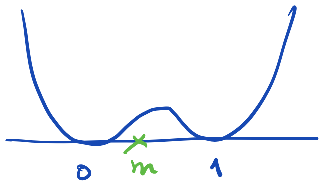
Recall:

$$I_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \right) dx, & u \in W^{1,2}(\Omega) \ (= H^1(\Omega)), \ \fint_\Omega u(x)\,dx = m \\ +\infty, & \text{——} \end{cases}$$

$\underbrace{\qquad\qquad}_{\frac{1}{\varepsilon} f(x, u, \varepsilon \nabla u)}$

$u \in W^{1,2}(\Omega)$: $u \in L^2(\Omega)$, $\int_\Omega |u(x)|^2 dx < +\infty$ **and** $\nabla u \in L^2(\Omega; \mathbb{R}^N)$, $\int_\Omega |\nabla u(x)|^2 dx < +\infty$

$m \in (0,1)$



$\Omega \subset \mathbb{R}^N$ open, bounded domain, Lipschitz

$\{W = 0\} = \{0, 1\}$

Fix $\varepsilon > 0$: Prove existence of minimizer of I_ε, u_ε

Use Direct Method of the Calculus of Variations:

Step 1 Consider infimizing sequence $\{u^n_\varepsilon\}_{n \in \mathbb{N}}$, i.e.,

$$\lim_{n \to +\infty} I_\varepsilon(u^\varepsilon_n) = \inf \left\{ I_\varepsilon(v) : v \in W^{1,2}(\Omega), \fint_\Omega v = m \right\}$$

Step 2 [Compactness] $\exists \ \{u^{n_k}\}_{k \in \mathbb{N}} \subset \{u^n_\varepsilon\}_{n \in \mathbb{N}}$

$\exists \ u_\varepsilon \in W^{1,2}(\Omega)$, $\fint_\Omega u_\varepsilon = m$,

$u^{n_k}_\varepsilon \rightharpoonup u_\varepsilon$ weakly in $W^{1,2}$, i.e.,

- $\int_\Omega |u^{n_k}_\varepsilon - u_\varepsilon|^2 dx \to 0$ ($u^{n_k}_\varepsilon \to u_\varepsilon$ in L^2 strong)

- $\nabla u^{n_k}_\varepsilon \rightharpoonup \nabla u_\varepsilon$ weakly in L^2, i.e.,

$\forall \ \varphi \in L^2(\Omega; \mathbb{R}^N)$

$$\int_\Omega \nabla u^{n_k}_\varepsilon \cdot \varphi \, dx \to \int_\Omega \nabla u_\varepsilon \cdot \varphi \, dx$$

TODAY: start with **STEP 3**, i.e.,

I_ε is sequentially weakly ($W^{1,2}$) lower semicontinuous, i.e.,

$v_n \rightharpoonup v \quad W^{1,2} \quad \Rightarrow \quad I_\varepsilon(v) \leq \varliminf_{n\to+\infty} I_\varepsilon(v_n)$

If so, then $\quad I_\varepsilon(u_\varepsilon) = \min \{ I_\varepsilon(v) : v \in W^{1,2}(\Omega), \ \fint_\Omega v = m \}$

DONE!

Why: $\quad v_n \rightharpoonup v \quad W^{1,2}$

$$\Downarrow \quad \begin{cases} v_n \to v & \text{strongly in } L^2 \\ \nabla v_n \rightharpoonup \nabla v & \text{weakly in } L^2 \end{cases}$$

Fact: $\quad f : \mathbb{R}^N \to [0, +\infty) \quad$ convex $\quad \overset{(\text{iff})}{\Rightarrow}$

$\xi_n \rightharpoonup \xi \quad$ in $L^2(\mathbb{R}^N)$ – weak

$$\int_\Omega f(\xi(x))\, dx \leq \varliminf \int_\Omega f(\xi_n(x))\, dx$$

$f = |\cdot|^2$ convex

Hence $$\int_\Omega |\nabla v(x)|^2\, dx \leq \varliminf \int_\Omega |\nabla v_n(x)|^2\, dx \qquad (1)$$

$v_n \to v$ in L^2 strong (up to a subsequence)

