Large-Scale Regularity in Elliptic Homogenization - Part II

Zhongwei Shen

University of Kentucky

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Elliptic Operators with Rapidly Oscillating Coefficients

Consider

$$\mathcal{L}_\varepsilon = -\mathrm{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0.$$

Let

$$A = A(y) = \big(a_{ij}(y)\big)$$
$$1 \le i, j \le d$$

Assume that

- A is real, bounded, and uniformly elliptic.
- A is 1-periodic.

Basic Assumptions

- Ellipticity: there exists $\mu > 0$ such that

$$\|A\|_\infty \leq \mu^{-1}$$
$$\mu|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j$$

for any $\xi \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$

- Periodicity:

$$A(y+z) = A(y) \quad \text{for any } z \in \mathbb{Z}^d$$

and for a.e. $y \in \mathbb{R}^d$

- All results hold for elliptic systems in divergence form

Large-Scale Lipschitz Estimate

Theorem (large-scale interior Lipschitz estimate)

Assume $A = A(y)$ is elliptic and periodic. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1 = B(0,1).$$

Then, for $\varepsilon \leq r \leq 1$,

$$⨍_{B_r} |\nabla u_\varepsilon|^2 \leq C ⨍_{B_1} |\nabla u_\varepsilon|^2,$$

where C depends only on d and μ.

- No smoothness assumption on $A(y)$ is needed.

Direct Approach by Convergence Rates (C. Smart - S.N. Armstrong)

- Advantage:
  No compactness theorem is needed
  Do not involve correctors
  Applicable in non-periodic settings (almost-period, random)

- Disadvantage:
  Require a Dini convergence rate (which may be obtained by using approximate correctors in the non-periodic settings)

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Lipschitz Estimates by Convergence Rates

Let

$$H(r) = \inf_{\substack{M\in\mathbb{R}^d \\ q\in\mathbb{R}}} \frac{1}{r} \left(\fint_{B_r} |u_\varepsilon - M\cdot x - q|^2 \, dx \right)^{1/2}$$

Show that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_1 and $0 < \varepsilon < (1/2)$, then

$$H(r) \le C \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2}$$

for $\varepsilon < r < (1/2)$.

Key Observation

$$H(\theta r) \leq$$

$$\inf_{\substack{M\in\mathbb{R}^d \\ q\in\mathbb{R}}} \frac{1}{\theta r}\left(\fint_{B_{\theta r}} |w - M\cdot x - q|^2\, dx\right)^{1/2} + \frac{1}{\theta r}\left(\fint_{B_{\theta r}} |u_\varepsilon - w|^2\right)^{1/2}$$

$$\leq \frac{1}{2}\inf_{\substack{M\in\mathbb{R}^d \\ q\in\mathbb{R}}} \frac{1}{r}\left(\fint_{B_r} |w - M\cdot x - q|^2\, dx\right)^{1/2} + \frac{1}{\theta r}\left(\fint_{B_{\theta r}} |u_\varepsilon - w|^2\right)^{1/2}$$

$$\leq \frac{1}{2}H(r) + \frac{C_\theta}{r}\left(\fint_{B_r} |u_\varepsilon - w|^2\right)^{1/2},$$

if θ is small and w is $C^{1,\alpha}$, e.g., $\mathcal{L}_0(w) = 0$ in B_r.

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\begin{aligned}
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Observation

- If u_ε is well approximated at each scale larger than ε by a function with $C^{1,\alpha}$ estimates, then u_ε is Lipschitz at all scales larger than ε.
- Let

$$H(r) = \inf_{\substack{M\in\mathbb{R}^d \\ q\in\mathbb{R}}} \frac{1}{r}\left(\fint_{B_r} |u_\varepsilon - M\cdot x - q|^2\,dx\right)^{1/2} = \inf_{q\in\mathbb{R}} \frac{1}{r}\left(\fint_{B_r} |u_\varepsilon - \widetilde{M}_r\cdot x - q|^2\,dx\right)^{1/2}$$

and $h(r) = |\widetilde{M}_r|$

-

$$\left(\fint_{B_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C \inf_{q\in\mathbb{R}} \frac{1}{r}\left(\fint_{B_{2r}} |u_\varepsilon - q|^2\right)^{1/2} \le C\left\{H(2r) + h(2r)\right\}$$

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A General Scheme (Armstrong - Smart, S.)

Let $H(r)$ and $h(r)$ be two nonnegative functions on $(0,1]$. Let $0 < \varepsilon < (1/4)$. Suppose that

$$\max_{r \le t \le 2r} H(t) \le C_0 H(2r),$$

$$\max_{r \le t, s \le 2r} |h(t) - h(s)| \le C_0 H(2r),$$

for any $r \in (\varepsilon, 1/2]$. Assume that for some $\theta \in (0, 1/4)$,

$$H(\theta r) \le (1/2) H(r) + C\omega(\varepsilon/r) \Big\{ H(2r) + h(2r) \Big\},$$

for any $r \in [\varepsilon, 1/2]$, where ω is increasing, $\omega(0) = 0$, and

$$\int_0^1 \frac{\omega(t)}{t} \, dt < \infty.$$

Then

$$\max_{\varepsilon \le r \le 1} \Big\{ H(r) + h(r) \Big\} \le C \Big\{ H(1) + h(1) \Big\}.$$

