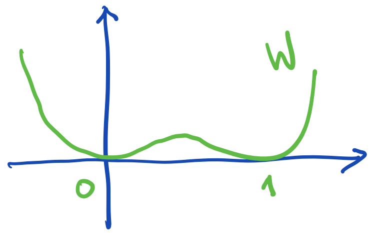
Energy $I_\varepsilon(u) := \int_\Omega \left( \frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \right) dx$, $u \in W^{1,2}(\Omega)$

$m \in (0,1)$, $\fint_\Omega u(x)\,dx = m$, $u : \Omega \to \mathbb{R}$

$\Omega \subset \mathbb{R}^N$ open, bounded, domain




$W^{1,2}(\Omega)$ (Sobolev space) $:= \{ u : \Omega \to \mathbb{R} : u \in L^2(\Omega), \nabla u \in L^2(\Omega; \mathbb{R}^N) \}$

$\int_\Omega |u(x)|^2 dx$, $\int_\Omega |\nabla u(x)|^2 dx$ $< +\infty$

Think about $\nabla u$ ... in the sense of distributions:

There exists a function $f \in L^2(\Omega; \mathbb{R}^N)$ such that

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\,dx = -\int_\Omega f_i(x)\varphi(x)\,dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

$f = (f_1, \dots, f_N)$

$$" -\int_\Omega \frac{\partial u}{\partial x_i}(x)\varphi(x)\,dx$$ if $u$ was "classically" differentiable

Then we set $\boxed{\frac{\partial u}{\partial x_i} := f_i}$

LAST TIME: For fixed $\varepsilon > 0$ use Direct Method of the Calculus of Variations to prove existence of minimizers for $I_\varepsilon(\cdot)$

DIRECT METHOD OF THE CALCULUS OF VARIATIONS

$I : (V, \tau) \to \mathbb{R}$ functional, $\tau$ topology on $V$

**Goal** Show that $\exists\, u \in V$ s.t. $I(u) = \min_{v \in V} I(v)$

**Step 1** Consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$

$$\inf_{v\in V} I(v) = \lim_{n\to+\infty} I(u_n)$$

**Step 2** [COMPACTNESS] Prove that $\{u_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$, i.e.,

$\exists\, u \in V$ s.t. $u_{n_k} \xrightarrow{\tau} u$

**Step 3** Prove that $I$ is sequentially lower semicontinuous, i.e., if $z_n \in V$, $z \in V$,

$$z_n \xrightarrow{\tau} z \Rightarrow I(z) \leq \varliminf_{n\to+\infty} I(z_n)$$

Then

$$\boxed{I(u) = \inf_{v\in V} (\min)\, I(v)}$$

**Why?**

$$I(u) \geq \inf_{v\in V} I(v) \underset{\text{step 1}}{=} \lim_{n\to+\infty} I(u_n)$$

$$\underset{\text{step 2}}{=} \lim_{k\to+\infty} I(u_{n_k})$$

$$= \varliminf_{k\to+\infty} I(u_{n_k})$$

$$\underset{\text{step 3}}{\geq} I(u)$$

So $\quad I(u) = \inf_{v \in V} I(v) \quad$ (min).

**Example for Step 2/3:** $\quad V = L^2(\Omega)$

$$I(v) := \int_\Omega |v(x)|^2 \, dx$$

Suppose $\quad \sup_{n \in \mathbb{N}} I(v_n) < +\infty.$

Then $\exists \ \{v_{n_k}\}_{k \in \mathbb{N}} \subset \{v_n\}_{n \in \mathbb{N}}$, $\ v \in L^2(\Omega)$ s.t.

$$v_{n_k} \xrightarrow{L^2} v, \quad \text{i.e.,}$$

$\forall \ \varphi \in L^2(\Omega)$

$$\int_\Omega v_{n_k}(x) \varphi(x) \, dx \longrightarrow \int_\Omega v(x) \varphi(x) \, dx.$$

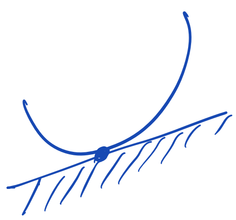
$\Sigma$ ... $L^2$- weak convergence

If $\quad \varphi : \mathbb{R} \to [0, +\infty) \quad (\varphi = |\cdot|^2)$

is convex, i.e.,

$$\varphi(\theta z_1 + (1-\theta) z_2) \leq \theta \varphi(z_1) + (1-\theta) \varphi(z_2)$$

$$\forall \theta \in (0,1), \quad z_1, z_2 \in \mathbb{R}$$



$$z_n \xrightarrow{L^2} z \Rightarrow \int_\Omega \varphi(z(x)) dx \leq \underline{\lim} \int_\Omega \varphi(z_n(x)) dx$$

Here $\quad \varphi(z) = |z|^2$ ... convex

$$\int_\Omega \underbrace{|v(x)|^2}_{\varphi(v(x))} dx \leq \underline{\lim} \int_\Omega \underbrace{|v_{n_k}(x)|^2}_{\varphi(v_{n_k}(x))} dx$$

**WARNING**  In general **STEP 3** fails.

In this case $\longrightarrow$ need to **RELAX** (RELAXATION) the problem

Replace $I(u)$ (original energy) by its **relaxed energy**:

$$R(u) := \inf_{\{u_n\}} \left\{ \varliminf_{n\to+\infty} I(u_n) : u_n \rightharpoonup u \right\}$$