$\Rightarrow \quad v_n(x) \to v(x) \quad \mathcal{L}^N$ a.e. $x \in \Omega$

W continuous, use Fatou's lemma ($W \geq 0$):

$$\int_\Omega W(v(x))\, dx = \int_\Omega \lim_{n\to+\infty} W(v_n(x))\, dx$$

$$= \int_\Omega \varliminf_{n\to+\infty} W(v_n(x))\, dx$$

$$\leq \varliminf_{n\to+\infty} \int_\Omega W(v_n(x))\, dx \qquad (2)$$

Therefore,

$$\lim_{n\to+\infty} I_\varepsilon(v_n) = \lim_{n\to+\infty} \int \left[\frac{1}{\varepsilon} W(v_n(x))dx + \varepsilon |\nabla v_n(x)|^2\right] dx$$

$$= \lim_{n\to+\infty} \left\{\frac{1}{\varepsilon}\int_\Omega W(v_n(x))\,dx + \varepsilon \int_\Omega |\nabla v_n(x)|^2 dx\right\}$$

$$\geq \frac{1}{\varepsilon} \lim_{n\to+\infty} \int_\Omega W(v_n(x))dx + \varepsilon \lim \int_\Omega |\nabla v_n(x)|^2 dx$$

$$\geq (\text{by } ① + ②) \ \frac{1}{\varepsilon}\int_\Omega W(v(x))dx + \varepsilon \int_\Omega |\nabla v(x)|^2 dx$$

$$= I_\varepsilon(v).$$

∎

Lecture #2 ... road map

② claim Up to a subsequence (not relabeled)

| $u_\varepsilon \longrightarrow u$ (some sense) some u
that implies pointwise a.e.
| $u \in BV(\Omega; \{0,1\})$

Recall we proved that

(i) $\sup_\varepsilon I_\varepsilon(u_\varepsilon) =: M < +\infty$ — OK for minimizers

(ii) $u_\varepsilon(x) \longrightarrow u(x)$ pointwise a.e.

(used prototype $W(t) := t^2(1-t)^2$)

then by Fatou's lemma $\Longrightarrow$ $u(x) \in \{0,1\}$ a.e. — OK

Recall also that we showed that (if) u_ε minimizes I_ε

then indeed $\sup_\varepsilon I_\varepsilon(u_\varepsilon) < +\infty$

To Do: our sequence of minimizers $\{u_\varepsilon\}$ converges (in particular,

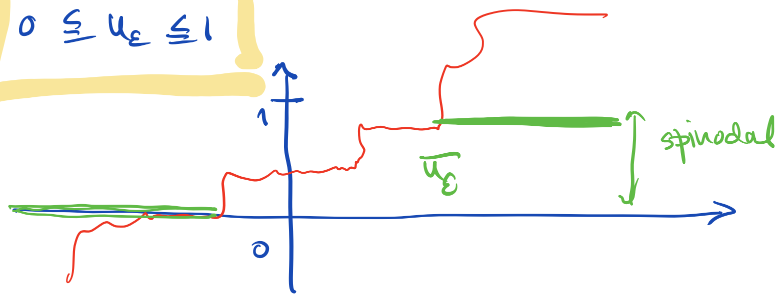
pointwise a.e., and after extracting a subsequence if necessary) to some $u \in BV(\Omega; \{0,1\})$

Define $\phi(t) := \int_0^t \sqrt{W(s)}\, ds$

$$\begin{cases} v_\varepsilon := \phi(u_\varepsilon) \\ \nabla v_\varepsilon = \phi'(u_\varepsilon)\nabla u_\varepsilon = \sqrt{W(u_\varepsilon)}\, \nabla u_\varepsilon \end{cases} \qquad (3)$$

WLOG (False!) $0 \leq u_\varepsilon \leq 1$

(truncation)



$$I_\varepsilon(\bar{u}_\varepsilon) = \int_{\{u_\varepsilon \leq 0\}} \frac{1}{\varepsilon} W(0) + \int_{\{0 < u_\varepsilon < 1\}} \frac{1}{\varepsilon} W(u_\varepsilon) + \int_{\{u_\varepsilon \geq 1\}} \frac{1}{\varepsilon} W(0)$$

$$+ \int_{\{u_\varepsilon \leq 0\}} \frac{1}{\varepsilon} |\nabla 0|^2 + \int_{\{0 < u_\varepsilon < 1\}} \frac{1}{\varepsilon} |\nabla u_\varepsilon|^2 + \int_{\{u_\varepsilon \geq 1\}} \frac{1}{\varepsilon} |\nabla 1|^2$$

$$\leq I_\varepsilon(u_\varepsilon) \quad \cdots \quad u_\varepsilon \text{ minimizers} \quad ⨯$$

