^d)$, the subspace of $L^2(\mathbb{R}^d)$ consisting of divergence free vector field functions. A function $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ satisfying $u \in L^\infty((0,T), L^2_\sigma(\mathbb{R}^d))$ and $\nabla u \in L^2(\mathbb{R}^d \times (0,T))$ is called a Leray-Hopf weak solution of (1.1) if the following hold.

(i) For every smooth compactly supported $\varphi : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ such that $\text{div}(\varphi) = 0$, we have

$$\int_0^T \int_{\mathbb{R}^d} \big[v \cdot \partial_t \varphi - \nabla u : \nabla \varphi + (u \otimes u) : \nabla \varphi\big] dxdt = 0.$$

(ii) The energy inequality

$$\sup_{t\in(0,\tau)} \int_{\mathbb{R}^d} |u(x,t)|^2 dx + 2\int_0^\tau \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dxdt \leq \int_{\mathbb{R}^d} |a_0(x)|^2 dx \tag{1.2}$$

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holds for a.e. $\tau \in [0,T)$.

(iii) For every $\varphi \in L^2(\mathbb{R}^d)$, the map

$$t \mapsto \int_{\mathbb{R}^d} u(x,t)\cdot\varphi(x)dx$$

is continuous on $[0,T)$.

(iv) The initial condition is satisfied in the $L^2(\mathbb{R}^d)$ sense

$$\lim_{t\to 0^+} \|u(\cdot,t) - a_0(\cdot)\|_{L^2(\mathbb{R}^d)} = 0.$$

The existence of a global time Leray-Hopf weak solution for (1.1) was first proved by J. Leray in the foundation work [23] (see also [22]). J. Leray's result can be summarized in the following theorem.

Theorem 1.2. *For each $a_0 \in L^2_\sigma(\mathbb{R}^3)$, there exists at least one Leray-Hopf weak solution of* (1.1) *in $\mathbb{R}^3\times(0,\infty)$.*

There are two open problems concerning Theorem 1.2.

(i) Given $a_0 \in L^2_\sigma(\mathbb{R}^3)$, is there a global time Leray-Hopf weak solution that is regular for all time?

(ii) Given $a_0 \in L^2_\sigma(\mathbb{R}^3)$, is the global time Leray-Hopf weak solution unique in the class of Leray-Hopf weak solution?

This paper concerns question (ii) on the uniqueness of Leray-Hopf weak solutions. We are particularly interested in the weak-strong uniqueness criterion for solutions in Lebesgue spaces that are invariant under the scaling. To put our work into perspectives, let us recall several known results in this direction. First of all, it should be noted that the solution set of the Navier-Stokes equation (1.1) is invariant under the scaling

$$u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \lambda > 0. \tag{1.3}$$

Then, it follows that

$$\|u^\lambda\|_{L^s((0,\infty),L^p(\mathbb{R}^d))} = \|u\|_{L^s((0,\infty),L^p(\mathbb{R}^d))}, \quad \forall\ \lambda > 0$$

if and only if

$$\frac{2}{s} + \frac{d}{p} = 1. \tag{1.4}$$

Under (1.4) and with $d = 3$, uniqueness of Leray-Hopf weak solutions which are in $L^s((0,T),L^p(\mathbb{R}^3))$ with $s \in [2,\infty)$ and $p > 3$ are proved in [29, 12, 21, 33]. Many other results in this direction with different functional spaces are also obtained, see [24, 2, 27] for examples, and [25, p. 361] and [35, p. 92] for discussion and more references.

The special case when $s = 2$ and $p = d = 3$ is non-trivial and it was resolved in the well-known work [28] by K. Masuda and [17] by H. Kozono and H. Sohr. See also [11] for some extension of this result. Indeed, [17] proves that as long as there is a Leray-Hopf weak solution u of (1.1) in $\mathbb{R}^3\times(0,T)$, $u \in L^\infty((0,T),L^3(\mathbb{R}^3))$ and $u : [0,T) \to L^3(\mathbb{R}^3)$ is right continuous, then there is only one Leray-Hopf weak solution of (1.1) on $\mathbb{R}^3\times(0,T)$. Later, the strong continuity of $u : [0,T) \to L^3(\mathbb{R}^3)$ is affirmed in the work [17] as long as u is a Leray-Hopf weak solution and $u \in L^\infty((0,T),L^3(\mathbb{R}^3))$. Among others, the results on the weak-strong uniqueness of solutions in [28, 17] are summarized in the following theorem.

Theorem 1.3. *Let u and v be Leray-Hopf weak solutions of* (1.1) *in* $\mathbb{R}^3 \times (0,T)$ *with the same initial data* $a_0 \in L^2_\sigma(\mathbb{R}^3) \cap L^3_\sigma(\mathbb{R}^d)$ *and with some* $T > 0$. *If* $u \in L^\infty((0,T), L^3(\mathbb{R}^3))$, *then* $u \equiv v$ *in* $\mathbb{R}^3 \times [0,T)$.

Note also that, similar to the uniqueness problem (ii) that we just mentioned, the problem (i) about the regularity of the Leray-Hopf weak solutions in critical scaling Lebesgue space sasitifying (1.4) has been studied by many mathematicians (see the books [25, 34, 35] for details and a complete list of references). Significantly, the smoothness of Leray-Hopf weak solution $u \in L^\infty((0,T), L^3(\mathbb{R}^3))$ was proved in the famous paper [20]. Recent extension of this result to larger functional spaces can be found in [32, 1], for examples.

This paper has two folds. On one hand, this work is inspired by Theorem 1.3 and we would like to revisit and highlight the beautiful ideas in the great work [28, 17]. On the other hand, we are interested in the anisotropic behavior of the initial data and solutions. This is specially motivated from the physical interpretation that the fluid behavior can be different in different directions. Therefore, understanding the solutions of Stokes, and Navier-Stokes equations in anisotropic functional spaces seem to be a topic of independent interest.

Throughout the paper, for a given $\vec{p} = (p_1, p_2, \ldots, p_d) \in [1,\infty)^d$, the mixed norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^d)$ is defined to be the space consisting of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that the norm

$$\|f\|_{L_{\vec{p}}(\mathbb{R}^d)} = \left(\int_{-\infty}^{\infty}\left(\cdots\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x)|^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\ldots\right)^{\frac{p_d}{p_{d-1}}}dx_d\right)^{\frac{1}{p_d}} \tag{1.5}$$
$$< \infty.$$

Similar definitions can be formulated if any of $\{p_1, p_2, \ldots, p_d\}$ is the same as ∞. We note that in case $p_1 = p_2 = \ldots = p_d = p$, the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^d)$ is reduced to the usual Lebesgue space $L^p(\mathbb{R}^d)$. Clearly, a significant feature of the mixed-norm $L^{\vec{p}}(\mathbb{R}^d)$ is that it captures functions that are singular or decay with different rates along different variable directions. Interested readers can read the classical work [3] for more details about this anisotropic mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^d)$.

For the Navier-Stokes equation (1.1), by a simple calculation, we can see that under this mixed-norm, and for u^λ defined in (1.3)

$$\|u^\lambda(t,\cdot)\|_{L^{\vec{p}}(\mathbb{R}^d)} = \|u(t,\cdot)\|_{L^{\vec{p}}(\mathbb{R}^d)}, \quad \forall\ \lambda > 0 \quad \forall\ t \in (0,T)$$

if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_d} = 1. \tag{1.6}$$

As before, $L^{\vec{p}}_\sigma(\mathbb{R}^d)$ denotes the subspace of $L^{\vec{p}}(\mathbb{R}^d)$ consisting of all divergence free vector field functions

$$L^{\vec{p}}_\sigma(\mathbb{R}^d) = \Big\{u \in L^{\vec{p}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} u(x)\cdot\nabla\varphi(x)dx = 0, \quad \forall\ \varphi \in C_0^\infty(\mathbb{R}^d)\Big\}.$$

In the setting of mixed-norm $L^{\vec{p}}(\mathbb{R}^d)$-spaces with (1.6), the main result of the paper is now stated in the following theorem on weak-strong uniqueness of Leray-Hopf weak solutions.

Theorem 1.4. *Let $\vec{p} = (p_1, p_2, \ldots, p_d) \in [2,\infty)^d$ such that* (1.6) *holds and $p_d \neq 2$. Suppose that u, v are two Leray-Hopf weak solutions of the Navier-Stokes* (1.1) *on $\mathbb{R}^d \times (0,T)$ with the same initial data $a_0 \in L^2_\sigma(\mathbb{R}^d) \cap L^{\vec{p}}_\sigma(\mathbb{R}^d)$ and with some $T \in (0,\infty]$. If $u \in L^\infty_{\rm loc}((0,T), L^{\vec{p}}(\mathbb{R}^d))$, then $u \equiv v$ in $\mathbb{R}^d \times [0,T)$.*

Observe that in the special case when $d = 3$ and $p_1 = p_2 = p_d = p$, (1.6) implies that $p = 3$. Therefore, Theorem 1.4 recovers the classical Masuda uniqueness theorem, Theorem 1.3. Note that Theorem 1.4 allows that some of $p_k, k = 1, 2 \ldots, d-1$ can be the same as 2, and therefore the condition on initial data a_0 and the solutions are more relaxed compared to those of Theorem 1.3. For example, consider a function in the form $g(x) = g_0(x_1)g_1(x')$ with $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and take $\vec{p} = (2, p_2, \ldots, p_d)$ satisfying (1.6). Then, the condition $g \in L^2(\mathbb{R}^d) \cap L^{\vec{p}}(\mathbb{R}^d)$ is equivalent to

$$g_0 \in L^2(\mathbb{R}) \quad \text{and} \quad g_1 \in L^2(\mathbb{R}^2) \cap L^{\vec{p}\,'}(\mathbb{R}^{d-1}), \quad \text{for} \quad \vec{p}\,' = (p_2, \ldots, p_d).$$

Meanwhile, $g \in L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^d)$ is equivalent to

$$g_0 \in L^2(\mathbb{R}) \cap L^3(\mathbb{R}) \quad \text{and} \quad g_1 \in L^2(\mathbb{R}^{d-1}) \cap L^{\vec{p}\,'}(\mathbb{R}^{d-1}).$$

In other words, in the setting of anisotropic spaces, no more requirement except the initial $L^2(\mathbb{R})$-condition is needed for the part g_0 of g.