$u_n \equiv u \Rightarrow R(u) \leq I(u)$

Now $R(\cdot)$ sequentially lower semicontinuous

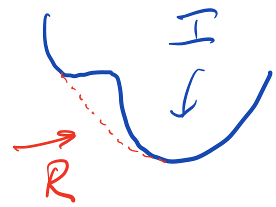
Apply the Direct Method of the Calculus of Variations to $R(\cdot)$

$\Downarrow$

$$\exists\, u \in V \text{ s.t. } R(u) = \min_{v\in V} R(v)$$

and

$$\boxed{\inf_{v\in V} I(v) = R(u)} \quad (*)$$



Why?

$$\inf_{v\in V} I(v) \geq \inf_{v\in V} R(v) = \min_{v\in V} R(v) = R(u)$$

$\downarrow$ because $I \geq R$

$$= \inf_{\{u_n\}} \left\{ \varliminf_{n\to+\infty} I(u_n) : u_n \rightharpoonup u \right\}$$

$$\geq \inf_{v\in V} I(v)$$

$$\geq \inf_{v\in V} I(v)$$

$$\geq \inf_{v\in V} I(v)$$

... Hence $(*)$!

Back to $I_\varepsilon$ ...  $I_\varepsilon(u) := \int_\Omega \left[ \frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \right] dx, \quad \int_\Omega u = m$

Fix $\varepsilon > 0$. Claim $I_\varepsilon$ admits a minimizer $u_\varepsilon$.

Use the Direct Method of Calc Var.:

Step 1 Let $\{u_\varepsilon^n\}_{n\in\mathbb{N}}$ be an infimizing sequence, i.e.,

$$\lim_{n\to+\infty} I_\varepsilon(u_\varepsilon^n) = \inf_{v\in W^{1,2}(\Omega),\ \fint_\Omega v(x)\,dx = m} I_\varepsilon(v) \quad (=: V)$$

Step 2 COMPACTNESS Lecture #1: $W(u) = u^2(1-u)^2$

$$\inf_{\delta>0,\ v\in V} I_\delta(v) =: M < +\infty$$

To show: $\exists\ \{u_\varepsilon^{n_k}\}_{k\in\mathbb{N}} \subset \{u_\varepsilon^n\}_{n\in\mathbb{N}}$ s.t.

$$u_\varepsilon^{n_k} \xrightarrow{\ ?\ } u, \quad u \in W^{1,2}(\Omega),\ \fint_\Omega u = m.$$

For $n \gg 1$, $0 < \varepsilon \ll 1$

$$\begin{cases} \int_\Omega \left(\frac{1}{\varepsilon} W(u_\varepsilon^n) + \varepsilon |\nabla u_\varepsilon^n|^2\right) dx \leq M+1 \\ \fint_\Omega u_\varepsilon^n(x)\,dx = m \end{cases}$$

Then

$$\begin{cases} \sup_{n,\,(n\gg 1)} \int_\Omega |\nabla u_\varepsilon^n(x)|^2\,dx \leq \frac{M+1}{\varepsilon} \\ \fint_\Omega u_\varepsilon^n(x)\,dx = m \end{cases}$$

Poincaré-Wirtinger Inequality:

$$\int_\Omega \Big| u_\varepsilon^n(x) - \underbrace{\fint_\Omega u_\varepsilon^n(y)\,dy}_{m} \Big|^2 dx \leq C \int_\Omega |\nabla u_\varepsilon^n(x)|^2 dx \leq \frac{C(M+1)}{\varepsilon}$$

So $\sup_{n \atop (n\gg 1)} \int_\Omega |u^n_\varepsilon(x)|^2\,dx < +\infty$

Thus $\sup_{n \atop (n\gg 1)} \int_\Omega \left( |u^n_\varepsilon(x)|^2 + |\nabla u^n_\varepsilon(x)|^2 \right) dx < +\infty$

Sobolev Spaces : $\exists \{u^{n_k}_\varepsilon\}_{k\in\mathbb{N}}$, $u_\varepsilon \in W^{1,2}(\Omega)$ s.t.

$u^{n_k}_\varepsilon \xrightarrow{W^{1,2}} u_\varepsilon$, i.e.,

- $u^{n_k}_\varepsilon \xrightarrow{L^2} u_\varepsilon$, i.e., $\int_\Omega |u^{n_k}_\varepsilon - u_\varepsilon|^2 dx \underset{k\to+\infty}{\longrightarrow} 0$

- $\nabla u^{n_k}_\varepsilon \xrightarrow{L^2} \nabla u_\varepsilon$, i.e., $\forall \varphi \in L^2(\Omega)$

$$\int_\Omega \nabla u^{n_k}_\varepsilon \cdot \varphi\, dx \to \int_\Omega \nabla u_\varepsilon \cdot \varphi\, dx$$

$$\left| \int_\Omega u_\varepsilon\, dx - m|\Omega| \right| = \left| \int_\Omega u_\varepsilon\, dx - \int_\Omega u^{n_k}_\varepsilon\, dx \right|$$

$$= \left| \int_\Omega (u_\varepsilon - u^{n_k}_\varepsilon)\, dx \right| \leq \Big( \underbrace{\int_\Omega |u_\varepsilon - u^{n_k}_\varepsilon|^2 dx}_{\downarrow 0} \Big)^{1/2} \cdot |\Omega|^{1/2} \to 0$$

$$\Rightarrow \int_\Omega u_\varepsilon\, dx - m|\Omega| = 0, \quad \text{i.e.,} \quad ⨍_\Omega u_\varepsilon(x)\, dx = m.$$

So $u_\varepsilon$ is admissible

$\leftarrow \ldots$ weak topology in $W^{1,2}(\Omega)$ $(=: H^1(\Omega))$

NEXT TIME : STEP 3