($<$)

Careful! $\fint_\Omega \bar{u}_\varepsilon(x)\,dx \neq m$

$M := \sup_{\varepsilon'} I_{\varepsilon'}(u_{\varepsilon'}) \geq$

$$+\infty > I_\varepsilon(u_\varepsilon) = \int_\Omega \Big(\underbrace{\tfrac{1}{\varepsilon} W(u_\varepsilon)}_{a^2} + \underbrace{\varepsilon |\nabla u_\varepsilon|^2}_{b^2}\Big)\, dx \quad \text{(Young's Inequality } a^2 + b^2 \geq 2ab\text{)}$$

[Modica–Mortola "trick"] $$\geq 2\int_\Omega \sqrt{W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx$$

$$\underset{(3)}{=} 2\int_\Omega |\nabla v_\varepsilon|\, dx \qquad (4)$$

$0 \leq u_\varepsilon \leq 1 \Rightarrow \{v_\varepsilon\}$ bounded in $L^\infty \Rightarrow \{v_\varepsilon\}$ bounded in $L^1(\Omega)$

(4) $\Rightarrow$ $\{\nabla v_\varepsilon\}$ is bounded in $L^1(\Omega; \mathbb{R}^N)$ $\quad \int_\Omega |\nabla v_\varepsilon| \leq \frac{M}{2}$

i.e., $\{v_\varepsilon\}$ bounded in $W^{1,1}(\Omega)$

$\Downarrow$

$\exists$ subsequence (not relabelled) $v_\varepsilon \to v$ in BV weak

- $v \in BV$ (Bounded Variation) — $v \in L^1(\Omega)$, Dv (distributional derivative) is a Radon measure

i.e., $\forall\, i \in \{1, \dots, N\}$ $\forall\, \varphi \in C_c^\infty(\Omega)$

$$\int_\Omega v \frac{\partial \varphi}{\partial x_i}\, dx \quad \left(= -\int_\Omega \frac{\partial v}{\partial x_i} \varphi \text{ (if) } v \text{ smooth} \right)$$

$$= -\int \varphi \, d\mu_i \quad , \mu_i \text{ Radon measure}$$

" $\frac{\partial v}{\partial x_i} \equiv \mu_i$ ", $Dv = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right)$

BV weak convergence
- $v_\varepsilon \to v$ in L^1 strong, wlog, $v_\varepsilon(x) \to v(x)$ pointwise convergence $\mathcal{L}^N$ a.e. $x \in \Omega$,
- $\nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega \xrightarrow{*} Dv$ weakly $*$ in the sense of measures

Remind you that $\Phi(t) := \int_0^t \sqrt{W(s)}\, ds$

$\Phi : [0,1] \to [0, +\infty)$ strictly increasing

$v_\varepsilon := \Phi(u_\varepsilon) \Rightarrow u_\varepsilon = \Phi^{-1}(v_\varepsilon) \to \Phi^{-1}(v) =: u$ pointwise a.e.

CORRECT PROOF Suppose that $\varepsilon_n \to 0^+$

$$M := \sup_n I_{\varepsilon_n}(u_n) < +\infty ,$$

claim $\exists$ subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ s.t.

$$u_{n_k} \xrightarrow{L^1} u, \quad \text{some } u \in L^1(\Omega) \text{ (actually, } \in BV(\Omega;\{0,1\})\text{)}$$

Assume (H1) $W : \mathbb{R} \to [0,+\infty)$ continuous

(H2) $W(t) \geq c|t|$ for all t, $|t| \geq L$, some $L > 0$, some $c > 0$

Next time: * finish this proof

* $I_\varepsilon \xrightarrow{\Gamma} I$ — $\Gamma\text{-}\underline{\lim}$ / $\Gamma\text{-}\overline{\lim}$