Theorem 1.4 is based on an establishment of some new results on properties of solutions of Stokes and Navier-Stokes equations in the anisotropic space $L^{\vec{p}}(\mathbb{R}^d)$. More precisely, several estimates of the Stokes semi-group in mixed-norm $L^{\vec{p}}(\mathbb{R}^d)$-spaces will be derived. Further regularity estimates of mild solutions in the mixed-norm space $L^{\vec{p}}(\mathbb{R}^d)$ obtained recently in [30] are investigated. These results seem to be new and and they strongly demonstrate the persistence of the anisotropic behavior of the initial data under the evolution of the Stokes and Navier-Stokes equations. Note that $L^{\vec{p}}(\mathbb{R}^d) \subset \dot{B}^{-1+\frac{d}{\bar{p}}}_{\bar{p},\infty}(\mathbb{R}^d)$, where $\dot{B}^{-1+\frac{d}{\bar{p}}}_{\bar{p},\infty}(\mathbb{R}^d)$ is the homogeneous Besov space with negative regularity index $-1+\frac{d}{\bar{p}}$ and $\bar{p} = \max\{p_1, p_2, \ldots, p_d\} > d$. When $d = 3$ and with a sufficiently small time, the weak-strong uniqueness of Leray-Hopf weak solution with initial data $a_0 \in \dot{B}^{-1+\frac{3}{\bar{p}}}_{\bar{p},\infty}(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ and $\bar{p} \in (3,\infty)$ is obtained in [2, Corollary 1.4]. See also [5, 6] for earlier results in slightly different spaces, and [7, Theorem 1.5] for another weak-strong uniqueness result with initial data in $\dot{H}^s \cap \widetilde{BMO}^{-1} \cap L^2_\sigma(\mathbb{R}^3)$ for some $s > 0$, where $\widetilde{BMO}^{-1}$ is the closure in BMO^{-1} of the set of compactly supported smooth functions. On the other hand, when $d \geq 4$, the weak-strong uniqueness of Leray-Hopf weak solutions in $L^\infty((0,T), \dot{B}^{-1+\frac{d}{\bar{p}}}_{\bar{p},\bar{q}}(\mathbb{R}^d))$ when $4 \leq d < \bar{p}, \bar{q} < \infty$ is proved in [27, Corollary 1.7], where $T > 0$ and may be large. Clearly, Theorem 1.4 is not covered by the mentioned results. Moreover, our analysis and approach here are slightly different from [2, 5, 7, 27] as we follow the spirits of Masuda work [28], and parts of this work focus on the anisotropic behavior of solutions of the Stokes and Navier-Stokes equations, which could be of physical interest as explained earlier. It seems to be an interesting problem to extend the Masuda weak-strong uniqueness result to the borderline cases such as $L^\infty((0,T), \dot{B}^{-1+\frac{d}{\bar{p}}}_{\bar{p},\infty}(\mathbb{R}^d)$ or $L^\infty((0,T), \mathrm{BMO}^{-1}(\mathbb{R}^3))$ given the existence results obtained in the well-known paper [16], where $T > 0$ and may be large. We plan to come back to this problem in the near future.

We would like to point out that the mixed-norm Lebesgue spaces were introduced in [3]. Due to various interests, the analysis theory on wellposedness and regularity estimates in mixed-norm Lebesgue spaces and weighted mixed-norm Lebesgue spaces is extensively developed for elliptic and parabolic equations. For examples, see [18, 8], the survey paper [19] and references therein. The local regularity estimates in weighted mixed-norm Lebesgue spaces for solutions to Stokes systems with variable coefficients are just recently developed in [10, 9]. Following this line of research and this paper, it seems to be an interesting problem to extend results on energy equality for Leray-Hofp weak solutions Navier-Stokes equations such as [26, 36, 14, 15] to the setting of mixed-norm Lebesgue spaces.

The rest of the paper is organized as follows. In Section 2, we recall and prove several analysis estimates in mixed-norm spaces. In particular, the Sobolev embedding theorem in mixed-norm spaces will be recalled. Several estimates of the Stokes semi-group will be introduced and proved. The existence of the mild solutions of Navier-Stokes equations in the critical mixed-norm spaces will be stated. The proof of Theorem 1.4 is provided in Section 3. We follow the approach used in [28, 17] and adapt it to the setting in mixed-norm spaces. The key step in this approach is to prove that the mild solution in mixed-norm spaces is a Leray-Hopf weak solution. This is done in Proposition 3.4 for which the estimates for mixed-norm spaces in Section 2 that we just mentioned are essential.

2. Preliminary inequalities and estimates in mixed-norm spaces

2.1. **Analysis inequalities and Sobolev embedding in mixed-norm spaces.**

We recall the following theorem which is a special case of the Sobolev embedding theorem in mixed-norm spaces proved in [4, p. 181].

Theorem 2.1 (Mixed-norm Sobolev Embedding). *Let* $\vec{q} = (q_1, q_2, \ldots, q_d) \in [2,\infty]^d$ *satisfy*

$$\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_d} = \frac{d}{2} - 1 \quad \text{and} \quad q_d \in (2, \infty).$$

Then there exist constants $N = N(\vec{q})$ *such that*

$$\|u\|_{L^{\vec{q}}(\mathbb{R}^d)} \leq N\Big[\|Du\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}\Big], \quad \forall\ u \in W^{1,2}(\mathbb{R}^d)$$

where the mixed-norm $L^{\vec{q}}(\mathbb{R}^d)$ *is defined as in* (1.5).

Next, we recall the following classical lemma about mixed-norm estimate of the Riesz transform, its proof is can be found in, for example, [30, Theorem 2.23] in which the ideas developed in [19, 8] were used.

Lemma 2.2. *Let* $\vec{p} = (p_1, p_2, \ldots, p_d) \in (1,\infty)^d$. *Then, there exists a constant* $N = N(\vec{p}) > 0$ *such that*

$$\|\mathcal{R}_i(f)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq N\|f\|_{L^{\vec{p}}(\mathbb{R}^d)} \quad \forall\ f \in L^{\vec{p}}(\mathbb{R}^d),$$

where $\mathcal{R}_i$ *is the i-th Riesz transform defined by* $\mathcal{R}_i(f) = \partial_{x_i}(-\Delta)^{-1/2}(f)$ *for* $i = 1, 2, \ldots, d$.

We now conclude this subsection with the following elementary result that will be used in the paper.

Lemma 2.3. *Let $f \in C^1((0,\infty))$ such that*

$$N = \sup_{s>0}[|f(s)| + s|f^{'}(s)|] < \infty.$$

Then,

$$|f(|\zeta|)\zeta - f(|\eta|)\eta)| \leq N|\zeta - \eta|, \quad \forall\ \zeta, \eta \in \mathbb{R}^d.$$

Proof. We provide the proof for completeness. For fixed $\eta, \zeta \in \mathbb{R}^d$, let $x(t) = \eta + t(\zeta - \eta)$. We note that

$$|\frac{d}{dt}|x(t)|| = |\frac{x(t)\cdot x'(t)}{|x(t)|}| \leq |x'(t)| \leq |\zeta - \eta|.$$

Then, it follows that

$$\begin{aligned}
|\zeta f(\zeta) - \eta f(\eta)| =& |f(|x(1)|)x(1) - f(|x(0)|)x(0)| \\
=& \Big| \int_0^1 [f(|x(t)|)x(t)]' dt \Big| \\
\leq& \int_0^1 |x' f(|x|) + x f(|x|)|x|'| dt \\
\leq& \int_0^1 |x'| f(|x|) dt + \int_0^1 |x| f'(|x|)||x|'| dt \\
\leq& |\zeta - \eta| \int_0^1 [f(|x|) + |x||f'(|x|)|] dt \\
\leq& N|\zeta - \eta|.
\end{aligned}$$

■

2.2. **Stokes and Navier-Stokes equations in mixed-norm spaces.** We recall and introduce several results and estimates of solutions for Stokes and Navier-Stokes equations in mixed-norm spaces that are needed in this paper. Let $\mathcal{P}(\cdot) = (\mathrm{Id} - \nabla\Delta^{-1}\nabla)(\cdot)$ be the Helmholtz-Leray projection onto the space of divergence-free vector fields. It is important to note that $\mathcal{P} : L^{\vec{p}}(\mathbb{R}^d) \to L^{\vec{p}}(\mathbb{R}^d)$ is bounded for all $\vec{p} \in (1,\infty)^d$, see Lemma 2.2 and also [30, Corollary 2.25]. Also, let us denote

$$\mathcal{A} = -\mathcal{P}\Delta = -\Delta\mathcal{P}.$$

We have the following estimate for the semi-group $e^{-\mathcal{A}t}$ of the Stokes equations in mixed-norm Lebesgue spaces, which is partially derived in [30, Lemma 3.5].