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Then

$$\max_{\varepsilon\le r\le 1} \Big\{H(r)+h(r)\Big\} \le C\Big\{H(1)+h(1)\Big\}.$$

Approximation and Convergence Rates

- Given $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_{2r} and $0 < \varepsilon < r$, find w such that $\mathcal{L}_0(w) = 0$ in B_r and

$$\left(\fint_{B_r} |u_\varepsilon - w|^2 \right)^{1/2} \le \omega(\varepsilon/r) \left(\fint_{B_{2r}} |u_\varepsilon|^2 \right)^{1/2},$$

  where

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

  By rescaling it suffices to prove this for $r = 1$.

- This gives

$$H(\theta r) \le \frac{1}{2} H(r) + C\omega(\varepsilon/r) \inf_{q\in\mathbb{R}} \frac{1}{r} \left(\fint_{B_{2r}} |u_\varepsilon - q|^2 \right)^{1/2}.$$

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Convergence Rates in L^2

Lemma

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega$$

Then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\,\varepsilon^\alpha \|f\|_{H^1(\partial\Omega)}$$

for some $\alpha > 0$. If A is symmetric, we may take $\alpha = 1/2$.

Proof - Step 1

Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)\eta_\varepsilon S_\varepsilon(\nabla u_0).$$

Show that

$$\|\nabla w_\varepsilon\|_{L^2(\Omega)} \le C\Big\{\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_\varepsilon)} + \varepsilon\|\nabla u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}\Big\}$$

where

$$\Omega_\varepsilon = \Big\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\Big\}$$

This gives

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)}$$
$$\le C\Big\{\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_\varepsilon)} + \varepsilon\|\nabla u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}\Big\}$$

Proof - Step 2

Use interior estimates for $\mathcal{L}_0$ to show

$$\|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{1}{2}-\frac{1}{p}} \|\nabla u_0\|_{L^p(\Omega)}$$

for $p > 2$. Thus,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\,\varepsilon^{\frac{1}{2}-\frac{1}{p}} \|\nabla u_0\|_{L^p(\Omega)}$$

By a Meyers-type estimate,

$$\|\nabla u_0\|_{L^p(\Omega)} \leq C\,\|f\|_{H^1(\partial\Omega)}$$

for some $p > 2$, we obtain

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\,\varepsilon^{\sigma} \|f\|_{H^1(\partial\Omega)}$$

where

$$\sigma = \frac{1}{2} - \frac{1}{p} > 0$$

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Approximation

Theorem

Assume A is elliptic and periodic. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_2.$$

Then there exists $w \in H^1(B_1)$ such that

$$\mathcal{L}_0(w) = 0 \quad \text{in } B_1$$

and

$$\left(⨍_{B_1} |u_\varepsilon - w|^2 \right)^{1/2} \le C\,\varepsilon^\alpha \left(⨍_{B_2} |u_\varepsilon|^2 \right)^{1/2}$$

Proof

- Suppose $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_2. By Cacciopoli's inequality,

$$\left(\fint_{B_{3/2}} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{B_2} |u_\varepsilon|^2 \right)^{1/2}$$

- By co-area formula, there exists $t \in (1, 3/2)$ such that

$$\left(\fint_{\partial B_t} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{B_2} |u_\varepsilon|^2 \right)^{1/2}$$

- Let w be the solution to $\mathcal{L}_0(w) = 0$ in B_t and $w = u_\varepsilon$ on ∂B_t. Then

$$\left(\fint_{B_1} |u_\varepsilon - w|^2 \right)^{1/2} \leq C \varepsilon^\alpha \|u_\varepsilon\|_{H^1(\partial B_t)}$$

$$\leq C \varepsilon^\alpha \left(\fint_{B_2} |u_\varepsilon|^2 \right)^{1/2}$$

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Boundary Regularity
- Localization

Let

$$D_r = \Big\{(x', x_d): \ |x'| < r \quad \text{and} \quad \psi(x') < x_d < 10(M+1)r\Big\}$$

$$\Delta_r = \Big\{(x', x_d): \ |x'| < r \quad \text{and} \quad x_d = \psi(x')\Big\},$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$, $\psi(0) = 0$, $\|\nabla\psi\|_{C^\sigma} \leq M$.

Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_2 \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \Delta_2$$

Boundary Regularity - Localization

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Set-up for the Dirichlet Problem

Let

$$H(t) = \inf_{\substack{E\in\mathbb{R}^d \\ q\in\mathbb{R}}} \left\{ \frac{1}{t} \left(\fint_{D_t} |u_\varepsilon - E\cdot x - q|^2 \right)^{1/2} + t \left(\fint_{D_t} |F|^p \right)^{1/p} + \|\nabla_{tan}(f - E\cdot x)\|_{L^\infty(\Delta_t)} + t^\sigma \|\nabla_{tan}(f - E\cdot x)\|_{C^{0,\sigma}(\Delta_t)} \right\}$$

Set-up for the Neumann Problem

Let

$$H(t) = \inf_{\substack{E\in\mathbb{R}^d \\ q\in\mathbb{R}}} \Bigg\{ \frac{1}{t} \left(\fint_{D_t} |u_\varepsilon - E\cdot x - q|^2 \right)^{1/2} + t \left(\fint_{D_t} |F|^p \right)^{1/p} + \left\| g - \frac{\partial}{\partial \nu_0}(E\cdot x) \right\|_{L^\infty(\Delta_t)} + t^\sigma \left\| g - \frac{\partial}{\partial \nu_0}(E\cdot x) \right\|_{C^{0,\sigma}(\Delta_t)} \Bigg\}$$

where

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_2 \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \Delta_2$$

Calderón-Zygmund Estimates

Suppose A is elliptic, periodic, and belongs to VMO. Let Ω be C^1. Consider

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \mathrm{div}(f) \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega$$

Then, for $1 < p < \infty$,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C_p \, \|f\|_{L^p(\Omega)}$$

Almost-Periodic Homogenization

- $A(x)$ is called a trigonometric polynomial in $\mathbb{R}^d$ if

$$A(x) = \sum_{\ell} a_\ell \exp(i\lambda_\ell \cdot x) \quad \text{(finite sum)} ,$$

  where $a_\ell \in \mathbb{C}$ (or $\mathbb{C}^m$) and $\lambda_\ell \in \mathbb{R}^d$.
  e.g.

$$A(x) = \sin(2\pi x) + \cos(\sqrt{2}\pi x)$$

- $A(x)$ is called *uniformly almost-periodic* (almost-periodic in the sense of Bohr) in $\mathbb{R}^d$, if A is the uniform limit of a sequence of trigonometric polynomials in $\mathbb{R}^d$.

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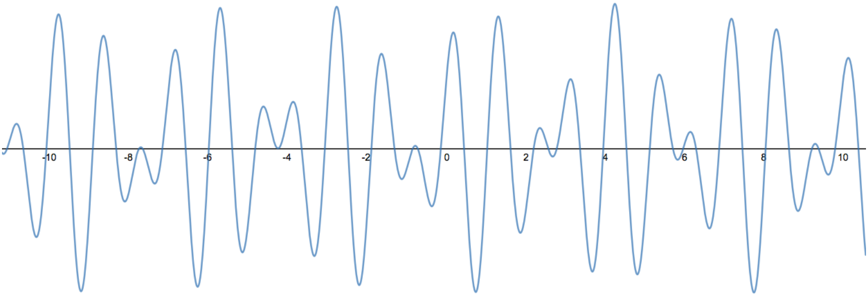

Almost-Periodic Homogenization

Theorem

Assume $A = A(y)$ is elliptic and almost-periodic. Let Ω be a bounded Lipchitz domain in $\mathbb{R}^d$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega,$$

where $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. Then

$$u_\varepsilon \to u_0 \quad \text{weakly in } H^1(\Omega)$$

and

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \quad \text{and} \quad u_0 = f \quad \text{on } \partial\Omega,$$

where $\mathcal{L}_0$ is an operator with constant coefficients.

The proof uses the Div-Curl Lemma and Weyl's decomposition

Approximate Correctors

Let

$$u = \chi_j^T$$

solve

$$\mathcal{L}(u) + T^{-2}u = -\mathcal{L}(y_j) \quad \text{in } \mathbb{R}^d.$$

and

$$\sup_x \int_{B(x,1)} \left(|\nabla u|^2 + |u|^2\right) dy < \infty$$

Study the behavior of χ^T, as $T \to \infty$

Quantify the Almost-Periodicity

Let

$$\rho(R) = \sup_{y\in\mathbb{R}^d} \inf_{\substack{z\in\mathbb{R}^d\\ |z|\le R}} \|A(\cdot + y) - A(\cdot + z)\|_\infty.$$

Observation:

Let $A(x)$ be a bounded continuous function in $\mathbb{R}^d$. Then

$A(x)$ is uniformly almost-periodic in $\mathbb{R}^d$
if and only if $\rho(R) \to 0$ as $R \to \infty$.

If $A(y)$ is periodic, then $\rho(R) = 0$ for R large.

Lipschitz Estimates in Almost-Periodic Homogenization

Theorem (S. Armstrong and S., 2016)

Suppose $A(y)$ is elliptic, uniformly almost-periodic, and Hölder continuous. Also assume that there exist $C_0 > 0$ and $N > 0$ such that

$$\rho(R) \le C_o \left[\log R\right]^{-N} \text{ for all } R > 2.$$

Let Ω be $C^{1,\alpha}$ and $p > d$. Then

1. *Lipschitz estimates hold for Dirichlet problem if $N > 5/2$,*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C\left\{\|F\|_{L^p(\Omega)} + \|u_\varepsilon\|_{C^{1,\sigma}(\partial\Omega)}\right\}.$$

2. *Lipschitz estimates hold for the Neumann problem if $N > 3$,*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C\left\{\|F\|_{L^p(\Omega)} + \|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\|_{C^{\sigma}(\partial\Omega)}\right\}.$$

Thank You