Lemma 2.4. *Let $\vec{p} = (p_1, p_2, \ldots, p_d), \vec{q} = (q_1, q_2, \ldots, q_d) \in (1,\infty)^d$ such that $q_k \geq p_k$ for all $k = 1, 2, \ldots, d$. Then, there exists $N = N(d, \vec{p}, \vec{q})$ such that*

$$\begin{aligned}
&\|e^{-\mathcal{A}t}\mathcal{P}a\|_{L^{\vec{q}}(\mathbb{R}^d)} \leq N t^{-\frac{\sigma}{2}} \|a\|_{L^{\vec{p}}(\mathbb{R}^d)} \\
&\|e^{-\mathcal{A}t}\mathcal{P}\partial_{x_k} a\|_{L^{\vec{q}}(\mathbb{R}^d)} \leq N t^{-\frac{1}{2}-\frac{\sigma}{2}} \|a\|_{L^{\vec{p}}(\mathbb{R}^d)}, \quad k = 1, 2, \ldots, d,
\end{aligned}$$

for all $a \in L^{\vec{p}}(\mathbb{R}^d)$, where

$$\sigma = \sum_{k=1}^d \Big[\frac{1}{p_k} - \frac{1}{q_k}\Big].$$

Proof. The proof is standard, for instance, see [30, Lemma 3.5]. In particular, for the second assertion, we write

$$\begin{aligned}\|e^{-\mathcal{A}t}\mathcal{P}\partial_{x_k}a\|_{L^{\vec{q}}(\mathbb{R}^d)} &= \|\mathcal{A}^{1/2}e^{-\mathcal{A}t}\mathcal{A}^{-1/2}\mathcal{P}\partial_{x_k}a\|_{L^{\vec{q}}(\mathbb{R}^d)} \\ &\leq \|\mathcal{A}^{1/2}e^{-\mathcal{A}t}\|_{L^{\vec{p}}_\sigma(\mathbb{R}^d)\to L^{\vec{q}}_\sigma(\mathbb{R}^d)}\|\mathcal{A}^{-1/2}\mathcal{P}\partial_{x_k}a\|_{L^{\vec{p}}(\mathbb{R}^d)} \\ &\leq Nt^{-\frac{1}{2}-\frac{\sigma}{2}}\|a\|_{L^{\vec{p}}(\mathbb{R}^d)},\end{aligned}$$

where in the last estimate, we used the second assertion and the fact that $\mathcal{A}^{-1/2}\mathcal{P}\partial_{x_k}: L^{\vec{p}}(\mathbb{R}^d) \to L^{\vec{p}}(\mathbb{R}^d)$ is bounded, which follows from Lemma 2.2. ■

Next, we introduce the following fundamental result about time smoothing estimate of the semi-group $e^{-\mathcal{A}t}$ in mixed-norm $L^{\vec{p}}(\mathbb{R}^d)$-spaces. The result seems to be new and it is important in the proof of Theorem 1.4.

Lemma 2.5. *Let $\vec{p} = (p_1, p_2, \ldots, p_d) \in (1,\infty)^d$ such that (1.6) holds. Let $\vec{q} = (q_1, q_2, \ldots, q_d) \in (1,\infty)^d$. Assume that*

$$q_k \geq p_k \quad \text{for all} \quad k \in \{1, 2, \ldots, d-1\}, \quad q_d > p_d,$$

and

$$1 - \frac{2}{p_d} < \delta := \frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_d}. \tag{2.1}$$

Then, for $l \in (2,\infty)$ so that

$$\frac{2}{l} + \delta = 1, \tag{2.2}$$

there exists $N = N(d, \vec{p}, \vec{q}, l) > 0$ such that

$$\|e^{-\mathcal{A}t}\mathcal{P}a\|_{L^l((0,\infty),L^{\vec{q}}(\mathbb{R}^d))} \leq N\|a\|_{L^{\vec{p}}(\mathbb{R}^d)}, \quad \forall\, a \in L^{\vec{p}}(\mathbb{R}^d).$$

Proof. We combine results in Lemma 2.4 with an interpolation. Recall that for $\alpha \in [1,\infty)$, a function $g : (0,\infty) \to \mathbb{R}$ is said to be in $L^\alpha_w(0,\infty)$, the weak-$L^\alpha(0,\infty)$ space, if its norm

$$\|g\|_{L^\alpha_w((0,\infty))} = \sup_{\tau>0} \tau|\{|g(t)| > \tau\}|^{1/\alpha} < \infty.$$

In particular, a simple calculation shows that

$$\|g\|_{L^\alpha_w((0,\infty))} = 1 \quad \text{for} \quad g(t) = t^{-1/\alpha}, \quad t \in (0,\infty). \tag{2.3}$$

Now, let us denote

$$U[a](t) = \|e^{-\mathcal{A}t}\mathcal{P}a\|_{L^{\vec{q}}}.$$

Then, for every $r \in (1, q_d]$ such that

$$\sigma = \sigma(r) = \frac{1}{r} + \sum_{k=1}^{d-1}\frac{1}{p_k} - \sum_{k=1}^{d}\frac{1}{q_k} = \frac{1}{r} + \big(1 - \frac{1}{p_d}\big) - \delta \geq 0, \tag{2.4}$$

it follows from Lemma 2.4 that

$$U[a](t) \leq Nt^{-\frac{\sigma}{2}}\|a\|_{L^r(\mathcal{X})}, \quad \forall\, a \in L^r(\mathcal{X}),$$

where

$$\mathcal{X} = \{\text{measurable } g : \mathbb{R} \to L^{\vec{p}'}(\mathbb{R}^{d-1})\}, \quad \vec{p}' = (p_1, p_2, \ldots p_{d-1}),$$

and $N = N(d, \vec{p}', r, \vec{q}) > 0$. This last estimate and (2.3) imply that

$$\|U[a]\|_{L^{\frac{2}{\sigma}}_w(0,\infty)} \leq N\|a\|_{L^r(\mathcal{X})}, \quad \forall\, a \in L^r(\mathcal{X}),$$

which means that U is of weak type $(r, \frac{2}{\sigma})$ if $\sigma = \sigma(r) > 0$. Observe that as $p_d < q_d$ we have $\delta < 1$ and then it follows from (2.4) that $\sigma(p_d) = 1 - \delta \in (0,1)$. Therefore, for every r_0, r_1 sufficiently close to p_d and $r_0 < p_d < r_1 < q_d$, by the continuity of σ in r, we see that $0 < \sigma(r_1) < \sigma(p_d) < \sigma(r_0) < 1$ and then

$$2 < l_0 := \frac{2}{\sigma(r_0)} < \frac{2}{\sigma(p_d)} < l_1 := \frac{2}{\sigma(r_1)}.$$

Note also that it follows from (2.4) that $l = \frac{2}{\sigma(p_d)}$. Moreover, for $y(r) = \frac{2}{\sigma(r)} - r$, we find from (2.1) and (2.4) that $y(p_d) = l - p_d > 0$. Then, by using the continuity of the function y at $r = p_d$, we can chose r_0, r_1 sufficiently close to p_d so that $y(r_0) > 0$ and $y(r_1) > 0$. Consequently, we have

$$r_0 < l_0 \quad \text{and} \quad r_1 < l_1.$$

From the above choice of r_0, r_1 and as U is simultaneously of weak types (r_0, l_0) and (r_1, l_1), we apply the Marcinkiewic's interpolation to conclude that $U : L^{p_d}(\mathcal{X}) \to L^l(0,\infty)$ is bounded and therefore

$$\|e^{-\mathcal{A}t}\mathcal{P}a\|_{L^l((0,\infty), L^{\vec{q}}(\mathbb{R}^d))} \leq N\|a\|_{L^{\vec{p}}(\mathbb{R}^d)}, \quad \forall\ a \in L^{\vec{p}}(\mathbb{R}^d).$$

The proof of the lemma is completed. ∎

Remark 2.6. It is possible that Lemma 2.5 is still true when the strict inequality in (2.1) is replaced by the equality. However, we do not pursue this direction as it is not needed in this paper. When $d = 3$ and with the un-mixed norm case, i.e. $p = p_1 = p_2 = p_d = 3$ and $q = q_1 = q_2 = q_d$, we see that (2.1) is equivalent to $q < 9$. Therefore, Corollary 2.1 recovers the classical result in the un-mixed norm case, see [35, Lemma 5.2]. It is important to point out that the un-mixed norm time smoothing estimate is still true for $q = 9$ (see [13, Acknowledgement]), but the case $q > 9$ seems to be open (see [35, Lemma 5.2 and p. 83]).

Next, we denote

$$G(u,v)(t) = -\int_0^t e^{-(t-s)\mathcal{A}} F(u(\cdot,s), v(\cdot,s))ds, \tag{2.5}$$

where

$$F(u,v) = \mathcal{P}((u(\cdot,s)\cdot\nabla)v(\cdot,s)).$$

We observe that the Navier-Stokes equation (1.1) can be recast as an integral equation as

$$u = u_1 + G(u,u), \quad \text{where} \quad u_1(t) = e^{-\mathcal{A}t}a_0. \tag{2.6}$$

Our next result gives the estimates of the bilinear form G.

Lemma 2.7. *Let $z_i \in (1,\infty)$ and $\alpha_i, \beta_i, \gamma_i \in (0,1]$ be given numbers satisfying*

$$\gamma_i \leq \alpha_i + \beta_i \leq z_i, \quad \forall\ i = 1,2,\ldots,d. \tag{2.7}$$

We write

$$\vec{\alpha} = \Big(\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}, \ldots, \frac{z_d}{\alpha_d}\Big), \quad \vec{\beta} = \Big(\frac{z_1}{\beta_1}, \frac{z_2}{\beta_2}, \ldots, \frac{z_d}{\beta_d}\Big), \quad \vec{\gamma} = \Big(\frac{z_1}{\gamma_1}, \frac{z_2}{\gamma_2}, \ldots, \frac{z_d}{\gamma_d}\Big),$$

and denote

$$\bar{\alpha} = \sum_{i=1}^d \frac{\alpha_i}{z_i}, \quad \bar{\beta} = \sum_{i=1}^d \frac{\beta_i}{z_i}, \quad \text{and} \quad \bar{\gamma} = \sum_{i=1}^d \frac{\gamma_i}{z_i}.$$

Then, there exists $N = N(d, \vec{\alpha}, \vec{\beta}, \vec{\gamma}) > 0$ *such that*

$$\|G(u,v)(t)\|_{L^{\vec{\gamma}}(\mathbb{R}^d)} \leq N \int_0^t (t-s)^{-\frac{\bar{\alpha}+\bar{\beta}-\bar{\gamma}}{2}} \|u(\cdot,s)\|_{L^{\vec{\alpha}}(\mathbb{R}^d)} \|\nabla v(\cdot,s)\|_{L^{\vec{\beta}}(\mathbb{R}^d)} ds, \quad (2.8)$$

and

$$\begin{aligned} &\|\nabla G(u,v)(t)\|_{L^{\vec{\gamma}}(\mathbb{R}^d)} \\ &\leq N \int_0^t (t-s)^{-\frac{1+\bar{\alpha}+\bar{\beta}-\bar{\gamma}}{2}} \|u(\cdot,s)\|_{L^{\vec{\alpha}}(\mathbb{R}^d)} \|\nabla v(\cdot,s)\|_{L^{\vec{\beta}}(\mathbb{R}^d)} ds, \end{aligned} \quad (2.9)$$

Moreover, for $u \in L_\sigma^{\vec{\alpha}}(\mathbb{R}^d)$ *and* $v \in L_\sigma^{\vec{\beta}}(\mathbb{R}^d)$, *we also have*

$$\begin{aligned} &\|G(u,v)(t)\|_{L^{\vec{\gamma}}(\mathbb{R}^d)} \\ &\leq N \int_0^t (t-s)^{-\frac{1+\bar{\alpha}+\bar{\beta}-\bar{\gamma}}{2}} \|u(\cdot,s)\|_{L^{\vec{\alpha}}(\mathbb{R}^d)} \|v(\cdot,s)\|_{L^{\vec{\beta}}(\mathbb{R}^d)} ds. \end{aligned} \quad (2.10)$$

Proof. Note that (2.8) and (2.9) are proved in [30, Lemma 3.4]. Indeed, we observe that [30, Lemma 3.4] is stated with the condition

$$\gamma_i \leq \alpha_i + \beta_i < z_i, \quad \forall\ i = 1, 2, \ldots, d,$$

meaning that the second inequality in (2.7) is required to be a strict inequality. However, by observing the proof in [30, Lemma 3.4], one can easily see that this strictness requirement is not needed as we only need to use Lemma 2.4 and Hölder's inequality. A similar result can be seen in [31, Lemma 3.1]. We therefore skip the details of the proof of (2.8) and (2.9). It then remains to prove (2.10). Observe that when u and v are divergence free, we can write

$$G(u,v)(t) = -\int_0^t e^{-(t-s)\mathcal{A}} \mathcal{P} \nabla \cdot (u(\cdot,s) \otimes v(\cdot,s)) ds.$$

Therefore, by the last assertion in Lemma 2.4, it follows that

$$\begin{aligned} &\|G(u,v)(t)\|_{L^{\vec{\gamma}}(\mathbb{R}^d)} \\ &\leq N \int_0^t (t-s)^{-\frac{1+\bar{\alpha}+\bar{\beta}-\bar{\gamma}}{2}} \|u(\cdot,s) \otimes v(\cdot,s)\|_{L^{\frac{z_1}{\alpha_1+\beta_1}, \ldots, \frac{z_d}{\alpha_d+\beta_d}}(\mathbb{R}^d)} ds. \end{aligned}$$

Then, as

$$\frac{\alpha_k + \beta_k}{z_k} = \frac{\alpha_k}{z_k} + \frac{\beta_k}{z_k}, \qquad \forall\ k = 1, 2, \ldots, d$$

we can apply Hölder's inequality to derive (2.10). The proof of the lemma is then completed. ∎

Now, for each $\vec{p} = (p_1, p_2, \ldots, p_d) \in (1, \infty)^d$ such that (1.6) holds, and for each $\vec{q} = (q_1, q_2, \ldots, q_d)$ such that $q_k \in [p_k, \infty)$ for $k = 1, 2, \ldots, d$, set

$$\delta := \frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_d} \in (0, 1). \quad (2.11)$$

Then, with each $T > 0$, let us denote $\mathcal{X}_{\vec{p}, \vec{q}, T}$ be the space consisting of all measurable vector field functions $f : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d$ such that for

$$g(x,t) = t^{\frac{1-\delta}{2}} f(x,t) \quad \text{and} \quad \tilde{g}(x,t) = t^{\frac{1}{2}} D_x f(x,t) \quad (x,t) \in \mathbb{R}^d \times (0,T)$$

we have

$$g \in C([0,T), L_\sigma^{\vec{q}}(\mathbb{R}^d)) \quad \tilde{g} \in C([0,T), L_\sigma^{\vec{p}}(\mathbb{R}^d)).$$

and the space $\mathcal{X}_{\vec{p},\vec{q},T}$ is endowed with the norm

$$\|f\|_{\mathcal{X}_{p,q,T}} = \sup_{t\in(0,T)} \left[\|g(\cdot,t)\|_{L^{\vec{q}}(\mathbb{R}^d)} + \|\tilde{g}(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)}\right] < \infty.$$

In a similar manner, $\mathcal{Y}_{\vec{p},T}$ consists of all measurable vector field functions $f : \mathbb{R}^d \times [0,T) \to \mathbb{R}^d$ such that

$$f \in C([0,T), L^{\vec{p}}_{\sigma}(\mathbb{R}^d)) \quad \text{in} \quad t^{\frac{1}{2}} D_x f \in C([0,T), L^{\vec{p}}_{\sigma}(\mathbb{R}^d))$$

and it is endowed with the norm

$$\|f\|_{\mathcal{Y}_{\vec{p},T}} = \sup_{t\in(0,T)} \left[\|f(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)} + t^{\frac{1}{2}}\|D_x f(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)}\right] < \infty.$$

We now conclude this subsection by recalling the following result on the existence of mild solutions of (1.1) in mixed-norm spaces which is due to [30, Theorem 1.9].

Theorem 2.8. *Let $\vec{p} = (p_1, p_2, \ldots, p_d) \in (1,\infty)^d$ and $\vec{q} = (q_1, q_2, \ldots, q_d) \in (1,\infty)^d$. Assume that $q_i \geq p_i$ for $i = 1, 2, \ldots, d$, (1.6) and (2.11) holds, and*

$$\frac{p_i}{q_i} + 1 \leq p_i, \quad i = 1, 2, \ldots, d. \tag{2.12}$$

Then for every $a_0 \in L^{\vec{p}}_{\sigma}(\mathbb{R}^d)$ there exists $T_0(\vec{p}, \vec{q}, a_0) > 0$ sufficiently small such that there is a unique local time mild solution $u \in \mathcal{X}_{\vec{p},\vec{q},T_0} \cap \mathcal{Y}_{\vec{p},T_0}$ to the Navier-Stokes equation (2.6). Moreover,

$$\|u\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}} \leq N\|a_0\|_{L^{\vec{p}}(\mathbb{R}^d)}$$

and

$$\|u\|_{\mathcal{Y}_{\vec{p},T_0}} \leq N\left[\|a_0\|_{L^{\vec{p}}(\mathbb{R}^d)} + \|a_0\|^2_{L^{\vec{p}}(\mathbb{R}^d)}\right],$$

where $N = N(d, \vec{p}, \vec{q})$.

Remark 2.9. We would like to point out that [30, Theorem 1.9] is stated under the restriction that $p_i \geq 2$ for $i = 1, 2, \ldots, d$. However, by following the proof of [30, Theorem 1.9], we see that the theorem still holds for $p_i \in (1,\infty)$ with an additional condition (2.12). Certainly, (2.12) is trivial if $p_i \geq 2$ for $i = 1, 2, \ldots, d$. A similar result for dissipative quasi-geostrophic equation can be found in [31, Theorem 1.1].

3. Proof of Theorem 1.4

This section provides the proof of Theorem 1.4. We need to better understand the mild solutions in mixed-norm space obtained in Theorem 2.8. We begin with the following lemma which shows that when the initial data $a_0 \in L^2(\mathbb{R}^d) \cap L^{\vec{p}}(\mathbb{R}^d)$, the solution u obtained in Theorem 2.8 is indeed in $L^\infty((0,T_0), L^2(\mathbb{R}^d))$.

Lemma 3.1. *Let $\vec{p} = (p_1, p_2, \ldots, p_d) \in [2,\infty)^d$ satisfy (1.6) and $a_0 \in L^{\vec{p}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then, for the solution u of (1.1) on $\mathbb{R}^d \times (0,T_0)$ obtained in Theorem 2.8, we have $u \in L^\infty((0,T), L^2(\mathbb{R}^d))$ for sufficiently small $T \in (0,T_0]$, and moreover $u : [0,T] \to L^2(\mathbb{R}^d)$ is continuous.*

Proof. We follow the standard approach using our mixed-norm estimates in Lemma 2.7. We provide it here for completeness. Let $\vec{q}$ be as in Theorem 2.8 and let δ be defined as in (2.11). Recall that $u \in \mathcal{X}_{\vec{p},\vec{q},T_0} \cap \mathcal{Y}_{\vec{p},\vec{q},T_0}$ and satisfies

$$u = u_1 + G(u,u),$$

where G and u_1 are defined in (2.5) and (2.6). Moreover, by the construction, u is the limit of the sequence $\{u_m\}_m$ in $\mathcal{X}_{\vec{p},\vec{q},T_0} \cap \mathcal{Y}_{\vec{p},\vec{q},T_0}$, where $\{u_m\}_m$ is defined by the Picard's iteration

$$u_m = u_1 + G(u_{m-1}, u_{m-1}), \quad m \geq 2, \quad \text{and} \quad u_1 = e^{-\mathbb{A}t}a_0.$$

To prove the claim, it is sufficient to show that $\{u_m\}_m$ is Cauchy in $L^\infty((0,T), L^2(\mathbb{R}^d))$ with sufficiently small $T \in (0, T_0]$. Observe that for all $m, n \in \mathbb{N}$

$$\begin{aligned} \|u_{m+1} - u_{n+1}\|_{L^2(\mathbb{R}^d)} =& \|(u_1 + G(u_m, u_m)) - (u_1 + G(u_n, u_n))\|_{L^2(\mathbb{R}^d)} \\ =& \|G(u_m, u_m) - G(u_m, u_n) + G(u_m, u_n) - G(u_n, u_n)\|_{L^2(\mathbb{R}^d)} \\ \leq& \|G(u_m, u_m - u_n)\|_{L^2(\mathbb{R}^d)} + \|G(u_m - u_n, u_n)\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.1}$$

Also, by using the assumption that $p_i \geq 2$, we can use (2.8) in Lemma 2.7 with $\alpha_i = \frac{2}{p_i} \leq 1$, $\beta_i = \gamma_i = 1$, and $z_i = 2$ to obtain

$$\begin{aligned} &\|G(u_m, u_m - u_n)\|_{L^2(\mathbb{R}^d)} \\ &\leq N \int_0^t (t-s)^{-\frac{1}{2}} \|u_m\|_{L^2(\mathbb{R}^d)} \|D_x(u_m - u_n)\|_{L^{\vec{p}}(\mathbb{R}^d)} ds \\ &\leq N \|u_m\|_{L^\infty((0,T_0), L^2(\mathbb{R}^d))} \|u_m - u_n\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \end{aligned}$$

By a simple calculation, we see that

$$\begin{aligned} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds &= \int_0^{t/2} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + \int_{t/2}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \\ &\leq 2 + \sqrt{2}, \quad \forall\, t > 0, \end{aligned}$$

Then, we have

$$\|G(u_m, u_m - u_n)\|_{L^2(\mathbb{R}^d)} \leq N \|u_m\|_{L^\infty((0,T_0), L^2(\mathbb{R}^d))} \|u_m - u_n\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}}. \tag{3.2}$$

On the other hand, for every $T \in (0, T_0]$, by applying Lemma 2.7 on u_{m+1} directly like what we just did, we have

$$\begin{aligned} \|u_{m+1}\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} &\leq \|u_1\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} + \|G(u_m, u_m)\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} \\ &\leq \|a_0\|_{L^2(\mathbb{R}^d)} + N \|u_m\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} \|u_m\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}}. \end{aligned}$$

Then, as $\|u_m\|_{X_{p,q,T_0}} \to 0$ when $T_0 \to 0^+$ and $\{u_m\}_m$ is Cauchy in $\mathcal{X}_{\vec{p},\vec{q},T_0}$, we can choose $T \in (0, T_0]$ sufficiently small such that

$$\|v_{m+1}\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} \leq \|a_0\|_{L^2(\mathbb{R}^d)} + \frac{1}{2} \|u_m\|_{L^\infty((0,T), L^2(\mathbb{R}^d))}, \quad \forall\, m \in \mathbb{N}.$$

By iterating this estimate, we obtain

$$\|u_m\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} \leq 2 \|a_0\|_{L^2(\mathbb{R}^d)}, \quad \forall\, m \in \mathbb{N}. \tag{3.3}$$

Then, we feed (3.3) into (3.2) to obtain

$$\|G(u_m, u_m - u_n)\|_{L^2(\mathbb{R}^d)} \leq N \|a_0\|_{L^2(\mathbb{R}^d)} \|u_m - u_n\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}}.$$

Similarly, we also have

$$\|G(u_m - u_n, u_n)\|_{L^2(\mathbb{R}^d)} \leq N \|a_0\|_{L^2(\mathbb{R}^d)} \|u_m - u_n\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}}.$$

Now, by combining the last two estimates with (3.1), we conclude that

$$\|v_{m+1} - v_{n+1}\|_{L^\infty((0,T), L^2(\mathbb{R}^d))} \leq N \|a_0\|_{L^2(\mathbb{R}^d)} \|u_m - u_n\|_{\mathcal{X}_{\vec{p},\vec{q},T_0}}.$$

As $\{u_m\}_m$ is Cauchy in $\mathcal{X}_{\vec{p},\vec{q},T}$, we see from this last estimate that $\{u_m\}_m$ is also Cauchy in $L^\infty((0,T),L^2(\mathbb{R}^d))$. Therefore, $u \in L^\infty((0,T),L^2(\mathbb{R}^d))$. Finally, we note that the continuity of $u:[0,T]\to L^2(\mathbb{R}^d)$ follows from the continuity of the semi-group $e^{\mathcal{A}t}$. The proof of the lemma is completed. ■

Next, we also need the following result about the time smoothing estimates of the solution u obtained in Theorem 2.8.

Lemma 3.2. *Let $\vec{p}=(p_1,p_2,\ldots,p_d)\in[2,\infty)^d$ satisfy (1.6) and $\vec{q}=(q_1,q_2,\ldots,q_d)\in(1,\infty)^d$ such that $q_k\geq p_k$ for all $k=1,2,\ldots,d-1$, $q_d>p_d$, and*

$$1-\frac{2}{p_d}<\delta:=\sum_{k=1}^{d}\frac{1}{q_k}.$$

For each $a_0\in L^{\vec{p}}(\mathbb{R}^d)$, let u be the solution of (1.1) obtained in Theorem 2.8. We have $u\in L^l((0,T),L^{\vec{q}}(\mathbb{R}^d))$ for sufficiently small $\in(0,T_0]$ and $l\in(2,\infty)$ such that

$$\frac{2}{l}+\delta=1.$$

Proof. Note that as $q_d>p_d$, we have $\delta\in(0,1)$. Let $\{u_m\}_m$ be defined as in the proof of Lemma 3.1. As in the proof of Lemma 3.1, it is sufficient to prove that $\{u_m\}_m$ is a Cauchy sequence in $L^l((0,T_0),L^{\vec{q}}(\mathbb{R}^d))$. Note that it follows from Lemma 2.5 that

$$\|u_1\|_{L^l((0,\infty),L^{\vec{q}}(\mathbb{R}^d))}\leq N\|a_0\|_{L^{\vec{p}}(\mathbb{R}^d)}.$$

Then, by following the proof of Lemma 3.1, it is sufficient to prove that the bilinear form G is bounded in $L^l((0,T_0),L^{\vec{q}}_\sigma(\mathbb{R}^d))$. For every $u,v\in L^l((0,T_0),L^{\vec{q}}_\sigma(\mathbb{R}^d))$, we apply (2.10) in Lemma 2.7 with $z_k=q_k,\gamma_k=\beta_k=\alpha_k=1$ to see that

$$\begin{aligned}\|G(u,v)(t)\|_{L^{\vec{q}}(\mathbb{R}^d)} &\leq N\int_0^t (t-s)^{-\frac{1+\delta}{2}}\|u(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}\|v(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}ds\\ &= N\int_0^t \frac{\|u(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}\|v(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}}{(t-s)^{1-\frac{1-\delta}{2}}}ds\end{aligned}$$

Then, as

$$\frac{1}{l}=\frac{1}{l/2}-\frac{1-\delta}{2}$$

it follows from the Hardy-Littlewood inequality and Hölder inequality that

$$\begin{aligned}\|G(u,v)\|_{L^l((0,T_0),L^{\vec{q}}(\mathbb{R}^d))}N&\big\|\|u(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}\|v(\cdot,s)\|_{L^{\vec{q}}(\mathbb{R}^d)}\big\|_{L^{\frac{l}{2}}(0,T_0)}\\ &\leq N\|u\|_{L^l((0,T_0),L^{\vec{q}}(\mathbb{R}^d))}\|u\|_{L^l((0,T_0),L^{\vec{q}}(\mathbb{R}^d))}\end{aligned}$$

and this completes the proof of the lemma. ■

Our next lemma is similar to Lemma 3.1 and Lemma 3.2 but for the gradient of the solution u.

Lemma 3.3. *Let $\vec{p}=(p_1,p_2,\ldots,p_d)\in[2,\infty)^d$ satisfy (1.6) and $a_0\in L^{\vec{p}}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Then, for the solution u of (1.1) on $\mathbb{R}^d\times(0,T_0)$ obtained in Theorem 2.8, we have $\nabla u\in L^2((0,T),L^2(\mathbb{R}^d))$ for sufficiently small $T\in(0,T_0]$.*

Proof. Let $\{u_m\}_m$ be the sequence defined as in Lemma 3.1. We prove that $\{\nabla u_m\}_m$ is Cauchy in $L^2(\mathbb{R}^d \times (0,T))$ with sufficiently small $T \in (0,T_0]$. The approach is the same as that of the proof of Lemma 3.1. Therefore, we just outline some important steps. Note that we have

$$\|\nabla u_1\|_{L^2(\mathbb{R}^d\times(0,\infty))} \le N(d)\|a_0\|_{L^2(\mathbb{R}^d)}.$$

Now, let $\vec{q}$ be as in Lemma 3.2. By using (2.9) in Lemma 2.7, we have

$$\|\nabla G(u_m, u_m - u_n)\|_{L^2(\mathbb{R}^d)} \le N\int_0^t (t-s)^{-\frac{1}{2}-\frac{\delta}{2}}\|u_m-u_n\|_{L^{\vec{q}}(\mathbb{R}^d)}\|\nabla u_m\|_{L^2(\mathbb{R}^d)}ds,$$

for all $m,n \in \mathbb{N}$ and for $\delta \in (0,1)$ defined in (2.11). Now, let $l \in (2,\infty)$ and $l_0 \in (1,2)$ as

$$l = \frac{2}{1-\delta} \in (2,\infty), \quad \frac{1}{l_0} = \frac{1}{l} + \frac{1}{2} = 1 - \frac{\delta}{2}.$$

We then apply the Hardy-Littlewood inequality to get

$$\begin{aligned}\|\nabla G(u_m, u_m-u_n)\|_{L^2(\mathbb{R}^d\times(0,T))} &\le N\|\|u_m-u_n\|_{L^{\vec{q}}(\mathbb{R}^d)}\|\nabla u_m\|_{L^2(\mathbb{R}^d)}\|_{L^{l_0}(0,T)} \\ &\le N\|u_m-u_n\|_{L^l((0,T),L^{\vec{q}}(\mathbb{R}^d))}\|\nabla u_m\|_{L^2(\mathbb{R}^d\times(0,T))},\end{aligned}$$

for every $T \in (0,T_0]$. We use (2.9) in Lemma 2.7 again but directly to u_{m+1} to see that

$$\begin{aligned}\|\nabla u_{m+1}\|_{L^2(\mathbb{R}^d\times(0,T))} \le& \|\nabla u_1\|_{L^2(\mathbb{R}^d\times(0,T))} + \|\nabla G(u_m,u_m)\|_{L^2(\mathbb{R}^d)} \\ \le& \|a_0\|_{L^2(\mathbb{R}^d)} + N\|\nabla u_m\|_{L^2(\mathbb{R}^d)\times(0,T))}\|u_m\|_{L^l((0,T),L^{\vec{q}}(\mathbb{R}^d))}.\end{aligned}$$

Recall that it follows from the proof of Lemma 3.2 that the sequence $\{u_m\}_m$ is bounded in $L^l((0,T),L^{\vec{q}}(\mathbb{R}^d))$, and $\|u_m\|_{L^l((0,T),L^q(\mathbb{R}^d))}$ is sufficiently small when T is sufficiently small. Therefore, with the choice of T sufficiently small, we obtain

$$\|\nabla u_{m+1}\|_{L^2(\mathbb{R}^d\times(0,T))} \le \|a_0\|_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla u_m\|_{L^\infty((0,T),L^2(\mathbb{R}^d))}.$$

Then, by iterating this estimate, we obtain

$$\|\nabla v_{m+1}\|_{L^2(\mathbb{R}^d\times(0,T))} \le 2\|\nabla u_1\|_{L^2(\mathbb{R}^d\times(0,T))} \le 2\|a_0\|_{L^2(\mathbb{R}^d)}$$

showing the uniform bound of $\|\nabla v_{m+1}\|_{L^2(\mathbb{R}^d\times(0,T))}$ as needed. The proof of the lemma is now completed. ■

Now, we are ready to prove that the solution u obtained in Theorem 2.8 is a Leray-Hopf weak solution as long as the initial data $a_0 \in L^2(\mathbb{R}^d) \cap L^{\vec{p}}(\mathbb{R}^d)$. The result is stated in the following proposition.

Proposition 3.4. *Let $\vec{p} = (p_1, p_2, \ldots, p_d) \in [2,\infty)^d$ satisfy (1.6) and $p_d \neq 2$. Let $a_0 \in L^{\vec{p}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and u be the solution of (1.1) on $\mathbb{R}^d \times (0,T_0)$ obtained by Theorem 2.8. Then, u is a $Leray-Hopf$ weak solution of (1.1) on $\mathbb{R}^d \times (0,T)$ for sufficiently small $T \in (0,T_0)$. Moreover, $u : [0,T] \to L^2(\mathbb{R}^d) \cap L^{\vec{p}}(\mathbb{R}^d)$ is continuous.*

Proof. By Lemma 3.1, Lemma 3.2 and Lemma 3.3, it remains to prove that u satisfies the energy inequality. We adapt the standard approach to our mixed-norm case. The procedure is standard, but careful analysis is needed in the mixed-norm space and we provide the proof here for completeness. Let $\vec{q} = (q_1, q_2, \ldots, q_d)$ be as in Lemma 3.2. In particular, $q_k \geq p_k$ for all $k = 1, 2$, $q_d > p_d$, and $\delta \in (1 - \frac{2}{p_d}, 1)$, where δ is defined in (2.11). By Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have

$$u \in L^\infty((0,T), L^2(\mathbb{R}^d)) \cap L^l((0,T), L^{\vec{q}}(\mathbb{R}^d)) \quad \text{and} \quad \nabla u \in L^2(\mathbb{R}^d \times (0,T)) \tag{3.4}$$

with sufficiently small $T \in (0, T_0]$. Our goal is to use u as a test function for the Navier-Stokes equation (1.1) on $\mathbb{R}^d \times (0,T)$. To proceed, we need to use an approximation. We split our proof into several steps.

Step 1. In this step, we smooth out the time variable for function u. Let $\phi \in C_0^\infty((-1,1))$ to be even, non-negative cut-off function and satisfy

$$\int_{-1}^{1} \phi(t)dt = 1.$$

For a fixed small $\tau \in (0,T)$ and for sufficiently small h let

$$u_h(x,t) = \int_0^\tau u(x,t')\phi_h(t-t')dt', \quad t \in (0,\tau), \tag{3.5}$$

where $\phi_h(t) = \frac{1}{h}\phi(\frac{t}{h})$ for $t \in \mathbb{R}$. We claim that

$$\begin{aligned} &\lim_{h\to 0^+} \|u_h - u\|_{L^l((0,\tau),L^{\vec{q}}(\mathbb{R}^d))} = 0, \\ &\lim_{h\to 0^+} \|u_h - u\|_{L^2(\mathbb{R}^d\times(0,\tau))} = 0, \quad \text{and} \\ &\lim_{h\to 0^+} \|\nabla u_h - \nabla u\|_{L^2(\mathbb{R}^d\times(0,\tau))} = 0. \end{aligned} \tag{3.6}$$

We only provide the proof of the first assertion in (3.6) as the proof of the others are similar. First of all, we note that

$$\begin{aligned} \|u_h - u\|_{L^l((0,h),L^{\vec{q}}(\mathbb{R}^d))} &\leq \|u_h\|_{L^l((0,h),L^{\vec{q}}(\mathbb{R}^d))} + \|u\|_{L^l((0,h),L^{\vec{q}}(\mathbb{R}^d))} \\ &\leq 2\|u\|_{L^l((0,h),L^{\vec{q}}(\mathbb{R}^d))} \to 0 \quad \text{as} \quad h \to 0^+. \end{aligned}$$

Similarly, we also have

$$\|u_h - u\|_{L^l((\tau-h,\tau),L^{\vec{q}}(\mathbb{R}^d))} \to 0 \quad \text{as} \quad h \to 0^+.$$

Therefore, it suffices to prove that

$$\|u_h - u\|_{L^l(I,L^{\vec{q}}(\mathbb{R}^d))} \to 0 \quad \text{as} \quad h \to 0^+, \quad \text{where} \quad I = I_h = (h, \tau - h).$$

Note that for $t \in I$, as $\int_0^\tau \phi_h(t-t')dt' = 1$, we find that

$$\begin{aligned} \|u_h(\cdot,t) - u(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)} &= \left\| \int_0^\tau [u(\cdot,t') - u(\cdot,t)]\, \phi_h(t-t')dt' \right\|_{L^{\vec{p}}(\mathbb{R}^d)} \\ &\leq \int_{-t/h}^{(\tau-t)/h} \|u(\cdot,t+hs) - u(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)}\phi(s)ds \\ &= \int_{-1}^{1} \|u(\cdot,t+hs) - u(\cdot,t)\|_{L^{\vec{p}}(\mathbb{R}^d)}\phi(\tau)ds. \end{aligned}$$

Then, we have

$$\|u_h - u\|_{L^l(I,L^{\vec{p}}(\mathbb{R}^d))} \leq \int_{-1}^{1} \|u(\cdot,\cdot+hs) - u(\cdot,\cdot)\|_{L^l(I,L^{\vec{p}}(\mathbb{R}^d))}\phi(\tau)ds.$$

Now, observe that by the triangle inequality, we have

$$\|u(\cdot,\cdot+hs) - u(\cdot,\cdot)\|_{L^l(I,L^{\vec{p}}(\mathbb{R}^d))} \leq 2\|u\|_{L^l((0,T),L^{\vec{p}}(\mathbb{R}^d))}.$$

Moreover, $\|u(\cdot,\cdot+hs)-u(\cdot,\cdot)\|_{L^l(I,L^{\vec{p}}(\mathbb{R}^d))}\to 0$ as $h\to 0^+$ by the continuity of the Lebesgue norm. Therefore, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{h\to 0^+}\|u_h-u\|_{L^l(I,L^{\vec{p}}(\mathbb{R}^d))}=0$$

which proves the claim.

Step 2. Note that the function $u_h+\frac{u_h(\cdot,0)-u_h(\cdot,\tau)}{\tau}t-u_h(\cdot,0)$ is zero at $t=0,\tau$. Hence, it can be represented as the limit of a sequence of functions $\{u_{hk}\}_k$ in $C^\infty_{0,\sigma}(\mathbb{R}^d\times[0,\tau])$. Then, by testing the equation (1.1) with $\{u_{hk}\}_k$ and taking the limit as $k\to\infty$, we obtain

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^d}\big[-u\cdot\partial_t u_h+\nabla u:\nabla u_d+(u\cdot\nabla)u\cdot u_h\big]dxdt\\&=(u(0,\cdot),u_h(\cdot,0))_{L^2(\mathbb{R}^d)}-(u(\cdot,\tau),u_h(\cdot,\tau))_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{3.7}$$

Step 3. In this step, we will pass through the limit as $h\to 0^+$ and obtain the energy inequality from (3.7). We will evaluate each of the three integrals on the left hand side of (3.7) separately. Observe that

$$\partial_t u_h(x,t)=\int_0^\tau u(x,t')\partial_t\phi_h(t-t')dt'.$$

Therefore,

$$\begin{aligned}\int_0^\tau u\cdot\partial_t u_h dxdt&=\int_0^\tau\int_{\mathbb{R}^d}\int_0^\tau u(x,t)\cdot u(x,t')\partial_t\phi_h(t-t')dt'dxdt\\&=\int_0^\tau\int_0^\tau(u(\cdot,t),u(\cdot,t'))_{L^2(\mathbb{R}^d)}\partial_t\phi_h(t-t')dt'dt\\&=0,\end{aligned}$$

for small h, as $\partial_t\phi_h$ is odd and the map $(t,t')\mapsto(u(\cdot,t),u(\cdot,t'))_{L^2(\mathbb{R}^d)}$ is symmetric.

Next, we show that the second term on the left hand side of (3.7) converges to $\int_0^\tau\int_{\mathbb{R}^d}|\nabla u|^2dxdt$ as $h\to 0^+$. In fact, we have

$$\begin{aligned}&\left|\int_0^\tau\int_{\mathbb{R}^d}\big(\nabla u_h\cdot\nabla u-\nabla u\cdot\nabla u\big)dxdt\right|\\&=\int_0^\tau\int_{\mathbb{R}^d}|\nabla(u_h-u)\cdot\nabla u|dxdt\\&\le\|\nabla u_h-\nabla u\|_{L^2(\mathbb{R}^d\times(0,\tau))}\|\nabla u\|_{L^2(\mathbb{R}^d\times(0,\tau))}\to 0\quad\text{as}\quad h\to 0^+\end{aligned}$$

where we used (3.6) in our last step.

We now consider the third term on the left hand side of (3.7). By the integration by parts, we find that

$$\int_0^\tau\int_{\mathbb{R}^d}(u\cdot\nabla)u_h\cdot u_h dxdt.$$

Then, we write

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot u dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot u_h dxdt + \int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot (u-u_h) dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot (u-u_h) dxdt.
\end{aligned}$$

Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in (2,\infty]^d$ such that

$$\frac{1}{\alpha_k} + \frac{1}{p_k} = \frac{1}{2}, \quad k = 1,2,\ldots,d.$$

Then, we have

$$\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot (u-u_h) dxdt\right| \\
&\leq \|\nabla u_h\|_{L^2(\mathbb{R}^d\times(0,\tau))} \left(\int_0^\tau \int_{\mathbb{R}^d} |u-u_h|^2|u|^2 dxdt\right)^{\frac{1}{2}} \\
&\leq N\|\nabla u\|_{L^2(\mathbb{R}^d\times(0,\tau))} \left(\int_0^\tau \|u-u_h\|_{L^{\vec{\alpha}}(\mathbb{R}^d)}^2 \|u\|_{L^{\vec{p}}(\mathbb{R}^d)}^2 dt\right)^{\frac{1}{2}} \\
&\leq N\|\nabla u\|_{L^2(\mathbb{R}^d\times(0,\tau))} \|u\|_{L^\infty((0,\tau),L^{\vec{p}}(\mathbb{R}^d))} \|\|u-u_h\|_{L^2((0,\tau),L^{\vec{\alpha}}(\mathbb{R}^d))},
\end{aligned}$$

where we have used the Hölder inequality in the last estimate. Now, observe that we have

$$\sum_{k=1}^d \frac{1}{\alpha_k} = \frac{d}{2} - 1 \quad \text{and} \quad \alpha_d \in (2,\infty).$$

Therefore, we can apply Theorem 2.1 about the mixed-norm Sobelev embedding and (3.6) to see that

$$\begin{aligned}
\|u-u_h\|_{L^2((0,\tau),L^{\vec{\alpha}}(\mathbb{R}^d))} &\leq N\Big[\|u-u_h\|_{L^2(\mathbb{R}^d\times(0,\tau))} + \|\nabla u - \nabla u_h\|_{L^2(\mathbb{R}^d\times(0,\tau))}\Big] \\
&\to 0, \quad \text{as} \quad h\to 0^+.
\end{aligned}$$

Hence,

$$\lim_{h\to 0^+} \int_0^\tau \int_{\mathbb{R}^d} (u\cdot\nabla)u_h\cdot u dxdt = 0.$$

It remains to find the limits of terms in the the right hand side of (3.7). As $\int_0^h \phi_h(s)ds = \frac{1}{2}$, it follows that when h is sufficiently small,

$$\frac{1}{2}\|u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)}^2 = \int_0^\tau \|u(\cdot,\tau)\|_2^2 \phi_h(s)ds = \int_0^\tau \|u(\cdot,\tau)\|_2^2 \phi_h(\tau - t')dt'.$$

Therefore,

$$\begin{aligned}
&\big|(u(\cdot,\tau), u_h(\cdot,\tau))_{L^2(\mathbb{R}^d)} - \frac{1}{2}\|u(\cdot,\tau)\|_2^2\big| \\
&\leq \int_0^\tau \|u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)} \|u(\cdot,t') - u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)} \phi_h(\tau-t')dt' \qquad (3.8) \\
&\leq \|u\|_{L^\infty((0,\tau),L^2(\mathbb{R}^d))} \int_{-1}^1 \|u(\cdot,\tau+hs) - u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)} \phi(s)ds.
\end{aligned}$$

By the strong continuity of u in $L^2(\mathbb{R}^d)$ on $[0,T]$, we see that

$$\|u(\cdot,\tau+hs)-u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)}\to 0 \quad \text{as} \quad h\to 0^+.$$

Moreover, we also have

$$\|u(\cdot,\tau+hs)-u(\cdot,\tau)\|_{L^2(\mathbb{R}^d)}\le 2\|u\|_{L^\infty((0,\tau),L^2(\mathbb{R}^d))}.$$

Then, by the Lebesgue dominated convergence theorem, we obtain from (3.8) that

$$\lim_{h\to 0^+}(u(\cdot,\tau),u_h(\cdot,\tau))_{L^2(\mathbb{R}^d)}=\frac{1}{2}\|u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^d)}.$$

By a similar proof, we also have

$$\lim_{h\to 0^+}(u(\cdot,0),u_h(\cdot,0))_{L^2(\mathbb{R}^d)}=\frac{1}{2}\|u(\cdot,0)\|^2_{L^2(\mathbb{R}^d)}.$$

In conclusion, when passing the limit as $h\to 0^+$, we obtain from (3.7) the energy equality

$$\frac{1}{2}\|u(\cdot,\tau)\|_2^2+\int_0^\tau\int_{\mathbb{R}^d}|\nabla u|^2dxdt=\frac{1}{2}\|u(\cdot,0)\|_2^2$$

showing that u is a Leray-Hopf weak solution. ■

Proposition 3.4 implies the following important result that will be useful in the proof of Theorem 1.4. In the special case with $p_1=p_2=p_d=3$, the result is due to H. Kozono and H. Sohr in [17].

Corollary 3.5. *Let $\vec{p}\in[2,\infty)^d$ and a_0 be as in Proposition 3.4. If u is a Leray-Hopf weak solution of (1.1) in $\mathbb{R}^d\times[0,T)$ and $u\in L^\infty_{\text{loc}}((0,T),L^{\vec{p}}(\mathbb{R}^d))$, then $u:[0,T)\to L^{\vec{p}}(\mathbb{R}^d)$ is weakly continuous, and strongly continuous from the right.*

Proof. We adapt the idea in [17] to our mixed-norm setting. Let $t_0\in[0,T)$ and $t_k\to t_0^+$. As $u\in L^\infty_{\text{loc}}((0,T),L^{\vec{p}}(\mathbb{R}^d))$, we may assume that

$$\|u(\cdot,t_k)\|_{L^{\vec{p}}(\mathbb{R}^d)}\le M, \quad \forall\ k=1,2,\ldots.$$

As in [3], we know that $L^{\vec{p}}(\mathbb{R}^d)$ is reflexive. Then, it follows from the Banach-Alaoglu theorem that there is a subsequence of $\{u(\cdot,t_k)\}_k$ converging weakly in $L^{\vec{p}}(\mathbb{R}^d)$ to some $\tilde{u}\in L^{\vec{p}}(\mathbb{R}^d)$. On the other hand, by the assumption that u is a Leray-Hopf weak solution, and so $\{u(\cdot,t_k)\}_k$ also converges weakly in $L^2(\mathbb{R}^d)$ to $u(\cdot,t_0)$. Then, it is not too hard to verify that $\tilde{u}(\cdot)=u(\cdot,t_0)$ and therefore $u(\cdot,t_0)\in L^2(\mathbb{R}^d)\cap L^{\vec{p}}(\mathbb{R}^d)$ and

$$\|u(\cdot,t_0)\|_{L^{\vec{p}}(\mathbb{R}^d)}\le M.$$

Now, we claim that $u(\cdot,t)$ converges weakly in $L^{\vec{p}}(\mathbb{R}^d)$ to $u(\cdot,t_0)$ as $t\to t_0$. Indeed, let $\vec{p}'=(p_1',p_2',\ldots,p_d')$ be the conjugate of $\vec{p}$, i.e.

$$\frac{1}{p_k}+\frac{1}{p_k'}=1, \quad \forall\ k=1,2,\ldots,d.$$

Also, let $\phi\in L^{\vec{p}'}(\mathbb{R}^d)$, we need to check that

$$\int_{\mathbb{R}^d}u(x,t)\phi(x)dx\to\int_{\mathbb{R}^d}u(x,t_0)\phi(x)dx, \quad \text{as} \quad t\to t_0. \tag{3.9}$$

To see this, fix $\epsilon > 0$ and let $\phi \in L^{\vec{p}'}(\mathbb{R}^d)$. By the density of $C_c^\infty(\mathbb{R}^d)$, we may choose $\psi \in L^2(\mathbb{R}^d) \cap L^{\vec{p}'}(\mathbb{R}^d)$ such that

$$\|\phi - \psi\|_{L^{\vec{p}'}(\mathbb{R}^d)} \leq \frac{\epsilon}{4\|u\|_{L^\infty_{\mathrm{loc}}((0,T),L^{\vec{p}}(\mathbb{R}^d))}}.$$

Then, it follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))\phi(x)dx\right| \\
&\leq \left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))\psi(x)dx\right| + \left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))(\phi - \psi)dx\right| \\
&\leq \left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))\psi(x)dx\right| + 2\|u\|_{L^\infty_{\mathrm{loc}}(0,T),L^{\vec{p}}(\mathbb{R}^d)}\|\phi - \psi\|_{L^{\vec{p}'}(\mathbb{R}^d)} \\
&\leq \left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))\psi(x)dx\right| + \frac{\epsilon}{2}.
\end{aligned}$$

Because $\psi \in L^2(\mathbb{R}^d)$ and u is weakly continuous in $L^2(\mathbb{R}^d)$, we see that

$$\left|\int_{\mathbb{R}^d} (u(x,t) - u(x,t_0))\psi(x)dx\right| \to 0 \quad \text{as} \quad t \to t_0.$$

From this, and as ϵ is arbitrary, it follows that

$$\int_{\mathbb{R}^d} u(x,t)\phi(x)dx \to \int_{\mathbb{R}^d} u(x,t_0)\phi(x)dx, \quad \text{as} \quad t \to t_0$$

and this proves the desired claim.

Finally, observe that we have shown that $u(\cdot, t_0) \in L^2(\mathbb{R}^d) \cap L^{\vec{p}}(\mathbb{R}^d)$. By Theorem 2.8 and Proposition 3.4 there exists a continuous Leray-Hopf weak solution $\tilde{u}$ of the Navier-Stokes equation in $\mathbb{R}^d \times [t_0, t_0 + T_0)$ for some $T_0 > 0$ sufficiently small. By the uniqueness of the solution as in Theorem 2.8, we have $\tilde{u} = u$ on $\mathbb{R}^d \times [t_0, t_0 + T_0)$. As $\tilde{u}$ is continuous on $L^{\vec{p}}(\mathbb{R}^d)$, we also obtain the right continuity of u on $[t_0, t_0 + T_0)$. ■

We now are ready to provide the proof of Theorem 1.4.

Proof of Theorem 1.4. We adapt the approach introduced in [28] to our mixed-norm case. We split the proof into two steps.

Step 1. Let $w = u - v$ and $t_0 \in [0, T)$ such that

$$w \equiv 0 \quad \text{on} \quad \mathbb{R}^d \times [0, t_0].$$

We claim that there exists a sufficiently small number $\nu > 0$ such that $w \equiv 0$ on $\mathbb{R}^d \times [0, t_0 + \nu]$. Without loss of generality, we prove the claim with $t_0 = 0$. By writing the equation for w and following the standard procedure for energy estimates, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |w(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^d} |\nabla w|^2 dxdt \\
&\leq I(t) := -2\int_0^t \int_{\mathbb{R}^d} u \cdot (w \cdot \nabla) w dxdt.
\end{aligned} \tag{3.10}$$

We claim that it suffices to show that there is some small positive ν such that

$$|I(t)| \leq \int_0^t \int_{\mathbb{R}^d} |\nabla w|^2 dxdt + N \int_0^t \int_{\mathbb{R}^d} |w|^2 dxdt \tag{3.11}$$

for all $t \in [0, \nu]$ and for some constant $N = N(u) > 0$. Indeed, if (3.11) holds, it follows from (3.10) that

$$\int_{\mathbb{R}^d} |w(t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^d} |\nabla w|^2 dxdt \leq 2 \int_0^t \int_{\mathbb{R}^d} |\nabla w|^2 dxdt + N \int_0^t \int_{\mathbb{R}^d} |w|^2 dxdt,$$

which yields

$$\int_{\mathbb{R}^d} |w(t)|^2 dx \leq N \int_0^t \int_{\mathbb{R}^d} |w|^2 dxdt. \tag{3.12}$$

From this and by Gronwall's inequality, it follows that $w \equiv 0$ in $\mathbb{R}^d \times [0, \nu]$.

It remains to prove (3.11). We follow the decomposition strategy introduced by K. Masuda in [28]. Choose $f \in C^1([0, \infty))$ such that $f \geq 0$, $f(s) = 1$ if $s < \frac{1}{2}$ and $f(s) = 0$ when $s > 1$. Let us define $f_\lambda(s) = f(\frac{s}{\lambda})$ for $\lambda > 0$. Then, we write $u = u_1 + u_2$, where

$$u_1 = f_\lambda(|u|)u \quad u_2 = (1 - f_\lambda(|u|))u.$$

Note that

$$\|u_2(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq \|1 - f_\lambda(|u(\cdot, 0)|)\|_{L^\infty(\mathbb{R}^d)} \|u(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)}.$$

Consequently, for each $\epsilon \in (0, 1)$, there exists $\lambda = \lambda(u(\cdot, 0), \epsilon)$ sufficiently large such that

$$\|u_2(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq \epsilon \quad \text{and} \quad \|u_1\|_{L^\infty(\mathbb{R}^d \times (0,T))} \leq \lambda.$$

Now, as $u : [0, T) \to L^{\vec{p}}(\mathbb{R}^d)$ is right continuous at $t_0 = 0$, there exists $\nu > 0$ small such that

$$\|u(\cdot, t) - u(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq \epsilon, \quad \forall\, t \in (0, \nu).$$

From this, and Lemma 2.3, it follows that

$$\|u_2(\cdot, t) - u_2(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq C\|u(\cdot, t) - u(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq \epsilon.$$

As a result, we have the estimate

$$\|u_2(\cdot, t)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq \|u_2(\cdot, t) - u_2(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} + \|u_2(\cdot, 0)\|_{L^{\vec{p}}(\mathbb{R}^d)} \leq 2\epsilon. \tag{3.13}$$

Now, let $\vec{\alpha} = (\alpha_2, \alpha_2, \ldots, \alpha_d) \in [2, \infty]^d$ satisfy

$$\frac{1}{\alpha_k} + \frac{1}{p_k} = \frac{1}{2}, \quad \forall\, k = 1, 2, \ldots, d.$$

Then, for $t_1 \in (0,\nu]$, we have

$$\begin{aligned}|I(t)| =&2\Big|\int_0^{t_1}\int_{\mathbb{R}^d} u_1\cdot(w\cdot\nabla)w dxdt + \int_0^{t_1}\int_{\mathbb{R}^d} u_2\cdot(w\cdot\nabla)w dxdt\Big| \\ \leq& 2\left[\|u_1\|_{L^\infty(\mathbb{R}^d\times(0,T))}\int_0^{t_1}\int_{\mathbb{R}^d}|w\cdot\nabla w|dxdt + \int_0^{t_1}\int_{\mathbb{R}^d}|u_2\cdot(w\cdot\nabla)w|dxdt\right] \\ \leq& 2\|u_1\|_{L^\infty(\mathbb{R}^d\times(0,T))}\left(\int_0^{t_1}\int_{\mathbb{R}^d}|w|^2dxdt\right)^{\frac{1}{2}}\left(\int_0^{t_1}\int_{\mathbb{R}^d}|\nabla w|^2dxdt\right)^{\frac{1}{2}} \\ &+ 2\int_0^{t_1}\left(\int_{\mathbb{R}^d}|u_2\cdot w|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\nabla w|^2dx\right)^{\frac{1}{2}}dt \\ \leq& 2q\|w\|_{L^2(\mathbb{R}^d\times(0,t_1))}\|\nabla w\|_{L^2(\mathbb{R}^d\times(0,t_1))} \\ &+ 2\|\nabla w\|_{L^2(\mathbb{R}^d\times(0,t_1))}\|u_2\|_{L^\infty((0,t_1),L^{\vec{p}}(\mathbb{R}^d))}\|w\|_{L^2((0,t_1),L^{\vec{\alpha}}(\mathbb{R}^d))}.\end{aligned}$$

Next, we note that

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_d} = \frac{d}{2}-1, \quad \text{and} \quad \alpha_d \in (2,\infty).$$

Then, we apply the Sobolev embedding in mixed-norm, Theorem 2.1, to find that

$$\|w\|_{L^2((0,t_1),L^{\vec{\alpha}}(\mathbb{R}^d))} \leq N\|w\|_{L^2((0,t_1),W^{1,2}(\mathbb{R}^d))}.$$

Using this, (3.13) and as λ is sufficiently large, we obtain

$$\begin{aligned}|I(t)| \leq& 2\lambda\|w\|_{L^2(\mathbb{R}^d\times(0,t_1))}\|\nabla w\|_{L^2(\mathbb{R}^d\times(0,t_1))} \\ &+ N\|\nabla w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))}\|u_2\|_{L^\infty((0,t_1),L^{\vec{p}}(\mathbb{R}^d))} \\ &+ N\|\nabla w\|_{L^2(\mathbb{R}^d\times(0,t_1))}\|w\|_{L^2(\mathbb{R}^d\times(0,t_1))}\|u_2\|_{L^\infty((0,t_1),L^{\vec{p}}(\mathbb{R}^d))} \\ \leq& N_1\|w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))} + \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))} \\ &+ N\|\nabla w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))}\|u_2\|_{L^\infty((0,t_1),L^{\vec{p}}(\mathbb{R}^d))} \\ \leq& N_1\|w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))} + (\frac{1}{2}+\epsilon N)\|\nabla w\|^2_{L^2(\mathbb{R}^d\times(0,t_1))},\end{aligned}$$

where $N_1 = N_1(\lambda,\vec{p}) > 0$ and $N = N(d,\vec{p}) > 0$. From this, and by choosing sufficiently small ϵ, we obtain (3.11).

Step 2. We prove that $w := u - v \equiv 0$ on $\mathbb{R}^d \times [0,T)$. Let us denote

$$\mathcal{U} = \Big\{\tau \in (0,T) : w \equiv 0 \quad \text{on} \quad \mathbb{R}^d \times [0,\tau]\Big\} \quad \text{and} \quad \overline{T} = \sup \mathcal{U}.$$

By **Step 1** and as $w(\cdot,0) = 0$, we see that $\overline{T} > 0$. If $\overline{T} = T$, we are done. Then, it remains to consider the case that $\bar{T} < T$. Let $\{t_k\}_k$ be an increasing sequence in $\mathcal{U}$ such that $t_k \to \bar{T}$ as $k \to \infty$. As u, v are both Leray-Hopf weak solutions of (1.1), we can assume that $\{u(\cdot,t_k)\}, \{v(\cdot,t_k)\}$ are in $L^2(\mathbb{R}^d)$ and they both converge weakly in $L^2(\mathbb{R}^d)$ to $u(\cdot,\bar{T})$ and $v(\cdot,\bar{T})$, respectively. Consequently, $\{w(\cdot,t_k)\}$ converges weakly in $L^2(\mathbb{R}^d)$ to $w(\cdot,\bar{T})$. However, from the definition of $\mathcal{U}$, we see that

$$w(\cdot,t_k) = u(\cdot,t_k) - v(\cdot,t_k) \equiv 0, \quad \text{for all} \quad k \in \mathbb{N}.$$

Therefore, $w(\cdot,\bar{T}) \equiv 0$. Then, by **Step 1**, we have $\nu > 0$ and sufficiently small such that $w \equiv 0$ on $\mathbb{R}^d \times [0,\bar{T}+\nu]$, which contradicts the definition of $\bar{T}$. The proof is then complete. ■

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